On sequences of polynomials and rational functions satisfying finite recurrence relations

See http://arxiv.org/abs/math.CA/0409353

Boris Shapiro
Department of Mathematics, Stockholm University

Joint work with Rikard Bøgvad and Julius Borcea, to appear in *Advances in Mathematics*

1. A scheme of rational approximation of algebraic functions: main results

2. Generalizations

3. Applications and related problems

4. (Padé approximation: short overview)

1. Algebraic functions

Let f be an algebraic function and let

$$P(y,z)=\sum_{i=0}^{k} P_{k-i}(z)y^{k-i} \tag{1}$$

denote the irreducible polynomial in (y,z) defining $f(z)$, i.e., $P(f(z),z)=0$. Rewrite (1) as

$$-y^k=\sum_{i=1}^{k}\frac{P_{k-i}(z)}{P_k(z)}y^{k-i} \tag{2}$$

and consider the associated recursion of length $k+1$ with rational coefficients

$$-q_n(z)=\sum_{i=1}^{k}\frac{P_{k-i}(z)}{P_k(z)}q_{n-i}(z). \tag{3}$$

Choosing any initial k-tuple of rational functions $IN = \{q_0(z),\dots,q_{k-1}(z)\}$ one can generate a family of rational functions $\{q_n(z)\}_{n\in\mathbb{N}}$ satisfying (3) for all $n \geq k$ and coinciding with the entries of IN for $0 \leq n \leq k-1$. We study the family $\{r_n(z)\}_{n\in\mathbb{N}}$, where $r_n(z) = \frac{q_n(z)}{q_{n-1}(z)}$.

Preliminaries

Consider first a recurrence relation of length $k + 1$ with constant coefficients

$$-u_n = \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + \ldots + \alpha_k u_{n-k}, \quad (4)$$

where $\alpha_k \neq 0$. The *asymptotic symbol equation* of recurrence (4) is given by

$$t^k + \alpha_1 t^{k-1} + \alpha_2 t^{k-2} + \ldots + \alpha_k = 0. \quad (5)$$

The left-hand side of the above equation is called the *characteristic polynomial* of recurrence (4). Denote the roots of (5) by $\tau_1, \ldots, \tau_k$ and call them the *spectral numbers* of the recurrence.

Definition 1. Recursion (4) and its characteristic polynomial are said to be of *dominant type* if there exists a unique (simple) spectral number τ_{max} of this recurrence relation satisfying $|\tau_{max}| = \max_{1 \leq i \leq k} |\tau_i|$. Otherwise (4) and (5) are said to be of *nondominant type*. The number τ_{max} will be referred to as the *dominant spectral number* of (4) or the *dominant root* of (5).

Definition 2. The hypersurface $\Xi_k \subset \mathcal{P}_k$ given by the closure of the set of all nondominant polynomials is called the *standard equimodular discriminant*. For any family

$$\Gamma(y, z_1, \ldots, z_q) = \Big\{a_k(z_1, \ldots, z_q)y^k + a_{k-1}(z_1, \ldots, z_q)y^{k-1} + \ldots + a_0(z_1, \ldots, z_q)\Big\}$$

of irreducible polynomials of degree at most k in y we define the *induced equimodular discriminant* Ξ_Γ to be the set of all parameter values $(z_1, \ldots, z_q) \in \mathbb{C}^q$ for which the corresponding polynomial in y is nondominant. Given an algebraic function $f(z)$ defined by (1) we denote by Ξ_f the induced equimodular discriminant of (2) considered as a family of polynomials in the variable y.

Example 1. For $k = 2$ the equimodular discriminant $\Xi_2 \subset \mathcal{P}_2$ is the hypersurface consisting of all solutions to $\epsilon a_1^2 - 4a_0a_2 = 0$, where ϵ is a real parameter with values in $[1, \infty)$.

Lemma 1. *Let* $(\alpha_1, \ldots, \alpha_k)$ *be a* k*-tuple of complex numbers with* $\alpha_k \neq 0$*. For any function* $u : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ *the following conditions are equivalent:*

(i) *For all* $n \geq k$ *the numbers* u_n *satisfy* $-u_n = \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + \ldots + \alpha_k u_{n-k}$

(ii) $\sum_{n \geq 0} u_n t^n = \frac{Q_1(t)}{Q_2(t)}$*, where* $Q_1(t)$ *is a polynomial in* t *whose degree is smaller than* k *and* $Q_2(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k$*.*

(iii) *For all* $n \geq 0$ *one has*

$$u_n = \sum_{i=1}^{r} p_i(n) \tau_i^n, \tag{6}$$

where $\tau_1, \ldots, \tau_r$ *are the distinct spectral numbers of* (4) *with multiplicities* $m_1, \ldots, m_r$*, respectively, and* $p_i(n)$ *is a polynomial in the variable* n *of degree at most* $m_i - 1$ *for* $1 \leq i \leq r$*.*

By Definition 1 the dominant spectral number τ_{max} of any dominant recurrence relation has multiplicity one.

Definition 3. An initial k-tuple of complex numbers $\{u_0, u_1, \dots, u_{k-1}\}$ is called *fast growing* with respect to a given dominant recurrence of the form (4) if the coefficient κ_{max} of τ_{max}^n in (6) is nonvanishing, that is, $u_n = \kappa_{max}\tau_{max}^n + \dots$ with $\kappa_{max} \neq 0$. Otherwise the k-tuple $\{u_0, u_1, \dots, u_{k-1}\}$ is said to be *slow growing*.

Definition 4. Given an algebraic function f defined by (1) and an initial k-tuple of rational functions $IN = \{q_0(z), \dots, q_{k-1}(z)\}$ we define the *pole locus* $\Upsilon_{f,IN}$ associated with the data (f, IN) to be the union between the zero set of the polynomial $P_k(z)$ and the sets of all poles of $q_i(z)$ for $0 \leq i \leq k-1$.

Main results

Theorem 1. *Let* $f(z)$ *be an algebraic function defined by* (1)*. For any fixed initial* k*-tuple of rational functions* $IN = \{q_0(z), \dots, q_{k-1}(z)\}$ *there exists a finite set* $\Sigma_{f,IN} \subset \mathbb{CP}^1 \setminus (\Xi_f \cup \Upsilon_{f,IN})$ *such that*

$$r_n(z) = \frac{q_n(z)}{q_{n-1}(z)} \rightrightarrows y_{dom}(z) \text{ in } \mathbb{CP}^1 \setminus \mathcal{D}_f \text{ as } n \to \infty,$$

where $y_{dom}(z)$ *is the dominant root of equation* (2)*,* $\mathcal{D}_f = \Xi_f \cup \Upsilon_{f,IN} \cup \Sigma_{f,IN}$*, and* $\rightrightarrows$ *stands for uniform convergence on compact subsets of* $\mathbb{CP}^1 \setminus \mathcal{D}_f$*.*

The set $\Sigma_{f,IN}$ consists precisely of those $z \in \Omega$ such that $IN = \{q_0(z), \dots, q_{k-1}(z)\}$ is slowly growing w.r.t. (3) evaluated at z.

$\Sigma_{f,IN}$ = the *set of slow growth* associated with the data (f, IN).

Given $0 < \epsilon \ll 1$ set $\Theta_\epsilon = \mathbb{CP}^1 \setminus \mathcal{O}_\epsilon$, where $\mathcal{O}_\epsilon$ is the ϵ-neighborhood of $\mathcal{D}_f = \Xi_f \cup \Upsilon_{f,IN} \cup \Sigma_{f,IN}$ in the spherical metric on $\mathbb{CP}^1$.

Theorem 2. *For any sufficiently small $\epsilon > 0$ the rate of convergence of $r_n(z) \rightrightarrows y_{dom}(z)$ in Θ_ϵ is exponential, that is, there exist constants $\mathfrak{M}_\epsilon > 0$ and $q_\epsilon \in (0, 1)$ such that $|r_n(z) - y_{dom}(z)| \leq \mathfrak{M}_\epsilon q_\epsilon^n$ for all $z \in \Theta_\epsilon$.*

Definition 5. Given a meromorphic function g in some open set $\Omega \subseteq \mathbb{C}$ we construct its (complex-valued) *residue distribution* ν_g as follows. Let $\{z_m \mid m \in \mathbb{N}\}$ be the (finite or infinite) set of all the poles of g in Ω. Assume that the Laurent expansion of g at z_m has the form $g(z) = \sum_{-\infty < l \leq l_m} \frac{A_{m,l}}{(z-z_m)^l}$. Then the distribution ν_g is given by

$$\nu_g = \sum_{m \geq 1} \left(\sum_{1 \leq l \leq l_m} \frac{(-1)^{l-1}}{(l-1)!} A_{m,l} \frac{\partial^{l-1}}{\partial z^{l-1}} \delta_{z_m} \right), \quad (7)$$

where δ_{z_m} is the Dirac mass at z_m and the sum in the right-hand side of (7) is meaningful as a distribution in Ω since it is locally finite in Ω.

Recall that an integrable complex-valued distribution ρ in $\mathbb{C}$ and its Cauchy transform $\mathcal{C}_\rho(z)$ satisfy

$$\mathcal{C}_\rho(z) = \int_{\mathbb{C}} \frac{d\rho(\xi)}{z-\xi}, \quad \rho = \frac{1}{\pi}\frac{\partial \mathcal{C}_\rho}{\partial \bar{z}},$$

and that any meromorphic function g defined in $\mathbb{C}$ is the Cauchy transform of its residue distribution ν_g if $\int_{\mathbb{C}} d\nu_g(\xi) < \infty$.

Definition 6. Given a family $\{\phi_n(z)\}_{n\in\mathbb{N}}$ of smooth functions defined in some open set $\Omega \subseteq \mathbb{C}$ one calls the limit $\Phi(z) = \lim_{n\to\infty} \frac{\phi_{n+1}(z)}{\phi_n(z)}$ the *asymptotic ratio* of the family, provided that this limit exists in some open subset of Ω. If $\{\phi_n(z)\}_{n\in\mathbb{N}}$ consists of analytic functions and ν_n denotes the residue distribution of the meromorphic function $\frac{\phi_{n+1}(z)}{\phi_n(z)}$ in Ω then the limit $\nu = \lim_{n\to\infty} \nu_n$ (if it exists in the sense of weak convergence) is called the *asymptotic ratio distribution* of the family.

Theorem 3. *Let $f(z)$ be an algebraic function defined by* (1) *and fix an initial k-tuple of rational functions $IN = \{q_0(z), \dots, q_{k-1}(z)\}$. If ν_n denotes the residue distribution of $r_n(z)$ and $\nu = \lim_{n\to\infty} \nu_n$ is the asymptotic ratio distribution of the family $\{q_n(z)\}_{n\in\mathbb{N}}$ then the following holds:*

(i) $\operatorname{supp} \nu$ *does not depend on the set of slow growth $\Sigma_{f,IN}$.*

(ii) Suppose there is a nonisolated point $z_0 \in \Xi_f$ such that equation (2) *considered at z_0 has the property that among its roots with maximal absolute value there are at least two with the same maximal multiplicity. Then* $\operatorname{supp} \nu = \Xi_f$ *provided that $\{r_n(z_0)\}_{n\in\mathbb{N}}$ diverges.*

(iii) One has

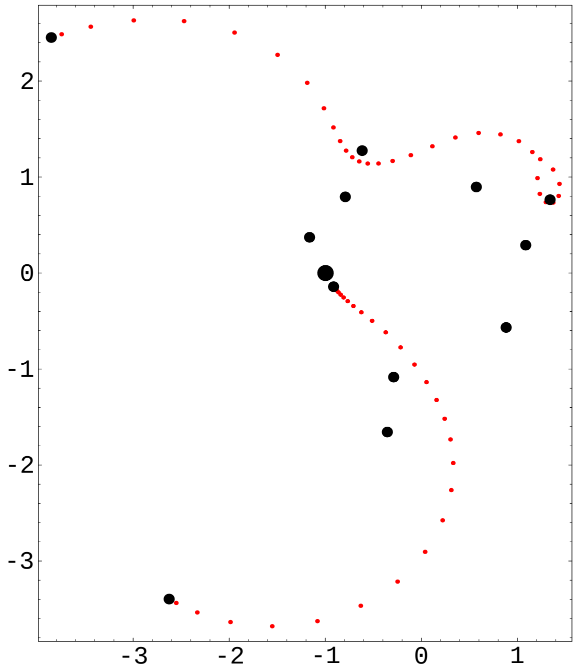
$$\nu = \frac{1}{\pi}\frac{\partial y_{dom}}{\partial \bar{z}} \iff y_{dom}(z) = \int_{\mathbb{C}} \frac{d\nu(\xi)}{z - \xi}.$$

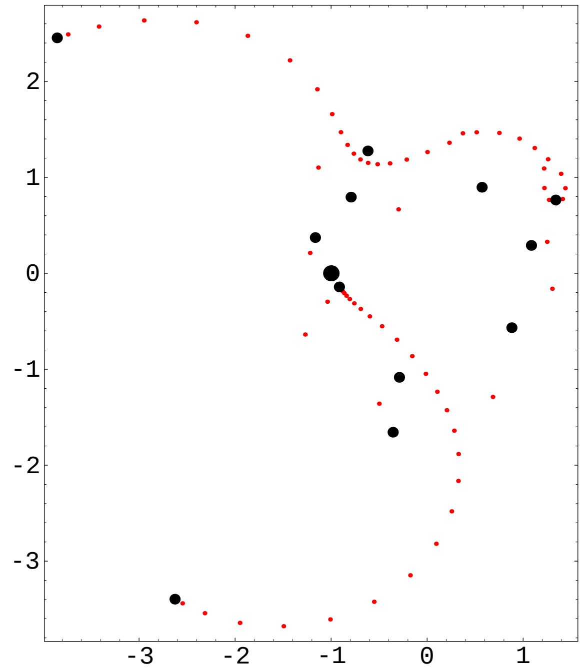
Corollary 1. *For any algebraic function $f(z)$ and any initial k-tuple IN of rational functions the set of all poles of the family $\{r_n(z)\}_{n\in\mathbb{N}}$ splits asymptotically into the following three types:*

(i) The fixed part consisting of a subset of $\Upsilon_{f,IN} \setminus (\Xi_f \cup \Sigma_{f,IN})$;

(ii) The regular part tending asymptotically to the finite union of curves Ξ_f, i.e., the induced equi-modular discriminant of (2) *(cf. Theorem 1);*

(iii) The spurious part tending to the (finite) set of slow growth $\Sigma_{f,IN}$.





Poles of $r_{31}(z)$ approximating the branch with maximal absolute value of the algebraic function $f(z)$ with defining equation $(z+1)y^3 = (z^2+1)y^2 + (z - 5I)y + (z^3 - 1 - I)$ for the initial triples $p_{-2}(z) = p_{-1}(z) = 0$, $p_0(z) = 1$ and $p_{-2}(z) = z^5 + Iz^2 - 5$, $p_{-1}(z) = z^3 - z + I$, $p_0(z) = 1$, respectively.

Theorem 4. *For any algebraic function $f(z)$ and any initial k-tuple IN of rational functions there exists a finite upper bound for the total number of spurious poles of the sequence of approximants $\{r_n(z)\}_{n\in\mathbb{N}}$ associated with the data (f, IN).*

Can actually show that $\exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$

$$Z(q_n) \leq \max\Big(Z(q_0), \ldots, Z(q_{N-1}), |\Sigma_{f,IN}|\Big).$$

Corollary 2. *If f is an algebraic nonrational function and $IN = \{0, 0, \ldots, 1\}$ then $\Sigma_{f,IN} = \emptyset$. In particular, the rational approximants $\{r_n(z)\}_{n\in\mathbb{N}}$ corresponding to the standard initial k-tuple have no spurious poles.*

2. Generalizations

I. Varying coefficients and Poincaré's theorem.

Theorem 5. *If the coefficients* $Q_i(x),\ i = 0, 1, \ldots, k-1$ *of a linear homogeneous difference equation*

$$f(x+k)+Q_{k-1}(x)f(x+k-1)+Q_{k-2}(x)f(x+k-2)+$$
$$+\ldots+Q_0(x)f(x) = 0 \quad (*)$$

have limits $\lim_{x\to\infty} Q_i(x) = a_i, \quad i = 0, 1, \ldots, k-1$ *and if the roots of the characteristic equation*

$$\lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_0 = 0 \quad (**)$$

have different absolute values then the limit of the ratio $\frac{f(x+1)}{f(x)}$ *for* $x \to \infty$ *of any solution* $f(x)$ *of the equation (*) equals one of the roots*

$$\lambda_1, \lambda_2, \ldots, \lambda_k$$

of the equation (**), i.e.

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \lambda_p.$$

A recent generalization of Poincare's theorem to the case of one dominant root of (**) obtained by S.Friedland claims that under some nondegeneracy assumptioms

there exists a hyperplane in the space of initial values such that in its complement one has

$$\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = \lambda_{dom}.$$

Corollary 3. *Most of our results are valid in the case of varying polynomial coefficients having well-defined polynomial limits.*

Remark 1. There are still unsettled problems with uniformity of convergence and smoothness of dependense of the latter hyperplane on parameters.

II. Multidimensional case.

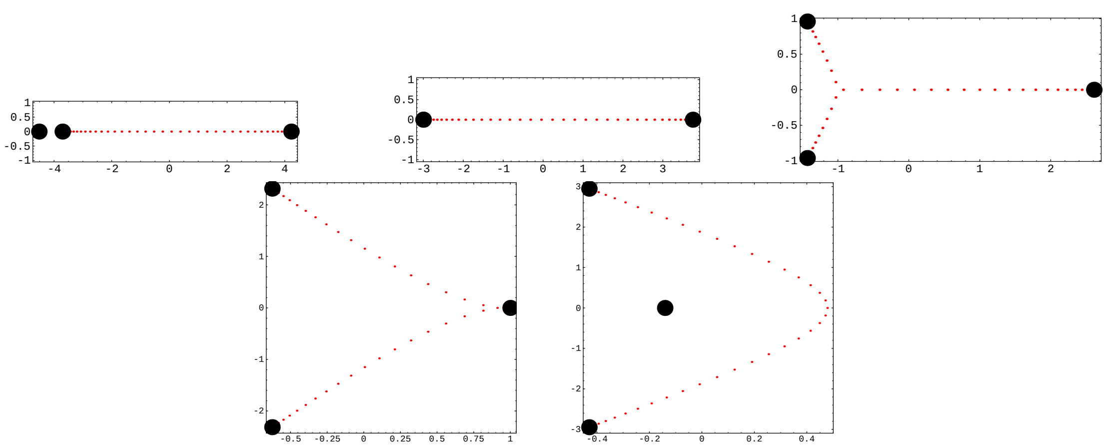
Similar statements for reccurence relations with fixed smooth (analytic, algebraic) coefficients in $\mathbb{R}^m$.

3. Applications and related problems

The 3-conjecture (Egecioglu, Redmond & Ryavec). *All polynomials in the sequence* $\{p_n(z)\}_{n\in\mathbb{N}}$ *defined by the 4-term recursion*

$$p_n(z) = zp_{n-1}(z) - Cp_{n-2}(z) - p_{n-3}(z), \text{ where}$$
$$p_{-2}(z) = p_{-1}(z) = 0,\ p_0(z) = 1 \text{ and } C \in \mathbb{R},$$

have real zeros if and only if $C \geq 3$*. If* $C > 3$ *then the zeros of* $p_{n+1}(z)$ *and* $p_n(z)$ *are interlacing* $\forall n \in \mathbb{N}$*.*



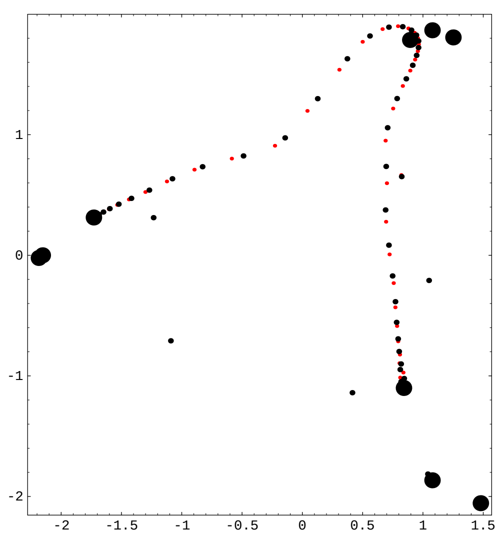
Zeros of $p_{41}(z)$ satisfying the above relation for $C = 4,\ 3,\ 1,\ -1,\ -2$.

Related problems

Problem 1. ("Interlacing along complex curves") Let $\{p_n(z)\}_{n\in\mathbb{N}}$ be any polynomial family satisfying

$$p_{n+1}(z) = \sum_{i=1}^{k} Q_i(z) p_{n-i}(z)$$

with deg $p_n(z) = n \ \forall n \in \mathbb{N}$. The zeros of $p_{n+1}(z)$ and $p_n(z)$ interlace along the curve Ξ_Q for all sufficiently high degrees n.



Zeros of $p_{40}(z)$ and $p_{41}(z)$ defined by $p_{n+1}(z) = (z+1-I)p_n(z) + (z+1)(z-I)p_{n-1}(z) + (z^3 + 10)p_{n-2}(z)$.

Problem 2. Find the relation between the asymptotic root counting measure μ and the asymptotic ratio distribution ν for general polynomial families. Is it true that ν depends only on μ, i.e., can two polynomial families with the same asymptotic root counting measure have different asymptotic ratio distributions?

4. Padé approximation

$\mathcal{P}_k = \{$complex polynomials of degree at most $k\}$.

Definition 7. Let $f(z)$ be a function analytic at ∞. For each pair $(m,n) \in \mathbb{N}^2$ there exist two polynomials $p_{m,n} \in \mathcal{P}_m$ and $q_{m,n} \in \mathcal{P}_n \setminus \{0\}$ such that

$$q_{m,n}\left(\frac{1}{z}\right) f(z) - p_{m,n}\left(\frac{1}{z}\right) = \mathcal{O}(z^{-m-n-1}) \quad (8)$$

as $z \to \infty$. The rational function

$$[m/n](z) := \frac{p_{m,n}(1/z)}{q_{m,n}(1/z)} \quad (9)$$

is called the (m,n)-Padé approximant to the function $f(z)$ expanded at ∞.

(8)$\Rightarrow$ Padé approximants are well-defined and exist uniquely for any such f.

Padé approximants to functions analytic near 0 are defined in similar fashion.

Padé approximants may be seen as a generalization of Taylor polynomials to the field of rational functions.

Applications to/related to continued fractions, (general) orthogonal polynomials, moment problems, quadratic differentials, physical problems, etc (see e. g. Baker & Graves-Morris). These connections are simpler and particularly useful in the case of *diagonal Padé approximants* $[n/n](z)$.

Nuttall-Pommerenke Theorem (1973). If f has no branching points in $\mathbb{CP}^1$ and analytic at ∞ then the sequence of diagonal Padé approximants $\{[n/n](z)\}_{n\in\mathbb{N}}$ converges to $f(z)$ in planar measure.

There are much deeper analogs of this theorem for functions with branch points due to H. Stahl ('98) which show that the Nuttall-Pommerenke Theorem holds if f has singularities of (logarithmic) capacity 0, and planar measure may be replaced by capacity.

A consequence of Stahl's results is that *most* of the poles of the diagonal Padé approximants $[n/n]$ to f tend to a set K_0 as $n \to \infty$, where K_0 is the union of trajectories of a certain quadratic differential and a (finite) set of isolated points.

However, not all of them do! Indeed, even in the case when $f(z)$ is algebraic one cannot hope for a better convergence type than convergence in (logarithmic) capacity. Uniform convergence in $\mathbb{CP}^1 \setminus K_0$ can fail in a rather dramatic way:

Example 2. Set $[n/n](z) = \frac{p_{n,n}(1/z)}{q_{n,n}(1/z)}$ and define the reverse denominator polynomial Q_n of $q_{n,n}$ by

$$Q_n(z) = z^n q_{n,n}\left(\frac{1}{z}\right).$$

Consider the (algebraic) function

$$f(z) = \frac{(z - \cos\pi\alpha_1)(z - \cos\pi\alpha_2)}{(z^2-1)^{\frac{3}{2}}} - z + (\cos\pi\alpha_1 + \cos\pi\alpha_2),$$

where $1, \alpha_1, \alpha_2$ are rationally independent numbers. One can show that the system $\{Q_n(z)\}_{n\in\mathbb{N}}$ is orthogonal w.r.t. $\frac{dx}{\pi\sqrt{1-x^2}}$ on $[-1,1]$ and that each polynomial $Q_n(z)$ has at most two zeros outside $[-1,1]$. However, one has $\cap_{n=1}^{\infty}\overline{\cup_{m\geq n} Z(Q_m)} = \mathbb{C}$, i.e., the zeros of $Q_n(z)$ cluster everywhere in $\mathbb{C}$ as $n \to \infty$.

This pathological behavior is due to the presence of so-called *spurious poles* for Padé approximants.

Definition 8. Let f be a function satisfying the conditions of Definition 7. Let further $\mathcal{N} \subseteq \mathbb{N}$ be an infinite sequence and $\{[n/n]\}_{n\in\mathcal{N}}$ be the corresponding subsequence of diagonal Padé approximants to f. We define *spurious poles* in two different situations:

(i) Assume that for each $n \in \mathcal{N}$ the approximant $[n/n]$ has a pole at $z_n \in \mathbb{CP}^1$ such that $z_n \to z_0$ as $n \to \infty$, $n \in \mathcal{N}$. If f is analytic at z_0 and the approximants $[n/n]_{n\in\mathcal{N}}$ converge in capacity to f in some neighborhood of z_0 then the poles of the approximants $[n/n]$ at z_n, $n \in \mathcal{N}$, are called *spurious*. In case $z_0 = \infty$ the convergence $z_n \to z_0$ has to be understood in the spherical metric.

(ii) Let f have a pole of order k_0 at z_0 and assume that for each $n \in \mathcal{N}$ the total order of poles of the approximant $[n/n]$ near z_0 is $k_1 = k_{1,n} > k_0$. Assume further that $[n/n]$ has poles at $z_{n,j}$, $j \in \{1, \ldots, m_n\}$, of total order $k_{1,n}$ and that for any selection of $j_n \in \{1, \ldots, m_n\}$ one has $z_{n,j_n} \to z_0$ as $n \to \infty$, $n \in \mathcal{N}$. Then poles of order $k_1 - k_0$ out of all poles of the approximants $[n/n]$ near z_0, $n \in \mathcal{N}$, are called *spurious*.

A naive definition of spurious poles: poles of Padé approximants which are not located near the set K_0.

The spurious poles phenomenon was observed already in the 60's by Baker *et al*: several of the approximants could have poles which in no way relate to those of the underlying function f.

However, spurious poles seem to affect the convergence pattern only in small neighborhoods. Moreover, there are apparently only very few "bad" approximants. After omitting these "bad" approximants one can expect to get convergence for the sequence of the remaining approximants.

Padé (Baker-Gammel-Wills) Conjecture (1961). *If f is meromorphic in the unit disk $\mathbb{D}$ and analytic at 0, and if $\{[n/n]\}_{n\in\mathbb{N}}$ denotes the sequence of Padé approximants to f expanded at the origin, then there exists an infinite subsequence $\mathcal{N} \subseteq \mathbb{N}$ such that*

$$[n/n](z) \rightrightarrows f(z) \text{ in } \mathbb{D} \setminus \{\textit{poles of f}\}$$

as $n \to \infty$, $n \in \mathcal{N}$, where $\rightrightarrows$ denotes uniform convergence *on compact sets.*

Theorem (D. Lubinsky, Ann. of Math. (2003)). Let H_q be the Rogers-Ramanujan continued fraction

$$H_q(z) = 1 + \frac{qz|}{|1} + \frac{q^2z|}{|1} + \frac{q^3z|}{|1} + \dots$$

and set $\frac{A_n(z)}{B_n(z)} = 1 + \frac{qz|}{|1} + \frac{q^2z|}{|1} + \dots \frac{q^nz|}{|1}$. If $q := \exp\left(\frac{4\pi i}{99+\sqrt{5}}\right)$ then H_q is meromorphic in $\mathbb{D}$ and analytic at 0. There does not exist any subsequence of $\{A_n/B_n\}_{n\in\mathbb{N}}$ that converges uniformly in all compact subsets of $\mathcal{A} := \{z : |z| < 0.46\}$. In particular, no subsequence of $\{[n/n]\}_{n\in\mathbb{N}}$ can converge uniformly in all compact subsets of $\mathcal{A}$ omitting poles of H_q.

However, the important question whether the Padé Conjecture could hold for certain classes of *algebraic* functions is still open. For this one needs a deeper understanding of the asymptotic distribution of spurious poles of diagonal Padé approximants.

Nuttall's Conjecture. *Let f be an algebraic function which is analytic at ∞. Then there exists an upper bound for the number of spurious poles for all Padé approximants $[n/n]$, $n \in \mathbb{N}$.*

In what follows we construct a new scheme of approximation of any multivalued algebraic function $f(z)$ by a sequence $\{r_n(z)\}_{n\in\mathbb{N}}$ of rational functions.

Compared to the usual Padé approximation this new scheme has a number of advantages that allow us to control the behavior of spurious poles and to prove natural analogs of the Padé Conjecture and Nuttall's Conjecture for the sequence $\{r_n(z)\}_{n\in\mathbb{N}}$ in the complement $\mathbb{CP}^1 \setminus \mathcal{D}_f$, where $\mathcal{D}_f$ is the union of a finite number of segments of real algebraic curves and finitely many isolated points.

Comparison of approximation schemes

1. For Padé one needs the Taylor expansion (at ∞) of an algebraic function $f(z)$, the above scheme uses only the defining algebraic equation for $f(z)$.

2. The regular poles of Padé approximants concentrate on the union of trajectories of a certain quadratic differential. The induced equimodular discriminant Ξ_f is easier to study.

3. Padé approximants have spurious poles with uncontrolled behavior (destroys uniform convergence). Our scheme has a finite number of spurious poles tending to a “nice” set $\Sigma_{f,IN}$ and a well-controlled (exponential) rate of convergence.

4. The denominators of the rational approximants constructed above satisfy a simple recursion with fixed rational coefficients. The denominators of Padé approximants satisfy a 3-term recurrence relation with varying coefficients (difficult to calculate & chaotic behavior).