On spectral asymptotics of quasi-exactly solvable quartic potential
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QES-quartic

A quasi-exactly solvable quartic oscillator was introduced by C. M. Bender and S. Boettcher and (in its restricted form) is a Schrödinger-type eigenvalue problem of the form

$$L_J(w) = w'' - (x^4/4 - ax^2/2 - Jx)w = \lambda w \qquad (1)$$

with the boundary conditions $w(t) \to 0$ and $w(te^{2\pi i/3}) \to 0$ as $t \to +\infty$, where $a \in \mathbb{C}$ and J are parameters of the spectral problem. With these boundary conditions, real a and J, (1) is not Hermitian but is PT-symmetric.

In case when $J = n + 1$ is a positive integer, then $L_{n+1}(w)$ maps the linear space of quasi-polynomials of the form

$$\{pe^{-x^3/6+ax/2} : \deg p \leq n\}$$

to itself where p runs over the linear space of polynomials of degree at most n. The restriction of L_{n+1} to the latter space is a finite-dimensional linear operator whose spectrum and eigenfunctions can be found explicitly using linear algebra. This part of the spectrum and eigenfunctions of (1) is usually referred to as *solvable*.

One can easily show that polynomial factors p in the quasi-exactly solutions $w = pe^h$ of (1) coincide with the polynomial solutions of the degenerate Heun equation

$$y'' - (x^2 - a)y' + (\alpha x - \lambda)y = 0, \tag{2}$$

where $a \in \mathbb{C}$ has the same meaning as above and (α, λ) are the spectral variables. Obviously, if equation (2) has a polynomial solution of degree n, then $\alpha = n$. Furthermore, to get a polynomial solution of (2) of degree n, the remaining spectral variable λ should be chosen as an eigenvalue of the operator

$$T_n(y) = y'' - (x^2 - a)y' + nxy$$

when it acts on the linear space of polynomials of degree at most n.

In the standard monomial basis $\{1, x, x^2, \ldots, x^n\}$, this action of the operator T_n is given by the 4-diagonal $(n+1) \times (n+1)$-matrix $M_n^{(a)}$ of the form

$$M_n^{(a)} := \begin{pmatrix} 0 & a & 2 & 0 & 0 & \cdots & 0 \\ n & 0 & 2a & 6 & 0 & \cdots & 0 \\ 0 & n-1 & 0 & 3a & 12 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 & 0 & (n-1)a & n(n-1) \\ 0 & 0 & \cdots & 0 & 2 & 0 & na \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (3)$$

We will call the bivariate polynomial $Sp_n(a, \lambda) := \det(M_n^{(a)} - \lambda I)$ the *n-th spectral polynomial of* (2). The degree of polynomial $Sp_n(a, \lambda)$ equals $n + 1$ which is also its degree with respect to the variable λ. (The maximal degree of the variable a in $Sp_n(a, \lambda)$ equals $\left[\frac{n+1}{2}\right]$). Additionally observe that if a is a real number then $Sp_n(a, \lambda)$ is a real polynomial in λ and therefore the spectrum of $M_n^{(a)}$ is symmetric with respect to the real axis.

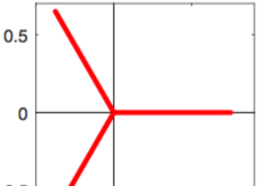
The main goal of this project is to study the asymptotics of the spectrum of $M_n^{(a_n)}$ in two different regimes. In the first regime we require that $\lim_{n\to\infty} \frac{a_n}{n^{2/3}} = 0$ while in the second regime we require that $\lim_{n\to\infty} \frac{a_n}{n^{2/3}} = A \neq 0$. (Our symbolic-numerical experiments indicate that if $\lim_{n\to\infty} \frac{a_n}{n^{2/3}}$ does not exist one can not expect any interesting limiting behavior of the spectrum).

First result

Theorem

(i) If $\lim_{n\to\infty} \frac{a_n}{n^{2/3}} = 0$, the maximal absolute value $r_n(a_n)$ of the eigenvalues of $M_n^{(a_n)}$ grows as $\frac{3}{4}n^{4/3}$.

(ii) If $\lim_{n\to\infty} \frac{a_n}{n^{2/3}} = 0$, the sequence $\{\mu_n^{(a_n)}\}$ of root-counting measures for the rescaled spectra of $\{M_n^{(a_n)}\}$ where each eigenvalue of $M_n^{(a_n)}$ is divided by $n^{4/3}$, weakly converges to the measure ν_0 supported on the union of three straight intervals connecting the origin with three cubic roots of $\frac{27}{64}$, see Figure 1.



Second result

Our second result is obtained on the physics level of rigor, i.e., modulo certain convergence assumptions. To formulate it, take the family of equations

$$\mathcal{C}^2 - (x^2 - A)\mathcal{C} + (x - \beta) = 0 \tag{4}$$

which we consider as quadratic equations in the variable $\mathcal{C}$ depending on the space variable x and parameters A and β.

Proposition

If $\lim_{n\to\infty} \frac{a_n}{n^{2/3}} = A \neq 0$, then (under three additional convergence assumptions) the sequence $\{\mu_n^{(a_n)}\}$ of root-counting measures for the rescaled spectra $\{Sp_n(a_n, \beta n^{4/3})\}$ of $M_n^{(a_n)}$ weakly converges to a special compactly supported probability measure ν_A. In particular, the support of ν_A consists of all values of the spectral parameter β for which there exists a compactly supported probability measure κ in the x-plane whose Cauchy transform $\mathcal{C}_\kappa(x)$ satisfies (4) almost everywhere.

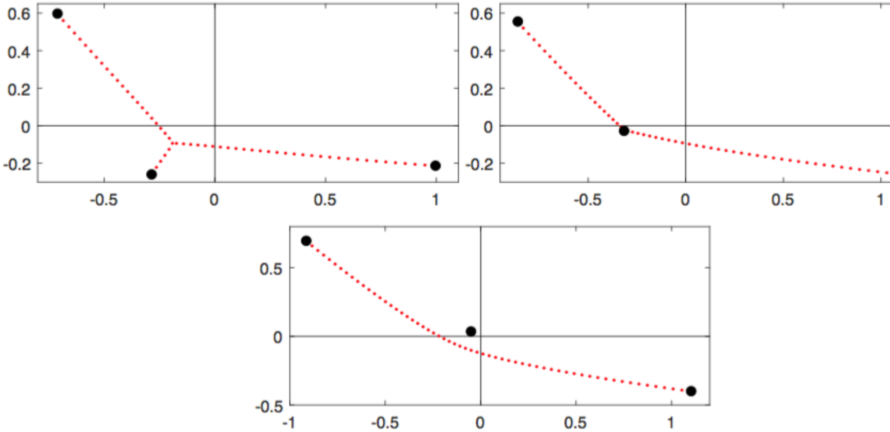


Figure: Root distributions of $Sp_{200}(An^{2/3}, \beta n^{4/3})$ in the β-plane for $A = (1 - i)/2$ (left), $A = 4/5 - 2i/3$ (right) and $A = 2/3 - i$ (down).

Conjectures

For a given positive integer n, denote by Σ_n the set of all branching points of the projection of the algebraic curve $\Gamma_n(a) : \{Sp_n(a, \lambda) = 0\}$ to the a-axis. In other words, Σ_n is the set of all values of the complex parameter a for which the matrix $M_n^{(a)}$ has a multiple eigenvalue, i.e., $Sp_n(a, \lambda)$ has a multiple root. In physics literature such points are called *level crossings*. Obviously, one can describe Σ_n as the zero locus of the univariate discriminant polynomial $D_n(a)$ which is the resultant of $Sp_n(a, \lambda)$ and $\frac{\partial Sp_n(a,\lambda)}{\partial \lambda}$ with respect to λ. One can show that the degree of $D_n(a)$ equals $\binom{n+1}{2}$.

Further recall that Yablonskii-Vorob'ev polynomials $\{Q_n\}$ are defined as follows. Set $Q_0 = 1,\ Q_1 = t$, and for $n \geq 1$, define

$$Q_{n+1} = \frac{t \cdot Q_n^2 - 4(Q_n \cdot Q_n'' - (Q_n')^2)}{Q_{n-1}}.$$

Although the latter expression a priori determines a rational function, Q_n is in fact a polynomial of degree $\binom{n+1}{2}$. The importance of Yablonskii-Vorob'ev polynomials is explained by the fact that all rational solutions of the second Painlevé equation

$$u_{tt} = tu + 2u^3 + \alpha, \alpha \in \mathbb{C},$$

are presented in the form

$$u(t) = u(t;n) = \frac{d}{dt}\left\{\ln\left[\frac{Q_{n-1}(t)}{Q_n(t)}\right]\right\},\ u(t,0) = 0,\ u(t;-n) := -u(t;n).$$

Denote by $\mathscr{Z}_n$ the zero locus of Q_n. Our conjectures below reveal an unexpected connection of Σ_n and $\mathscr{Z}_n$.

Remark

One can show that the maximal absolute value of points in $\mathcal{Z}_n$ grows as $\frac{3}{\sqrt[3]{2}}n^{2/3}$. Similarly, the maximal absolute value of points in Σ_n grows as $\frac{3}{\sqrt[3]{4}}n^{2/3}$.

Conjecture

Given a positive integer ℓ, define $\Upsilon_\ell := \Sigma_{3\ell} \cup \Sigma_{3\ell+1} \cup \Sigma_{3\ell+2}$. For $R > 0$, let K_R be the square with the side $2R$ centered at the origin. Then for any fixed $R > 0$ and $\ell \to \infty$, the points in Υ_ℓ converge inside K_R to the nodes of a certain fixed hexagonal lattice.

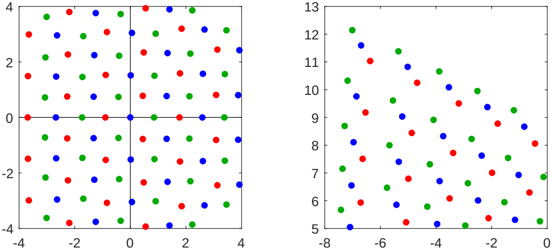


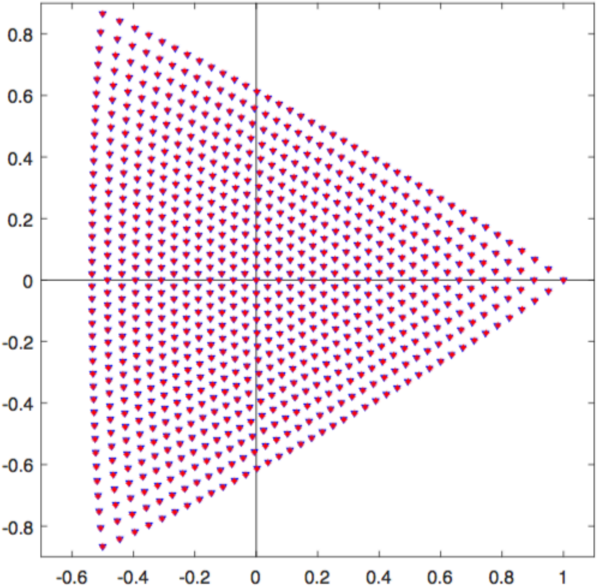
Figure: The points of Υ_5 inside the square K_4 (left) and close to the upper left corner (right). The points of Σ_{15} are shown in red, Σ_{16} in blue, and Σ_{17} in green.

Conjecture

Set $\widetilde{\Sigma}_n = \sqrt[3]{2} \cdot \Sigma_n$, *i.e., multiple every point in* Σ_n *by* $\sqrt[3]{2}$. *Then every point in* $\widetilde{\Sigma}_n$ *lies very close to the unique point in* $-\mathcal{Z}_n$ *and vice versa. Fixing* n, *define* $d(n) := max_{p\in\widetilde{\Sigma}_n} \min_{q\in-\mathcal{Z}_n} d(p,q)$, *i.e.,* $d(n)$ *is the maximal distance between points in* $\widetilde{\Sigma}_n$ *and their respective nearest points in* $-\mathcal{Z}_n$.
The sequence $\{d(n)\}$ *is very slowly growing with* n, *see Example below. It might even have a limit when* $n \to \infty$. *Moreover for any fixed* $R > 0$, *the sequence* $d_R(n)$ *converges to* 0 *where* $d_R(n)$ *is a similar maximin of the pairwise distances taken over all points in* $\widetilde{\Sigma}_n$ *and* $-\mathcal{Z}_n$ *which lie inside the square* K_R.

Example

Numerical experiments show that for $n = 10, 15, 20, 25, 30, 35, 40$, the corresponding values of $d(n)$ are approximately $0.03016, 0.04160, 0.051156, 0.05837, 0.06378, 0.06863, 0.07272$ respectively.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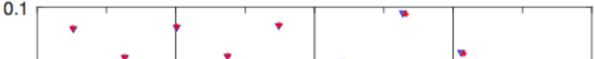

0.1

Conjecture

When $n \to \infty$, the points in the sequences $\frac{\sqrt[3]{4}}{3n^{2/3}}\{\Sigma_n\}$ and $\{-\frac{\sqrt[3]{2}}{3n^{2/3}}\mathcal{Z}_n\}$ asymptotically fill the same curvilinear triangular shape $\mathfrak{F}$, see Fig 4.
The interior of $\mathfrak{F}$ consists of all values $a \in \mathbb{C}$ for which the support of measure ν_a introduced in Proposition 2 is a tripod, i.e., consists of three smooth segments with a common point, see Figure 2 (left).
The complement of $\mathfrak{F}$ consists of all values $a \in \mathbb{C}$ for which the support of ν_a is a single smooth segment, see Figure 2 (down).
The boundary of $\mathfrak{F}$ consists of those $a \in \mathbb{C}$ for which the support of ν_a is a single curve with a singular point, i.e., a belongs to the boundary between the domain where this support is a tripod and the domain where the support is a smooth single curve, see Figure 2 (right).

For the sequence $\{\mathcal{Z}_n\}$ parts of the latter conjecture have been settled.

Observe that, for any positive integer n and generic values of parameter a, the roots of $Sp_n(a, \lambda)$ with respect to λ are simple. The latter roots are called the *quasy-exactly solvable spectrum* of the quartic oscillator under consideration.
Moreover, for any given n, and any sufficiently large positive a, these roots are real and distinct. The set $\Sigma_n \subset \mathbb{C}$ of branching points of $Sp_n(a, \lambda)$, i.e., the set of all values of a for which two eigenvalues coalesce, has cardinality $\binom{n+1}{2}$. When plotted these branching points form a regular pattern in the complex plane shown in Figures 4 and 5.

Below we present our (mostly) numerical results and conjectures about the monodromy of the roots of $Sp_n(a, \lambda)$ when a runs along different closed paths in the complement to Σ_n in the a-plane. We start with the following statement.

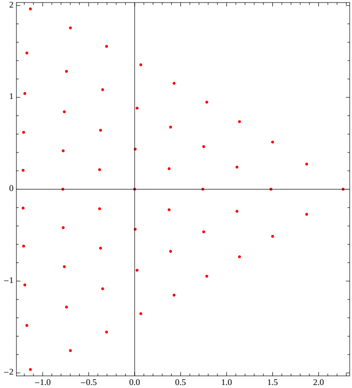
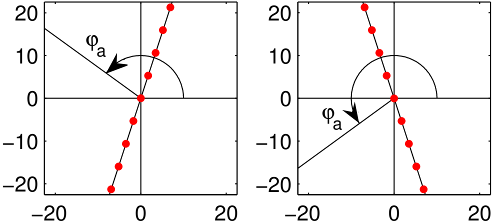


Figure: The curvilinear triangle of the branching points for $Sp_{10}(a, \lambda)$.



Proposition

For any given n, if $|a| \to \infty$ *with* $\arg a = \phi$ *fixed, then the roots of* $Sp_n(a, \lambda)$ *divided by* $n^{4/3}$ *will be asymptotically uniformly distributed on the straight segment* $[-\sqrt{a}, +\sqrt{a}]$*, see Figure 6. In particular, if a runs over the circle* $Re^{2\pi it},\ t \in [0, 1]$ *for any sufficiently large R, then the resulting monodromy of roots of* $Sp_n(R, \lambda)$ *(which are all real) is the complete reversing of their order, i.e., the leftmost and the rightmost roots change places, the second from the left and the second from the right change places etc.*

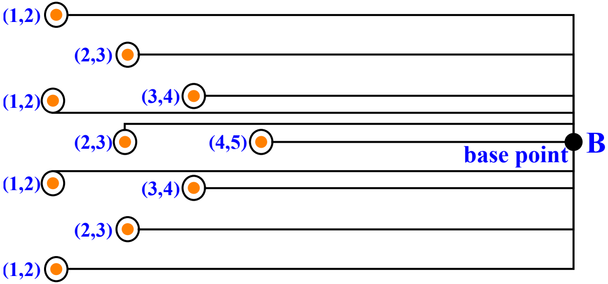


Figure: The system of standard paths and the monodromy (transpositions) of the spectrum which they produce for $n = 4$.

To describe (our conjecture on) the monodromy of the spectrum, let us introduce a system of standard paths connecting a base point chosen as a sufficiently large positive number with every branching point, see Fig. 7. Based on our numerical experiments, we see that Σ_n form a triangular shape with points regularly arranged into columns and rows in $\mathbb{C}$. There are n columns (enumerated from left to right) where the j-th column consists of $n-j$ branching points with approximately the same real part and there are n rows (enumerated from bottom to top) where the i-th row consists of points with approximately the same imaginary part. We denote the branching points $\sigma_{i,j} \in \Sigma_n$ where $i = 1, \ldots, 2n-1$ is the row number and $j = 1, \ldots, n$ is the column number.

Fixing a base point B as a sufficiently large positive number, connect B with every $\sigma_{i,j}$ by a "vertical hook" $\mathcal{P}_{i,j}$, i.e., move from B vertically to the imaginary part of $\sigma_{i,j}$, then move horizontally to the left untill you almost hit $\sigma_{i,j}$, then circumgo $\sigma_{i,j}$ counterclockwise along a small circle centered at $\sigma_{i,j}$ and return back to B along the same path. Conjecturally, along such a path one will never hit any other branching points unless $\sigma_{i,j}$ lies on the real axis. In other words, the imaginary parts of all branching points except for the real ones are all distinct. In case when $\sigma_{i,j}$ is real one can slightly deform the suggested path (which is a real interval) in an arbitrary way to move it away from the real axis. The resulting monodromy will (conjecturally) be independent of any such small deformation, see below. Finally we can state our surprisingly simple guess.

Conjecture

For any $\sigma_{i,j} \in \Sigma_n$ and any sufficiently large positive base point B, the monodromy corresponding to the standard path $\mathcal{P}_{i,j}$ is a simple transposition $(j, j+1)$ of the roots of $Sp_n(B, \lambda)$ ordered from left to right. (Recall that by our choice of B all roots of $Sp_n(B, \lambda)$ are real and therefore naturally ordered.)

This conjecture has been numerically checked for all $n \leq 10$. Observe that since the system of standard paths gives a basis of the fundamental group $\pi_1(\mathbb{C} \setminus \Sigma_n)$, then knowing the monodromy for the standard paths, one can calculate the monodromy along any loop in $\mathbb{C} \setminus \Sigma_n$ based at B.

Part of the symbolic-numerical experiments have been carried out in Macaulay 2 on Linux, Dell OptiPlex 790, 3392 MHz processor, 8 processors (54275 BogoMIPS), 16 GB RAM. Other numerical results have been obtained using Wolfram Mathematica 12, version 12.2.0.0 on Mac OS x86 (64 bits) platform.

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