Return of the plane evolute

Singularities in the North 2025, Sundsvall

Boris Shapiro, Stockholm University, joint with C. Riener and R. Piene

August 25, 2025

Topics to discuss

Main references

(i) C. Huygens, Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricae, (1673).
https://archive.org/details/B-001-004-158/page/n59/mode/2up
(English translation, Richard J Blackwell, Christiaan Huygens' the pendulum clock, or, Geometrical demonstrations concerning the motion of pendula as applied to clocks.)

(ii) G. Salmon, A treatise on the higher plane curves: intended as a sequel to “A treatise on conic sections". 3rd ed. Chelsea Publishing Co., New York 1960 xix+395 pp.

(iii) R. Piene, C. Riener, B. Shapiro, Return of the plane evolute, Annales de l'institut Fourier, Febr. 2025, 67 pp.

As we usually teach our students in calculus classes, the *evolute* of a curve in the Euclidean plane is the *locus of its centers of curvature*.

The following intriguing information about evolutes can be found on Wikipedia.

“Apollonius (c. 200 BC) discussed evolutes in Book V of his treatise "Conics".

However, Huygens is usually credited with being the first to study them. Huygens formulated his theory of evolutes sometime around 1659 to help solve the problem of finding the tautochrone curve, which in turn helped him construct an isochronous pendulum. This was because the tautochrone curve is a cycloid, and the cycloid has the unique property that its evolute is also a cycloid. The theory of evolutes, in fact, allowed Huygens to achieve many results that would later be found using calculus."

Figure: Christiaan Huygens, 1629–1695

Among several dozens of books on (plane) algebraic curves available now only very few by Coolidge, Hilton and Salmon mention evolutes at all, the best of them being Salmon's book first published more than one and half century ago.

Some properties of evolutes have been studied in connection with the so-called 4-vertex theorem of Mukhopadhyaya-Kneser as well as its generalizations. Their definition has been generalized from the case of plane curves to that of plane fronts and also from the case of Euclidean plane to that of Poincaré disk.

Singularities of evolutes and involutes have been discussed in details by V. Arnold and his school and more recently by a group of Brazilian mathematicians (F. Scalco Dias and F. Tari).

From the computational point of view the most useful presentation of the evolute of a plane curve is as follows. Using a local parametrization of a curve Γ in $\mathbb{R}^2$ one can parameterize its evolute E_Γ as

$$E_\Gamma(t) = \Gamma(t) + \rho(t)\bar{n}(t), \tag{1}$$

where $\rho(t)$ is its curvature radius at the point $\Gamma(t)$ (assumed non-vanishing) and $\bar{n}(t)$ is the unit normal at $\Gamma(t)$ pointing towards the curvature center. In Euclidean coordinates, for $\Gamma(t) = (x(t), y(t))$ and $E_\Gamma(t) = (X(t), Y(t))$, one gets the following explicit expression

$$\begin{cases} X(t) = x(t) - \frac{y'(t)(x'(t)^2+y'(t)^2)}{x'(t)y''(t)-x''(t)y'(t)} \\ Y(t) = y(t) + \frac{x'(t)((x'(t))^2+(y'(t))^2)}{x'(t)y''(t)-x''(t)y'(t)} \end{cases} . \tag{2}$$

If a curve Γ is given by an equation $f(x,y)=0$, then the equation of its evolute can be obtained as follows. Consider the system

$$\begin{cases} f(x,y)=0 \\ X=x+\frac{f'_x((f'_x)^2+(f'_y)^2)}{2f'_xf'_yf''_{xy}.-(f'_y)^2f''_{xx}-(f'_x)^2f''_{yy}} \\ Y=y+\frac{f'_y((f'_x)^2+(f'_y)^2)}{2f'_xf'_yf''_{xy}.-(f'_y)^2f''_{xx}-(f'_x)^2f''_{yy}} \end{cases} . \tag{3}$$

defining the original curve and the family of centers of its curvature circles. Then eliminating the variables (x,y) from (3) one obtains a single equation defining the evolute in variables (X,Y). For concrete bivariate polynomials $f(x,y)$ of lower degrees, such an elimination procedure can be carried out in Macaulay 2.

Example

Two classically known explicit examples of the evolutes are as follows.

For the parabola $\Gamma = (t, t^2)$, its evolute is given by $E_\Gamma = (-4t^3, \frac{1}{2} + 3t^2)$ which is a semicubic parabola satisfying the equation $27X^2 = 16(Y - \frac{1}{2})^3$, see Figure below, left.
For the ellipse $\Gamma = (a\cos t, b\sin t)$, the evolute is given by

$$E_\Gamma = (\tfrac{a^2-b^2}{a}\cos^3 t, \tfrac{b^2-a^2}{b}\sin^3 t),$$

which is an astroid satisfying the equation $(aX)^{2/3} + (bY)^{2/3} = (a^2 - b^2)^{2/3}$, see Figure below, right.

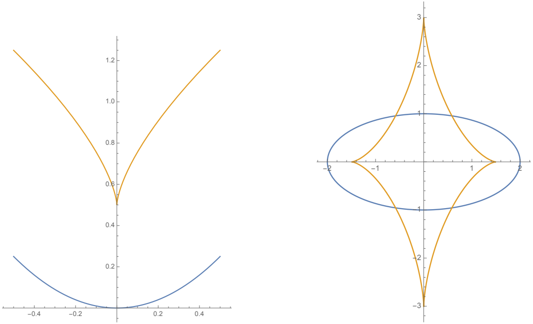


Figure: The evolutes of a parabola and an ellipse

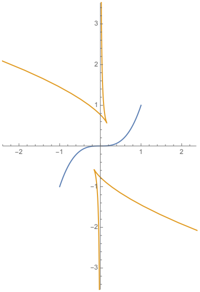


Figure: The evolute near an inflection point goes to ∞.

Observe that if Γ is a rational algebraic curve, then the above recipe provides the global parametrization of E_Γ. Given a plane curve Γ, the alternative definition of its evolute E_Γ which will be particularly useful for us is that E_Γ is the *envelope of the family of normals to* Γ, where a normal of Γ is an affine line perpendicular to Γ at some point.

In other words, each normal to Γ is a tangent line to E_Γ and each tangent to E_Γ is a normal to (the analytic continuation) of Γ. From the definition it follows that the evolute $E_\Gamma \subset \mathbb{R}^2$ is the caustic=critical locus of the projection of the cotangent bundle $T^*\Gamma \subset T^*\mathbb{R}^2$ to the initial curve Γ to the (phase) plane $\mathbb{R}^2$. This circumstance explains, in particular, why the singularities of the evolutes behave differently from those of (generic) plane algebraic curves.

Definition

For a plane algebraic curve $\Gamma \subset \mathbb{R}^2 \subset \mathbb{R}P^2$, define its *curve of normals* $\widetilde{N}_\Gamma \subset (\mathbb{R}P^2)^*$ as the curve on the dual projective plane whose points are the normals of Γ. (We start with the quasiprojective curve N_Γ of all normals to Γ and take its projective closure in $(\mathbb{R}P^2)^*$.)

Similarly to the above, for a (locally) parameterized curve $\Gamma(t) = (x(t), y(t))$ and $N_\Gamma(t) = (U(t), V(t))$, one gets

$$\begin{cases} U(t) = \frac{x'(t)}{y'(t)} \\ V(t) = -\frac{x(t)x'(t)+y(t)y'(t)}{y'(t)} \end{cases} . \qquad (4)$$

(Here we assume that the equation of the normal line to Γ at the point $(x(t), y(x))$ is taken in the form $Y + U(t)X + V(t) = 0$.)

Let us first summarize some complex-algebraic facts about the evolute and the curve of normals mainly borrowed from the classical treatise [Sa].

Proposition (see Art. 111, 112, p. 94–96, in [Sa])

For an affine real-algebraic curve $\Gamma \subset \mathbb{R}^2$ of degree d, in general position with respect to the line at infinity, and having only δ nodes and κ ordinary cusps as singularities, the curves $\widetilde{\Gamma}^{\mathbb{C}}$, $\widetilde{E}_\Gamma^{\mathbb{C}}$ and $\widetilde{N}_\Gamma^{\mathbb{C}}$ are birationally equivalent. The degree of $\widetilde{E}_\Gamma^{\mathbb{C}}$ equals $3d(d-1) - 6\delta - 8\kappa$, while the degree of $\widetilde{N}_\Gamma$ equals $d^2 - 2\delta - 3\kappa$. (Here $\widetilde{\Gamma}^{\mathbb{C}}$, $\widetilde{E}_\Gamma^{\mathbb{C}}$ and $\widetilde{N}_\Gamma^{\mathbb{C}}$ stand for projectivized and complexified curves.)

The genericity assumption for the birationality can be substantially weakened, but not completely removed.

Lemma (see Art. 113, p. 96, in [Sa])

For a generic affine real-algebraic curve $\Gamma \subset \mathbb{R}^2$ of degree d, $\widetilde{E}_\Gamma^{\mathbb{C}}$ has no inflection points.

Proposition (see Art. 113, p. 97, of [Sa])

For an affine real-algebraic curve $\Gamma \subset \mathbb{R}^2$ as in the above Proposition, the only singularities of $\widetilde{E}_\Gamma^{\mathbb{C}}$ and $\widetilde{N}_\Gamma^{\mathbb{C}}$ are nodes and cusps, except that $\widetilde{N}_\Gamma^{\mathbb{C}}$ has an ordinary d-uple point (the line at infinity). There are $\#_c(E) = 3(d(2d-3) - 4\delta - 5\kappa)$ cusps on $\widetilde{E}_\Gamma^{\mathbb{C}}$.
If Γ is nonsingular, there are $\#_n^E = \frac{d}{2}(3d-5)(3d^2-d-6)$ nodes on $\widetilde{E}_\Gamma^{\mathbb{C}}$ and $\#_n^N = \binom{d}{2}(d^2+d-4)$ nodes on $\widetilde{N}_\Gamma^{\mathbb{C}}$. There are no cusps on $\widetilde{N}_\Gamma^{\mathbb{C}}$ (since $\widetilde{E}_\Gamma^{\mathbb{C}}$ has no inflection points).

Recall that real nodes of real-algebraic curves are classically subdivided into *crunodes* and *acnodes* the former being transversal intersections of two real branches and the latter being transversal intersections of two complex conjugate branches.

Remark

Notice that a crunode of N_Γ (i.e., the real node with two real branches) corresponds to the *diameter* of Γ which is a straight segment connecting pairs of points on Γ and which is perdendicular to the tangent lines to Γ at these endpoints.

Observe also that a real cusp of E_Γ (resp. an inflection point on N_Γ) corresponds to a *vertex* of Γ which is a critical point of Γ's curvature.

As we mentioned above, vertices of plane curves appear, for example, in the classical 4-vertex theorem and its numerous generalizations.

Theorem (Syamadas Mukhopadhyaya, 1909)

The curvature along a simple, closed, smooth plane curve has at least four local extrema (specifically, at least two local maxima and at least two local minima).

Beautiful lower bounds on the number of diameters of plane curves, plane wavefronts as well as their higher dimensional generalizations have been obtained in symplectic geometry in the late 90's by P. Pushkar.

We need to remind of the following notion originally introduced by R. Thom in the 60's and which deserves to be better known.

Definition

Given a closed semi-analytic hypersurface $H \subset \mathbb{R}^n$ without boundary, we define its $\mathbb{R}$-*degree* as the supremum of the cardinality of $H \cap L$ taken over all lines $L \subset \mathbb{R}^n$ such that L intersects H transversally. (Observe that we count points in $H \cap L$ without multiplicity.) In what follows, we denote the $\mathbb{R}$-degree of H by $\mathbb{R}\deg(H)$.

For a real-algebraic (or piecewise real-algebraic) hypersurface $H \subset \mathbb{R}^n$, one has $\mathbb{R}\deg(H) \leq \deg(H)$ where $\deg(H)$ is the usual degree of H (respectively the degree of the Zariski closure of H). In particular, the $\mathbb{R}$-degree of a real-algebraic hypersurface is always finite which is, in general, not the case for real-analytic hypersurfaces.

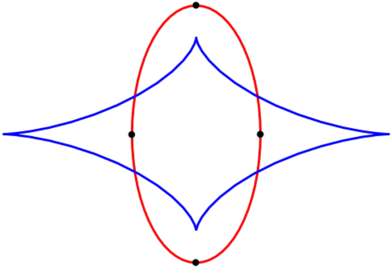


Figure: What is the value of $\mathbb{R}$-degree of an astroid?

We discuss four real-algebraic questions related to the evolutes and curves of normals of plane real-algebraic curves.

Problem (1)

For a given positive integer d, what are the maximal possible $\mathbb{R}$-degrees of the evolute E_Γ and of the curve of normals N_Γ where Γ runs over the set of all real-algebraic curves of degree d?

The usual degree of E_Γ is typically $3d(d-1)$ and the usual degree of N_Γ is d^2.

Problem (2)

For a given positive integer d, what its the maximal possible number of real cusps on E_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal number of vertices a real-algebraic curve Γ of degree d might have?

To make Problem (2) well-defined we have to assume that Γ does not have a circle as its irreducible component.

The number ot complex cusps of E_Γ is typically $3d(2d-3)$.

Problem (3)

For a given positive integer d, what its the maximal possible number of crunodes on N_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal number of (real) diameters Γ might have?

Here we again have to assume that Γ does not have a circle as its irreducible component.

The number of complex nodes on $\widetilde{N}_\Gamma$ generically equals $\binom{d}{2}(d^2 + d - 4)$.

Problem (4)

For a given positive integer d, what its the maximal possible number of crunodes (self-intersections) on E_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal possible number of points in $\mathbb{R}^2$ which are the centers for at least two distinct (real) curvature circles of Γ?

The number of complex nodes of E_Γ is typically $\frac{d}{2}(3d-5)(3d^2-d-6)$.

Remark

As we mentioned above, questions similar to Problems (2) and (3) have been studied in the classical differential geometry and symplectic theory.

They can also be connected to the study of plane curves of constant breadth which has been carried out by such celebrities as L. Euler, A. Hurwitz, H. Minkowski and W. Blaschke.

To the best of our knowledge, Problems (1) and (4) have not been previously discussed in the literature.

Proposition

For any $d \geq 3$, the maximal $\mathbb{R}$-degree among the evolutes of algebraic curves of degree d is not less than $d(d-2)$. (Typically its degree is $3d(d-1)$.)

Proof.

Each real inflection point of a real curve corresponds to its evolute going to infinity. By Klein's theorem at most one third of complex inflection points can be real and this bound is achieved. The number of complex inflection points of a generic curve of degree d equals $3d(d-2)$. Thus there exists a real curve of degree d with $d(d-2)$ real inflection points. The evolute of such curve hits the line at infinity (transversally) at $d(d-2)$ real points. Thus its $\mathbb{R}$-degree is at least $d(d-2)$. □

Remark

The above lower bound is apparently not sharp. For $d = 2$ the sharp bound is 4 although the degree of the evolute of a conic is 6. For $d = 3$, taking a small deformation of three lines creating a compact oval one gets an example with $\mathbb{R}$-degree of the evolute greater than or equal to 6 while the number of real inflections is 3.

The complex number is $3d(d-1)$ which has leading coefficient 3 while our bound has leading coefficient 1. The correct leading coefficient at d^2 is unknown at the moment.

Our second result solves the second part of Problem (1) about the maximal $\mathbb{R}$-degree of the curve of normals.

Proposition

There exists a real-algebraic curve Γ of degree d and a point $p \in \mathbb{R}^2$ such that all d^2 complex normals to Γ through p are, in fact, real. In other words, the maximal $\mathbb{R}$-degree of N_Γ equals d^2 which is the usual degree of N_Γ.

Proof: A crunode (which is a transversal intersection of two smooth real local branches) admits two types of real smoothing. By a classical theorem of Brusotti, any (possibly reducible) plane real-algebraic curve with only nodes as singularities admits a small real deformation which realizes independently prescribed smoothing types of all its crunodes.

Given a crunode and a point z such that the line L through this point and through the crunode is not tangent to the real local branches at the crunode, there exists a smoothing type of the crunode which we call *admissible* such that, slightly rotating the line L around z, one obtains two real normals to this smoothing.

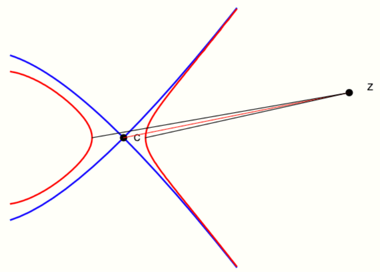
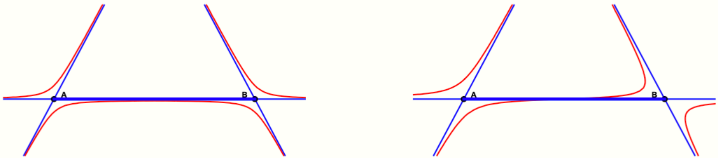


Figure: Admissible resolution of a crunode with respect to a point

Now take an arrangement $\mathcal{A} \subset \mathbb{R}^2$ of d real lines in general position and a point z outside these lines. By Brusotti, smoothing all $d(d-1)/2$ nodes in an appropriate way we obtain $d(d-1)$ normals close to the lines joining z with the nodes of $\mathcal{A}$. Additional d normals are obtained by small deformations of the altitudes connecting z with each of the d given lines. Thus, there exist d^2 real normals through z implying the $\mathbb{R}$-degree of the curve of normals for the obtained curve is a least d^2. But its usual degree is d^2. The result follows.

Useful notion for Problem (2)



(a) The smoothening respects the bounded edge. *(b)* The smoothening twists the bounded edge.

Proposition

The number of real cusps for the evolute of an arbitrary small deformation $\mathcal{R}$ of any generic line arrangement $\mathcal{A} \subset \mathbb{R}^2$ consisting of d lines equals $d(d-1)$ plus the number of bounded edges of $\mathcal{A}$ respected by $\mathcal{R}$.

All bounded edges of $\mathcal{A}$ are respected by $\mathcal{R}$ if and only if the small deformation $\mathcal{R}(\mathcal{A})$ is a convex curve; in this case the total number of real cusps on its evolute equals $d(2d-3)$ which is the maximal possible number among small deformations of line arrangements and is exactly 1/3 of $\#_c^E(d) = 3d(2d-3)$.

Proof: Consider the complement $\mathbb{R}^2 \setminus \mathcal{A}$. It consists of $2d$ infinite convex polygons and $\binom{d-1}{2}$ bounded convex polygons. Now take any small deformation $\mathcal{R}$ of $\mathcal{A}$. Locally near any vertex v of $\mathcal{A}$ the smooth curve $\mathcal{R}(\mathcal{A})$ will consists of two convex branches for each of which the curvature has a local maximum near v. These local maxima will correspond to two cusps on the evolute of $\mathcal{R}(\mathcal{A})$ which gives totally $2\binom{d}{2} = d(d-1)$ cusps corresponding to local maxima of curvature. Let us now show that every bounded edge of $\mathcal{A}$ respected by $\mathcal{R}$ corresponds to the unique point on $\mathcal{R}(\mathcal{A})$ where the curvature attains its minimum.

Moreover all extremal points of the curvature belong either to the first or to the second types. On the other hand, every twisted edge corresponds to an inflection point on $\mathcal{R}(\mathcal{A})$ which means that the evolute goes to infinity. The total number of bounded edges of any generic arrangement with d lines equals $d(d-2)$.

Clearly there exist exactly two small deformations for which all bounded edges will be respected, i.e. the deformed curve has no inflection points. In such case we get

$$d(d-1) + d(d-2) = d(2d-3)$$

extrema of curvature on $\mathcal{R}(\mathcal{A})$.

Remark

It is not clear that Klein's bound $1/3$ is valid for evolutes which are highly singular curves. We tried to apply Klein's equation to the evolute, but have not got any definite conclusion yet. Thus it is not clear at the moment whether our lower bound is optimal.

Discussing Problem (3)

Assume that we have a generic arrangement $\mathcal{A}$ of d lines in $\mathbb{R}^2$ meaning that no lines are parallel and no three intersect at the same point. As we mentioned above we can find a small real deformation of $\mathcal{A}$ which resolves each of $\binom{d}{2}$ nodes of $\mathcal{A}$ in a prescribed way. (There are 2 possible types of resolutions of each node and therefore $2^{\binom{d}{2}}$ types of resolutions of $\mathcal{A}$.)

It turns out that under some additional generality assumptions one can exactly count the number of real diameters of any such small resolution $\widetilde{\mathcal{A}}$. To move further we need several notions related to line arrangements and their small resolutions.

Notation: - A line arrangement $\mathcal{A} \subset \mathbb{R}^2$ is called *strongly generic* if in addition to the requirements that no two lines are parallel and no three lines intersect at the same point we require that no two lines are *perpendicular*.

By an *altitude* of a given line arrangement $\mathcal{A} \subset \mathbb{R}^2$ we call a straight segment connecting a vertex of $\mathcal{A}$ with a point on a line belonging to $\mathcal{A}$ perpendicular to this line. (Notice that if $\mathcal{A} \subset \mathbb{R}^2$ is strongly generic then no altitude of $\mathcal{A}$ connects its two vertices.)

Finally, we call a segment of a line belonging to $\mathcal{A}$ and connecting its two vertices a *side* of the arrangement $\mathcal{A}$.

Given two intersecting lines in $\mathbb{R}^2$, we say that a pair of opposite sectors of its complement form a *cone*. Thus the complement to the union of two lines consists of two disjoint cones. (The closure of a cone will be called a *closed cone*.)
Assume that we have chosen some type $\mathcal{R}$ of resolution of a given strongly generic arrangement $\mathcal{A}$ which means that at each vertex v of $\mathcal{A}$ we have (independently of other nodes) chosen one of two cones bounded by the lines whose intersection coincides with v. (Two sectors of this cone will stay disjoint under a local deformation while the two sectors of the other cone will merge together.) We will call the first cone *persistant* and the second one *vanishing*.

For a given line $\ell \subset \mathbb{R}^2$ and point $p \in \ell$, denote by $\ell^{\perp}(p)$ the line passing through p and orthogonal to ℓ. For a cone bounded by two lines ℓ_1 and ℓ_2 intersecting at some vertex v, define its *dual cone* as the union of all lines passing through v and such that every line is orthogonal to some line passing through v and belonging to the initial cone. (The dual cone is bounded by $\ell_1^{\perp}(v)$ and $\ell_2^{\perp}(v)$.)

Given a generic line arrangement $\mathcal{A} \subset \mathbb{R}^2$ of degree d, define its *derived arrangement* $\mathcal{DA} \subset \mathbb{R}^2$ of degree $d(d-1)$ as follows.

For any pair of lines ℓ_1 and ℓ_2 from $\mathcal{A}$, let v denote their intersection point. Then $\mathcal{DA}$ consists of all lines $\ell_1^\perp(v)$ and $\ell_2^\perp(v)$ where ℓ_1 and ℓ_2 run over all pairs of distinct lines in $\mathcal{A}$. If we choose some resolution $\mathcal{R}$ of $\mathcal{A}$ then at each vertex v of $\mathcal{A}$ we get the persistent cone $\mathcal{C}_v(\mathcal{R})$ and its dual persistent cone $\mathcal{C}_v^\perp(\mathcal{R})$ bounded by two lines of $\mathcal{DA}$ which are perpendicular to the lines of $\mathcal{A}$ those intersection gives v.

If *al* is an altitude starting at v and $\mathcal{R}$ is some resolution, we say that *al* is *admissible* w.r.t. $\mathcal{R}$ if it lies inside $\mathcal{C}_v(\mathcal{R})$ and *non-admissible* otherwise.

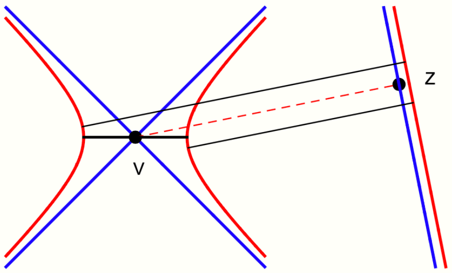
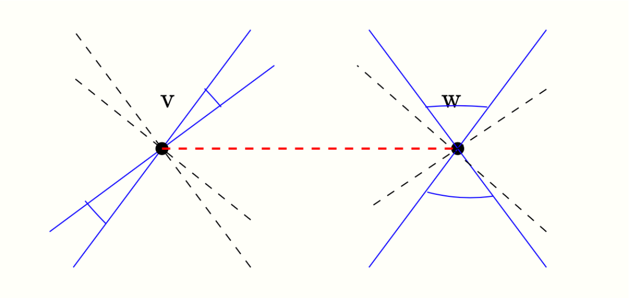


Figure: An altitude admissible with respect to a small deformation.

Finally, for a given strongly generic $\mathcal{A}$, its two vertices v_1 and v_2 and any resolution $\mathcal{R}$, we say that v_1 and v_2 *have each other in sight w.r.t.* $\mathcal{R}$ if $v_2 \in \mathcal{C}^{\perp}_{v_1}(\mathcal{R})$ and $v_1 \in \mathcal{C}^{\perp}_{v_2}(\mathcal{R})$.



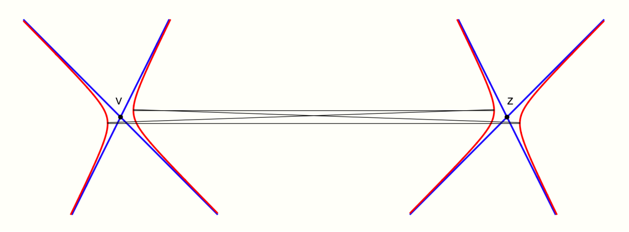


Figure: Two vertices having each other in sight w.r.t to $\mathcal{R}$ and 4 diameters created by $\mathcal{R}$.

Lemma

Given a strongly generic line arrangement $\mathcal{A}$, the following holds:

(i) Any small resolution of a vertex of $\mathcal{A}$ creates one diameter;

(ii) If an altitude al is admissible w.r.t. a small deformation $\mathcal{R}$ then $\mathcal{R}$ creates two diameters close to al;

(iii) If v_1 and v_2 have each other in sight w.r.t. a small deformation $\mathcal{R}$ then $\mathcal{R}$ creates four diameters close to the segment (v_1, v_2).

Proposition

Given any small resolution $\mathcal{R}$ of a strongly generic arrangement $\mathcal{A}$ consisting of d lines, the number of diameters of the obtained smooth curve $\mathcal{R}(\mathcal{A})$ equals

$$\#_{diam}(\mathcal{R}(\mathcal{A})) = \#_{ver} + 2\#_{adm.alt} + 4\#_{pairs\ in\ sight},$$

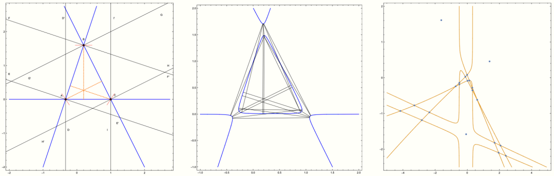
where $\#_{ver} = \binom{d}{2}$ is the number of vertices of $\mathcal{A}$, $\#_{adm.alt}$ is the number of admissible altitudes w.r.t. $\mathcal{R}$, and $\#_{pairs\ in\ sight}$ is the number of pairs of vertices having each other in sight.

Proof.

Indeed, firstly, any small resolution of a strongly generic line arrangements creates a short diameter connecting the two sectors of every persistent cone. Thus there are $\#_{ver} = \binom{d}{2}$ such short diameters.

Then each admissible altitude will split into two nearby diameters connecting the two sectors of the persistent cone with a point close to the base point of the altitude.

Finally, each pairs of dual persistent sectors in proper position centered at the vertices v_1 and v_2 creates 4 diameters close to the straight segment v_1, v_2. Such diameters connect each of the two sectors close to v_1 with each of two sectors close to v_2. No other diameters are possible. □



(a) Three lines (shown in blue), their altitudes (shown dotted in orange), and their derived arrangement (shown in black). Red lines indicate the smoothening.

(b) A smoothening creating a smooth triangle-shaped oval with 3 infinite arcs having 21 diameters, which are drawn in black.

(c) The curve of normals corresponding to the smoothening has 24 real singularities in the affine plane. 21 of these are crunodes, of which 18 are visible in this picture – there are two more out left and one more to the right.

Figure: The considered small resolution results in the maximal possible number of diameters in a small deformation of 3 lines.

Corollary

The number of diameters of any small resolution of any line arrangement of degree d does not exceed $\kappa(d)$ given by

$$\kappa(d) = \binom{d}{2} + 2\binom{d}{2}(d-2) + 4\binom{\binom{d}{2}}{2} = \frac{d^4}{2} - 3d^2 + \frac{5d}{2}. \quad (5)$$

Proof.

This upper bound is obtained if all altitudes and all pairs of persistent cones will contribute. □

The above Proposition implies the following lower bound for $\mathbb{R}Diam(d)$.

Proposition

In the above notation,

$$\mathbb{R}Diam(d) \geq \frac{d^4}{2} - d^3 + \frac{d}{2}. \tag{6}$$

To settle this Proposition we need to introduce some class of arrangements. We say that an arrangement $\mathcal{A}$ is *oblate* if the slopes of all lines in $\mathcal{A}$ are close to each other. As a particular example, one can take d lines tangent to the graph of $\arctan x$ for d values of the variable x of the form $101, 102, 103, \dots, 100 + d$.

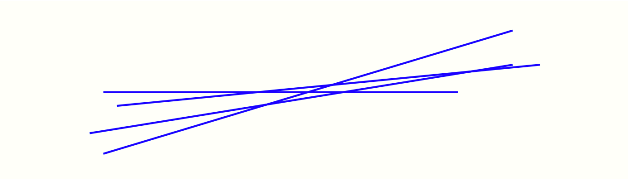


Figure: Example of an oblate arrangement of lines.

Sketch of proof.

For the special small resolution of an oblate arrangement for which is that we will making narrow cones at each vertex as the persistent ones the following diameters will be present. Every pair of persistent cones will be in proper position and contribute 4 diameters and each vertex will contribute 1 diameter. On the other hand, all altitudes will be non-admissible. Thus we get $4\binom{\binom{d}{2}}{2} + \binom{d}{2} = \frac{d^4}{2} - d^3 + \frac{d}{2}$ diameters for this resolution. □

Discussing Problem (4)

Recall that the number of complex nodes of the evolute of a generic curve of degree d is given by

$$\delta_E = \frac{1}{2}d(3d-5)(3d^2-d-6).$$

Denote by $\delta_E^{\mathrm{cru}}(d)$ the maximal number of crunodes for the evolutes of real-algebraic curves of degree d. At the moment we have only a rather weak lower bound for this number of crunodes, which we again obtain by resolving a line arrangement.

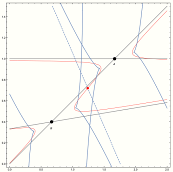
Proposition

We have $\delta_E^{\mathrm{cru}}(d) \geq (\lfloor \frac{d-2}{2} \rfloor + d - 2)^4 - \frac{1}{2}$.

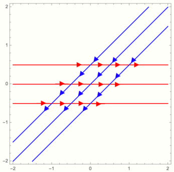
Proof. Take a line arrangement $\mathcal{A}$ and assume that in this arrangement we have a line which intersects two other lines in acute angles such that there is a bounded edge e between the two resulting nodes A and B. By Brusotti's theorem there exists a resolution $\mathcal{R}$ in which A and B are resolved in such a way that the long component of the resulting local branch $\mathcal{R}_e$ has curvatures of different signs near the two nodes, the tangent direction changes by more than 90 degrees near each of the nodes, and such that it twists e. We call such a resolution a "zig-zag". We know that the long component of $\mathcal{R}_e$ will have at least one inflection point as well as two maxima of the absolute value of curvature close to the vertices.

Considering the part of the evolute corresponding to $\mathcal{R}_e$, we have that these two maxima correspond to two cusps of the evolute, which, due to the different signs of curvature, are oriented towards each other. Furthermore, the existence of the inflection point results in a line, perpendicular to the compact line segment, which is an asymptote of the evolute, in such a way that from each of the two cusps one branch of the evolute approaches this asymptote. Furthermore, since the resolution can be chosen in such a way to ensure that the curvature radius becomes as large as necessary, the remaining two branches are guaranteed to intersect with the other branches (the described situation can also be seen in Figure below.

Now it follows that the two branches which tend to the asymptotical line are connected via the other two branches, and that the part of the evolute corresponding to $\mathcal{R}_e$ therefore contains a pseudo-line, Moreover, $\mathcal{R}$ can be chosen in such a way that every bounded edge that can be resolved in a zig-zag, contributes a pseudo-line in the evolute. For $d \geq 3$, consider a set of $\lfloor \frac{d}{2} \rfloor$ parallel lines as well as another set of $\lfloor \frac{d+1}{2} \rfloor$ parallel lines which are almost parallel to the first set of lines. The union of these two sets will yield a line arrangement with $(d-2) + \lfloor \frac{(d-2)^2}{2} \rfloor$ many bounded edges, and there exists a resolution of the line arrangement which resolves each of the bounded edges in a zig-zag. The resulting evolute thus has $(d-2) + \lfloor \frac{(d-2)^2}{2} \rfloor$ pseudo-lines, each of which will intersect pairwisely, yielding the announced lower bound of crunodes.



(a) An arrangement of three lines (in gray) together with a zigzag deformation in red, with the inflection point on the compact segment between **A** and **B**. The resulting evolute (in blue) has an asymptote (dashed in blue).



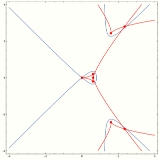
(b) Two sets of 3 lines. By a smoothening of the 6 lines with changing the colours at every vertex and following the orientation indicated by the arrows one obtains the desired smoothening.

Figure: An arrangement of three lines (in gray) together with a zigzag deformation in red.

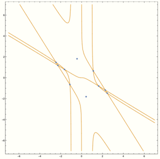
Final remarks

There is apparently a lot of space for improvement of the suggested bounds (which are very naive) as well as for other real-algebraic problems related to the evolutes, curves of normals and their high-dimensional analogs!

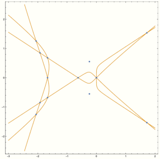
Some examples



(a) The nodal cubic in blue and its evolute in red with marked singular points.

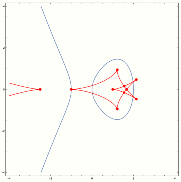


(b) The curve of normals for the latter nodal cubic in the (u, v) chart, with marked singular points.

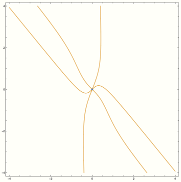


(c) The curve of normals in the (u, w) chart, with marked singular points.

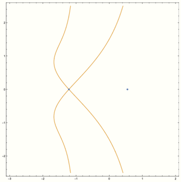
Figure: Nodal cubic.



(a) The Weierstrass cubic in blue and its evolute in red.

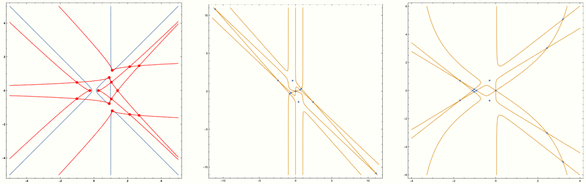


(b) The curve of normals in the (u, v) chart.



(c) The curve of normals in the (v, w) chart.

Figure: Weierstrass cubic.

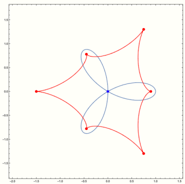


(a) The cubic in blue and its evolute in red.

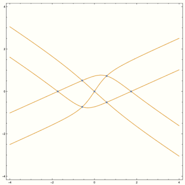
(b) The curve of normals in the (u, v) chart, with marked singular points.

(c) The curve of normals in the (u, w) chart, with marked singular points.

Figure: General cubic.

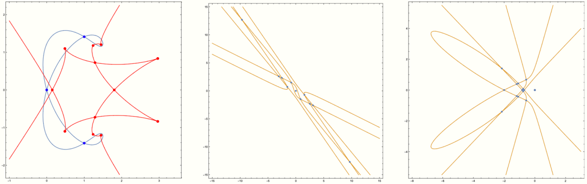


(a) The trifolium in blue with its evolute in red.



(b) The curve of normals in the (u, v) chart.

Figure: Trifolium.

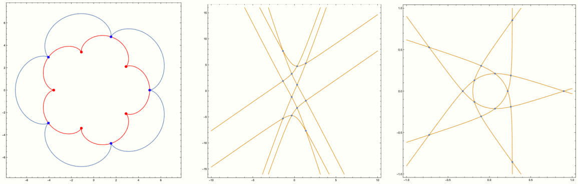


(a) The ampersand curve in blue with its evolute in red.

(b) The curve of normals in the (u, v) chart.

(c) The curve of normals in the (u, w) chart, with marked singular points.

Figure: Ampersand curve.



(a) The ranunculoid in blue with its evolute in red. *(b)* The curve of normals in the (u, v) chart. *(c)* The curve of normals in the (u, w) chart.

Figure: Ranunculoid.

Many thanks for your patience!

Stort tack för ert tålamod!

Muito obrigado pela sua paciência.!

Dziękuję bardzo za cierpliwość.!

V. I. Arnold - Hyugens and Barrow, Newton and Hooke, Birkhäuser (1990).

V. I. Arnold. Topological Invariants of Plane Curves and Caustics. AMS Univ. Lect. Ser., 5, Providence, 1994.

M. Baillif, Curves of constant diameter and inscribed polygons, Elem. Math. 64 (2009), 102–108.

L. Brusotti, Sulla “piccola variazione" di una curva piana algebrica reali. Rend. Rom. Ac. Lincei (5), **30**, 375–379 (1921).

F. Catanese, C. Trifogli, Focal loci of algebraic varieties, Commun. Algebra 28 (2000) 6017–6057.

J. L. Coolidge, A treatise on algebraic plane curves. Dover Publications, Inc., New York 1959 xxiv+513 pp.

- S. Di Rocco, Sandra, D. Eklund, M. Weinstein, The bottleneck degree of algebraic varieties, SIAM J. Appl. Algebra Geom., vol, 4(1), 2020, pp. 227–253.
- J. Draisma, E. Horobet, G. Ottaviani, B. Sturmfels, R. Thomas, The Euclidean Distance Degree of an Algebraic Variety, Found Comput Math (2016) 16:99–149.
- Dm. Fuchs, Evolutes and Involutes of Spatial Curves, The American Mathematical Monthly, Vol. 120, No. 3 (March 2013), pp. 217–231.
- T. Fukunaga and M. Takahashi, Evolutes of fronts in the Euclidean plane, Journal of Singularities Volume 10 (2014), 92-107.

M. Garai, On vanishing inflection points of plane curves, Ann. de l'institut Fourier, v. 52(3), 2002, 849–880.

D. A. Gudkov, The topology of real projective algebraic varieties, Uspekhi Mat. Nauk 29 (1974), 3–79 (Russian), English transl., Russian Math. Surveys 29:4 (1974), 1–79.

H. Hilton, Plane algebraic curves, Oxford University Press (1932), pp. xv+390.

C. Huygens, Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricae, (1673). https://archive.org/details/B-001-004-158/page/n59/mode/2up (English translation, Richard J Blackwell, Christiaan Huygens' the pendulum clock, or, Geometrical demonstrations concerning the motion of pendula as applied to clocks. Ames : Iowa State University Press, 1986.)

- A. Josse, F. Péne, On the degree of caustics by reflection, Comm. Algebra 42 (2014), 2442–2475.
- A. Josse, F. Pène, Normal class and normal lines of algebraic hypersurfaces. hal-00953669v4, 2014, 24.
- L. Lang, B. Shapiro, E. Shustin, On the number of intersection points of the contour of an amoeba with a line, Indiana Univ. Math. J., to appear.
- P. E. Pushkar', Diameters of immersed manifolds and of wave fronts. C. R. Acad. Sci. Paris Sér. Math. 326 (1998), no. 2, 201–205.

R. Piene, Polar classes of singular varieties, Ann. Sci. École Norm. Sup. (4) 11 (1978), no. 2, 247–276.

R. Piene, Higher order polar and reciprocal polar loci, to appear in *Facets of Algebraic Geometry: A Volume in honour of William Fulton's 80th Birthday,* Cambridge University Press, 2021; arXiv:2002.03691.

P. E. Pushkar', A generalization of Chekanov's theorem. Diameters of immersed manifolds and wave fronts. (Russian) Tr. Mat. Inst. Steklova 221 (1998), 289–304; translation in Proc. Steklov Inst. Math. 1998, no. 2(221), 279–295.

G. Salmon, A treatise on the higher plane curves: intended as a sequel to "A treatise on conic sections". 3rd ed. Chelsea Publishing Co., New York 1960 xix+395 pp.

- M. Salarinoghabi and F. Tari, Flat and round singularity theory of plane curves, Quart. J. Math. 68 (2017), 1289–1312.
- F. Scalco Dias and F. Tari, On vertices and inflections of plane curves, Journal of Singularities, Volume 17 (2018), 70-80
- S. Tabachnikov, The Four-Vertex Theorem Revisited–Two Variations on the Old Theme, The American Mathematical Monthly, Vol. 102, No. 10 (Dec., 1995), pp. 912–916.
- R. C. Yates, A Handbook on Curves and Their Properties, J. W. Edwards, Ann Arbor, Mich., 1947. x+245 pp.

- J. G. Yoder, Unrolling Time: Christiaan Huygens and the Mathematization of Nature. Cambridge University Press. (2004).
- O. Ya. Viro, Some integral calculus based on Euler characteristic. Topology and geometry – Rohlin Seminar, Lect. Notes in Math. 1346, Springer Verlag 1988, 127–138.
- Wall, C. T. C.: Duality of real projective plane curves: Klein's equation. Topology 35 (1996), 355–362.
- https://en.wikipedia.org/wiki/Evolute