First steps towards total reality of meromorphic functions

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Conjecture on total reality for rational curves, 93.

Any rational curve $\gamma : \mathbb{CP}^1 \to \mathbb{CP}^n$ such that the inverse images of all its flattening points lie on the real line $\mathbb{RP}^1 \subset \mathbb{CP}^1$ is real algebraic up to a Möbius transformation of the image $\mathbb{CP}^n$. (By a *flattening* point p on γ we mean a point at which the Frenet n-frame $(\gamma', \gamma'', \dots, \gamma^{(n)})$ is degenerate.)

At the present moment it is supported by a large number of partial results and extensive numerical evidence. At the same time the only case of this conjecture which is completely settled is the case $n = 1$, i.e. the case of the usual rational functions. (The authors were recently informed by Prof. A. Eremenko and A. Gabrielov that they have managed to prove the above conjecture in the first nontrivial case of plane rational curves of degree 5.)

Theorem 1 (Eremenko-Gabrielov, 2000)**.** *For any given* $(2d-2)$*-tuple of distinct real numbers there exist at least* $\mathbf{Cat}_d = \frac{1}{d}\binom{2d-2}{d-1}$ *real rational functions (considered up to a real Möbius transformation of the image* $\mathbb{CP}^1$*) with these critical points.*

The above theorem together with the result of L. Goldberg claiming that for any $(2d-2)$-tuple of distinct complex numbers the number of complex rational functions (considered up to a complex Möbius transformation of the image $\mathbb{CP}^1$) with these critical points is at most $\mathbf{Cat}_d$ gives the proof in the case $n = 1$.

The main idea of the proof of Theorem 1 is the explicit construction of such functions using the notion of *garden* which is the graph on the source $\mathbb{CP}^1$ obtained as the inverse image of $\mathbb{RP}^1$ under a real rational function. The number of topologically different gardens for generic real rational functions of degree d with all real, simple and distinct critical points turned out to coincide with $\mathbf{Cat}_d$. The difficult part of the proof of Theorem 1 is then to show that for a given topological type of a garden there exists a real rational function with this garden and having $(2d-2)$ prescribed real critical points.

The purpose of this talk is to discuss a (conjectural) generalization of Theorem 1 to the case of the source curves of higher genera, i.e. to the case of meromorphic functions. Existence of real meromorphic functions with all real (and closely located) critical points on real curves of positive genus was recently proved by B. Osserman.

Some notation.

Definition. A pair $(\mathcal{C}, \sigma)$ consisting of a compact Riemann surface $\mathcal{C}$ and its antiholomorphic involution σ is called a *real algebraic* curve. It is well-known that if $\mathcal{C}$ is a compact Riemann surface of genus g then for any σ the set $\mathcal{C}_\sigma$ (if nonempty) consists of at most $g+1$ disjoint smooth closed non-selfintersecting loops called the *ovals* of $\mathcal{C}_\sigma$. The set $\mathcal{C}_\sigma \subset \mathcal{C}$ of all fixed points of σ is called the *real part* of $(\mathcal{C}, \sigma)$.

If $(\mathcal{C}, \sigma)$ and $(\mathcal{D}, \tau)$ are real curves (varieties) and $f : \mathcal{C} \to \mathcal{D}$ a holomorphic map, then we shall use the notation $\overline{f}$ for $\tau \circ f \circ \sigma$ which is another holomorphic map. The map f is *real* if $\overline{f} = f$. We call a real algebraic curve $(\mathcal{C}, \sigma)$ with $\mathcal{C}$ compact of genus g an M*-curve* if its $\mathcal{C}_\sigma$ consists of exactly $g+1$ ovals.

Main question.

Problem 1. Given a meromorphic function $f : (\mathcal{C}, \sigma) \to \mathbb{CP}^1$ such that

i) all its critical points and values are distinct;

ii) all its critical points belong to $\mathcal{C}_\sigma$

is it true that that f becomes a real meromorphic function after a choice of a real structure of $\mathbb{CP}^1$?

Definition. A pair of positive integers (g, d) is said to satisfy *the total reality property* if any degree d meromorphic function $f : (\mathcal{C}, \sigma) \to \mathbb{CP}^1$ on a real genus g curve $\mathcal{C}$ with all real critical points is real up to a Möbius transformation of $\mathbb{CP}^1$.

Notice that Problem 1 has the following modification. Since the number of critical points/values of a generic degree d meromorphic function from a genus g curve equals $2d-2+2g$ one has that the dimension of the space of corresponding linear systems equals $2d-2-g$, i.e. one can arbitrarily assign the position of $2d-2-g$ critical points and the remaining $3g$ critical points will be determined by the latter ones.

Problem 2. Given a meromorphic function $f : (\mathcal{C}, \sigma) \to \mathbb{CP}^1$ of degree d such that

i) all its critical points and values are distinct;

ii) its $2d-2-g$ critical points belong to $\mathcal{C}_\sigma$

is it true that that f becomes a real meromorphic function after a choice of a real structure of $\mathbb{CP}^1$?

Main results.

Theorem 2. *Any pair (g, d) where d is a prime or the square of a prime and which additionally satisfies the inequality: $g > \frac{d^2-4d+3}{3}$ satisfies the total reality property.*

Proposition 1. *For any prime d the validity/nonvalidity of the total reality property for a pair (g, d) is equivalent to the problem of nonexistence/existence of a plane real algebraic curve $\gamma \subset (\mathbb{CP}^2, \tau)$ of degree $2d - 2$ with the following properties:*

a) the geometric genus of γ equals g;

b) it contains 2d-3+2g real cusps;

c) it contains a pair of complex conjugate singular points p_1 and p_2 such that the line l_{p_1,p_2} through p_1 and p_2 is tangent to the real part γ_τ of the curve γ at its smooth point and, additionally, $\nu(p_1) = \nu(p_2) = d-2$ where $\nu(P)$ stands for the local multiplicity of intersection *of a plane singularity P with a generic line passing through P;*

d) its other singularities (real and complex) are the unions of smooth (and not necessarily transversal) local branches.

Unfortunately, the known results about the realizability of singularities by real algebraic curves are not strong enough to cover this situation.

Corollary 1. *The property on total reality for meromorphic functions holds for all functions of degrees $2, 3$. For degree 4 the property holds if and only if there is no real sextic with 7 usual cusps and 2 complex conjugate nodes such that the line thru them intersects the sextic in a smooth real point.*

On the other hand, we show that the answer to the Problem 2 is negative.

Proposition 2. *There exists a real elliptic curve $(\mathcal{C}, \sigma)$ with a nonempty real part $\mathcal{C}_\sigma$ and a meromorphic function $f : \mathcal{C} \to \mathbb{CP}^1$ of degree 3 with 3 of its 6 critical points lying on $\mathcal{C}_\sigma$ and which can not be made real by a Möbius transformation of the image $\mathbb{CP}^1$.*

Remark 1. Note that a further (naive) generalization of the above problem to the case of maps between two real curves definitely has a negative answer. The counterexample exists already for a map between two elliptic curves. Note that by the Riemann-Hurwitz formula such a map has no ramification locus. If our conjecture holds then any map $f : E_1 \to E_2$ from a real elliptic curve (E_1, σ) to an elliptic E_2 is real, i.e., there exists an involution $\tau : E_2 \to E_2$ such that $\bar{f} = \tau f \sigma = f$. However, it is evidently false. Namely, let $E_1 = E_2 = (\mathbb{C}/\mathbb{Z} + i\mathbb{Z}, \sigma)$ where σ is the standard complex conjugation. Let $\Phi : \mathbb{C} \to \mathbb{C}$ be the linear map $z \mapsto (2+i)z$, and $\phi : E_1 \to E_2$ be degree 5 map induced by Φ. Existence of τ means, in particular, that the pushforward of σ is well defined. Put $\xi = \frac{i}{2+i} = \frac{1+2i}{5} = \Phi^{-1}(i) \in \phi^{-1}(\mathbb{Z} + i\mathbb{Z})$. $\Phi(\bar{\xi}) = \frac{4-3i}{5} \not\in \mathbb{Z} + i\mathbb{Z}$, which shows that the push forward of σ under ϕ is not well defined. Hence, there is no antiholomorphic involution on E_2 which makes ϕ into a real map.

Proofs

Proposition 3. *If* $(\mathcal{C}, \sigma)$ *is a proper irreducible real curve and* $f : \mathcal{C} \to \mathbb{CP}^1$ *a non-constant holomorphic map (with* $\mathbb{CP}^1$ *provided with its standard real structure), then* f *is real for some real structure on* $\mathbb{CP}^1$ *precisely when there is a Möbius transformation* $\varphi : \mathbb{CP}^1 \to \mathbb{CP}^1$ *such that* $\overline{f} = \varphi \circ f$*.*

Assume now that $(\mathcal{C}, \sigma)$ is a proper irreducible real curve and $f : \mathcal{C} \to \mathbb{CP}^1$ a non-constant holomorphic map. It defines a holomorphic map

$$\mathcal{C} \xrightarrow{(f,\overline{f})} \mathbb{CP}^1 \times \mathbb{CP}^1$$

and if $\mathbb{CP}^1 \times \mathbb{CP}^1$ is given the real structure that takes (x, y) to $(\tau(y), \tau(x))$, which we shall call the *involutive real structure*, then it is clearly a real map.

Proposition 4. *1. The image* $\mathcal{D}$ *of the curve* $\mathcal{C}$ *under the map* $(f, \overline{f})$ *is of type* (δ, δ) *for some positive integer* δ *and if* ∂ *is the degree of the map* $\mathcal{C} \to \mathcal{D}$

we have that $d = \delta\partial$, where d is the degree of the original f.

2. *f is real for some real structure on $\mathbb{CP}^1$ precisely when $\delta = 1$.*

3. *Assume that $\mathcal{C}$ is smooth and all the critical points of f are real. Then all the critical points of $g : \widetilde{\mathcal{D}} \longrightarrow \mathbb{CP}^1$, the composite of the normalization map $\widetilde{\mathcal{D}} \rightarrow \mathcal{D}$ and the restriction of the projection of $\mathbb{CP}^1 \times \mathbb{CP}^1$ has all its critical points real.*

Part (2) of the above Proposition gives another reformulation of the conjecture on total reality for meromorphic functions.

Corollary 2. *If a degree d function $f : (\mathcal{C}, \sigma) \longrightarrow \mathbb{CP}^1$ has all real critical points then the map $\mathcal{C} \xrightarrow{(f,\overline{f})} \mathcal{D} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ must have degree d as well.*

The above Proposition motivates the following definition.

Definition. A meromorphic function $f : (\mathcal{C}, \sigma) \to \mathbb{CP}^1$ is called *real-factorizable* if one can find a real algebraic map $\psi : (\mathcal{C}, \sigma) \to (E, \mu)$ of degree greater than 1 and a meromorphic function $g : (E, \mu) \to \mathbb{CP}^1$ such that $f = g \circ \psi$.

Proposition 5. *A degree d meromorphic function $f : (\mathcal{C}, \sigma) \to \mathbb{CP}^1$ is real-factorizable if and only if the image curve $\mathcal{D} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ (see Proposition 4) has bidegree (δ, δ) where δ is a nontrivial factor of d (i.e. $\delta < d$).*

Remark 2. The value of $\delta \leq d$ in the latter Proposition might serve as a measure of closeness of a meromorphic function on a real curve to the class of real functions. Namely, if $\delta = d$ the function itself is real (up to a Möbius transformation) and if $\delta < d$ then one can find a nontrivial real factor. The opposite situation when $\delta = 1$, i.e. $\mathcal{C}$ is mapped onto $\mathcal{D}$ birationally is called *purely complex*.

If $\mathcal{C}$ is a curve and $p_1, \ldots, p_k$ are its smooth points then we let $\pi : \mathcal{C} \to \mathcal{C}(p_1, \ldots, p_k)$ be the finite map which is a homeomorphism and for which $\mathcal{O}_{\mathcal{C}(p_1,\ldots,p_k)} \to \pi_*\mathcal{O}_{\mathcal{C}}$ is an isomorphism outside of $\{p_1, \ldots, p_k\}$ with $\mathcal{O}_{\mathcal{C}(p_1,\ldots,p_k),\pi(p_i)} \to \mathcal{O}_{\mathcal{C},p_i}$ having image the inverse image of $\mathbb{C}$ in $\mathcal{O}_{\mathcal{C},p_i}/\mathfrak{m}_{p_i}^2$. Then π has the following two (obvious) properties:

Lemma 1. *1. A holomorphic map $f : \mathcal{C} \to X$ which is not an immersion at all the points $p_1, \ldots, p_k$ factors through π.*

2. If $\mathcal{C}$ is proper, then the arithmetic genus of $\mathcal{C}(p_1, \ldots, p_k)$ is k plus the arithmetic genus of $\mathcal{C}$.

Proposition 6. *Assume that $(\mathcal{C}, \sigma)$ is a smooth and proper real curve and let $f : \mathcal{C} \to \mathbb{CP}^1$ be a holomorphic map of degree d. If there are k real points $p_1, \ldots, p_k$ on $\mathcal{C}$ which are critical points for f and if $(f, \overline{f})$ gives a map of degree 1 from $\mathcal{C}$ to its image*

$\mathcal{D}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$, then $g(\mathcal{C}) + k \leq (d-1)^2$. If $g(\mathcal{C}) + k = (d-1)^2$, then the map $h : \mathcal{C} \to \mathcal{D}$ factors to give an isomorphism $\mathcal{C}(p_1, \ldots, p_k) \xrightarrow{\sim} D$.

The case of a prime degree d in Theorem 2 is now a simple corollary of Proposition 6. Indeed, if we assume that all the critical points of a generic meromorphic function $f : (\mathcal{C}, \sigma) \to \mathbb{CP}^1$ are real then k in the above Proposition equals $2d - 2 + 2g(\mathcal{C})$. Under the assumption $g(\mathcal{C}) > \frac{d^2-4d+3}{3}$ one gets $g(\mathcal{C}) + k = 2d - 2 + 3g(\mathcal{C}) > (d-1)^2$. Thus, the case when $\mathcal{C}$ maps birationally to $\mathcal{D}$ is impossible by the above Proposition. Since d is prime the only other possible case is when the degree of the map $\mathcal{C} \to \mathcal{D}$ equals d and therefore, the degree of the map $\mathcal{D} \to \mathbb{CP}^1$ equals 1 which by (2) of Proposition 4 satisfies the conjecture on total reality. To settle the case $d = p^2$ we need some additional lemmas.

Lemma 2. *Let f be a meromorphic function on $\mathcal{C}$ of $\deg\ f = p^2 > 1$ for some prime p with only simple critical points then either $\deg\ h = p^2$ or $\deg\ h = 1$.*

Corollary 3. *Let f be a meromorphic function of degree $d = p^2 > 1$ for some prime p on a Riemann surface of genus $g > \frac{d^2-4d+3}{3}$. If f has only real simple critical points then f is real.*

Thus Theorem 2 is completely settled.

Now we settle Proposition 1.

Indeed, by Proposition 6 for d prime there are only two cases to consider, namely, the case when the degree of the map $\mathcal{C} \longrightarrow \mathcal{D}$ equals d in which the conjecture on total reality holds and the second case when the map $\mathcal{C} \longrightarrow \mathcal{D}$ is birational. In the latter case the image $\mathcal{D} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is of bidegree (d, d) and has (by assumptions of simplicity and reality of all the criticial points of f) exactly $2d-2+2g(\mathcal{C})$ ordinary real cusps and no other singularities. Notice that given an arbitrary Möbius transformation φ, the holomorphic map $\Phi\colon (x, y) \mapsto (\varphi(x), \overline{\varphi(y)})$ is a real automorphism of $\mathbb{CP}^1 \times \mathbb{CP}^1$ with involutive real structure. Projection on the first factor identifies the real points of $\mathbb{CP}^1 \times \mathbb{CP}^1$ (still with its involutive structure) with $\mathbb{CP}^1$ and under this identification Φ acts on the real points as φ acts on $\mathbb{CP}^1$.

Now, the complete linear system of type $(1,1)$ embeds $\mathbb{CP}^1 \times \mathbb{CP}^1$ as a quadric Q in $\mathbb{CP}^3$. This system has a real structure with respect to the involutive real structure and realizes $\mathbb{RP}^1 \times \mathbb{RP}^1$ as a quadric of signature $(+1, -1, -1, -1)$. It has the property that there are real points on it but the two rulings on it are not defined over the reals, instead through each real point on it there are two complex conjugate lines on Q passing though it. Projecting Q from a real point p gives a map from Q with p blown up to the real projective plane $\mathbb{RP}^2$ which gives an isomorphism from Q with p blown up and the strict transform of the two lines passing through it blown down, taking the exceptional curve to a line through the two blown down curves. Conversely, given two complex conjugate points q and $\overline{q}$ in the projective plane, the linear system of quadrics passing through them gives a map from the plane with q and $\overline{q}$ blown up onto a quadric in $\mathbb{CP}^3$ which contracts exactly the line through q and $\overline{q}$.

Now, assume that $\mathcal{D} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is a real curve of type (d, d) with $2d - 2 + 2g$ real ordinary cusps. We now project from one of the cusps p. The strict transform $\bar{\mathcal{D}}$ of $\mathcal{D}$ in the blowing up Q of $\mathbb{CP}^1 \times \mathbb{CP}^1$ will then meet the exceptional curve in one??? real point. The strict transform of the two lines through p, E_1 and E_2 are complex conjugate in Q and meet $\bar{\mathcal{D}}$ transversally in one??? point each (which are each other's complex conjugate). Mapping to the projective plane gives a curve $\mathcal{D}'$ of degree $2d - 2$ with $2d - 3 + 2g$ real ordinary cusps and no other real singular points as well as one smooth real point whose tangent intersects $\mathcal{D}'$ in that point and two complex conjugate points. Conversely, suppose $\mathcal{D}'$ is a plane curve of degree $2d - 2$ with $2d - 3 + 2g$ real ordinary real cusps, no other singularities, and a smooth real point whose tangent intersects the curve in that point and two complex conjugate points. Blowing up those two complex conjugate points and blowing down the tangent gives a curve on $\mathbb{CP}^1 \times \mathbb{CP}^1$ with $2d - 2 + 2g$ ordinary real cusps. CHECK! □

Case $d = 2$. Suppose that the degree d of the map $f : (\mathcal{C}, \sigma) \to \mathbb{CP}^1$ is equal to 2. That only leaves two possibilities: The first is that the map $\mathcal{C} \to \mathcal{D}$ has degree 2 and then by Proposition 4 f is real for some real structure $(\mathbb{CP}^1, \tau)$ on $\mathbb{CP}^1$. In particular, if the set $\mathcal{C}_\sigma$ of real points is nonempty then so does $(\mathbb{CP}^1, \tau)$ which means that it is equivalent to the standard real structure. The second is that the map $\mathcal{C} \to \mathcal{D}$ is birational and then by Proposition 6 we get $g(\mathcal{C}) + k \leq 1^2 = 1$, where k is the number of real critical points of f. In particular if $g(\mathcal{C}) > 0$ then there are no real critical points. Thus a hyper-elliptic map from a real curve $(\mathcal{C}, \sigma)$ is real if one of its critical points is real.

Case $d = 3$. In this case again we have only two possibilities; either f is real for a real structure on $\mathbb{CP}^1$

or $\mathcal{C} \to \mathcal{D}$ is birational in which case we have $g(\mathcal{C}) + k \leq 2^2 = 4$. The case $g(\mathcal{C}) = 0$ was settled earlier. Recall that the total number of critical points equals $2d-2+2g(\mathcal{C})$. But if $g(\mathcal{C}) > 0$ then $2 \cdot 3 - 2 + 3g > 4$ and this case of Theorem 1 is settled. Analogously to the case $d = 2$ a function f with the degree $d = 3$ is real if it has more than $\max(4 - g(\mathcal{C}), 1)$ real critical points.

Case $d = 4$. We have three possibilities; $\mathcal{D}$ has degree $(1,1)$, $(2,2)$, or $(4,4)$. In the first case f can be made real. In the second case, by Proposition 4, the projection on the first factor will give a map from the normalization $\widetilde{\mathcal{D}}$ of $\mathcal{D}$. The arithmetic genus $p_a(\mathcal{D}) = 1$, and the geometric genus $g(\widetilde{\mathcal{D}})$ of the normalization $\widetilde{D}$ does not exceed 1. Let $\widetilde{h} : \mathcal{C} \to \widetilde{\mathcal{D}}$ be the lift of $h : \mathcal{C} \to \mathcal{D}$. Note that if $p_i \in \mathcal{C}$ is a critical point of f then either its image $h(p_i)$ is a cusp of $\mathcal{D}$ or p_i is a ramification point of $\widetilde{h}$. The ramification divisor $R(\widetilde{h}) = 2g(\mathcal{C}) + 2 - 4g(\widetilde{\mathcal{D}})$. The number of cusps of $\mathcal{D}$ does not exceed 1, whereas the number of distinct critical points of f is $2g(\mathcal{C})+6$. Therefore, we must have $2g(\mathcal{C})+6-(2g(\mathcal{C})+2-4g(\widetilde{\mathcal{D}}))\leq 1$ which is impossible.

We are hence left with the case when $\mathcal{D}$ has degree $(4,4)$. The only case when $2\cdot 4-2+3g(\mathcal{C})\leq 9$ for $g(\mathcal{C})>0$ is the case of $g(\mathcal{C})=1$. If all the critical points $p_1,\ldots,p_8$ of $f:\mathcal{C}\to\mathbb{CP}^1$ are real, then we get a birational map $\mathcal{C}(p_1,\ldots,p_8)\to\mathcal{D}$ and as then both $\mathcal{C}(p_1,\ldots,p_8)$ and $\mathcal{D}$ have arithmetic genus 9, this map is an isomorphism. Hence $\mathcal{D}$ is a curve with 8 ordinary real cusps and no other singularities. Projecting from a cusp p gives us a real plane curve $\mathcal{D}'$ of degree 6. Apart from the 7 surviving real cusps, $\mathcal{D}'$ will have a pair of complex conjugate singularities obtained by contracting the strict transforms of the two lines through p. These strict transforms will have intersection number two with the strict transform of the curve and hence the singularities will be ordinary nodes (when they intersect in two points) or cusps (when they are tangent). The line through these two points will then be a tangent to $\mathcal{D}'$. Conversely, if we have an irreducible plane curve of degree 6 with 7 real ordinary cusps and two complex conjugate ordinary nodes or cusps such that the line through them is tangent to the curve we can go backwards and get a curve of degree $(4, 4)$ with the desired properties.

Note also that if one composes the map from the plane blown up at two points to a quadric with the projection onto one of the rulings one obtains the projection from one of the two blown up points. Hence, given a plane curve $\mathcal{D}'$ of degree 6 as above, the map f is obtained by composing the normalization map $\mathcal{C} \to \mathcal{D}'$ with the projection from one of the non-real singularities.

Remark 3. As a curiosity, if one has a, not necessarily real, irreducible plane curve $\mathcal{D}'$ of degree 6 with 7 real ordinary cusps and one tangent at a smooth real point that intersects $\mathcal{D}'$ in two complex conjugate singular points, then the curve is real. Indeed, if not it is distinct from it is complex conjugate $\overline{\mathcal{D}'}$ and by the Bezout theorem the total intersection multiplicity $\mathcal{D}' \cdot \overline{\mathcal{D}'} =$

36, whereas the nine singular points of $\mathcal{D}'$ are also singular points of $\overline{\mathcal{D}'}$ giving a total contribution of at least $9 \cdot 4 = 36$ and the smooth real point gives an extra contribution of at least 1 (at least 2 actually as its tangent is common to both curves).

To finish the proof we have to show that real algebraic curves in $\mathbb{CP}^2$ of degree 6 with the following singularities: 7 real cusps and either 2 complex conjugate cusps or nodes with additional property that the line through those is tangent to the real part at a smooth point do not exist. The situation with 7 real and 2 complex conjugate cusps is easy to reject. Namely, any complex degree 6 curve has at most 9 cusps. If it is additionally real and has exactly 9 cusps then it is dual to a nonsingular real curve of degree 3 with exactly 3 real inflection points. But then the original curve has exactly 3 (and no more) real cusps.

The remaining case is that of 7 real cusps and 2 complex conjugate nodes...

Finally, let us settle Proposition 2.

We will show that there exist curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ real in the involutive real structure, having degree $(3, 3)$ and 3 real ordinary cusps and no other singularities.

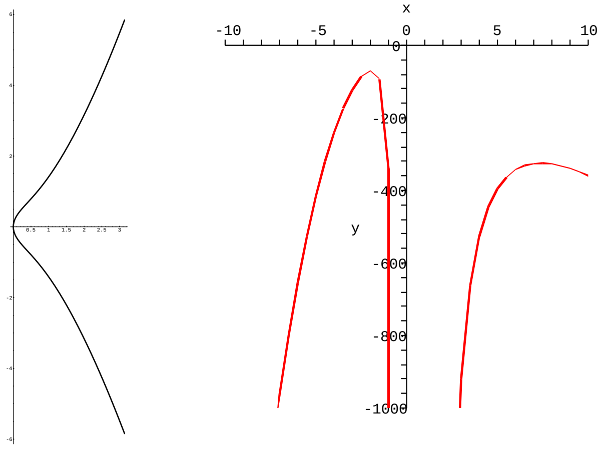
Proposition 7. *The space of degree 4 plane curves with two ordinary cusps and no other singularities is a smooth subvariety of the space of all quartics. In the real locus of this space, the conditions that the two cusps are both real and that there is a smooth point whose tangent intersects the curve in the point and two complex conjugated points is open and nonempty.*

Proof. For the non-real part it will be enough to show that for a given curve $\mathcal{D} = \{f = 0\}$ in the space, the map from the linear space of quartics to the product of the tangent spaces to mini-versal deformations of the two singularities is surjective. That product can be identified with the product of the "Milnor spaces" $\mathcal{O}_{D,p}/(f_x, f_y, f_z)$ at the two singular points. As the points are ordinary cusps we have that $\mathfrak{m}^2_{\mathbb{CP}^2,p} \subset (f_x, f_y, f_z)$ but it is clear that the quartics fill out $\mathcal{O}_{\mathbb{CP}^2,p}/\mathfrak{m}_p^2 \oplus \mathcal{O}_{\mathbb{CP}^2,q}/\mathfrak{m}_q^2$ for any two points of p and q of $\mathbb{CP}^2$.

For the real part, the openness is clear and hence it is enough to show that it is nonempty. It is easily verified that $f = xz^3 - yz^3 + xyz^2 + x^2y^2$ has ordinary cusps at $(1 : 0 : 0)$ and $(0 : 1 : 0)$ and no other singularities. $(0 : 0 : 1)$ lies on the curve and its tangent is given by $x = y$ which intersects the curve in $(0 : 0 : 1)$, $(i : i : 1)$, and $(-i : -i : 1)$. $\square$

Using Proposition 1 we get from the latter statement that there exist real curves of bidegree $(3, 3)$ with exactly 3 real ordinary cusps and no other singularities which settles Proposition 2. $\square$

A more explicit example of a degree 3 function from a real elliptic curve with 3 real critical points but which can not be made real is presented below.



The curve $y^2 = x^3 + x$ and the graph of the determinant of the three tangent lines at $P_0 + P,\ P_1 + P, P_2 + P$ computed in Maple.

Example. It is well known that any degree 3 meromorphic function on an elliptic curve $\mathcal{C}$ realized in $\mathbb{CP}^2$ by the standard equation $y^2 = P_3(x)$ where $P_3(x)$ is a cubic polynomial can be represented as the composition of the group shift of the whole $\mathcal{C}$ by some fixed point on it with the projection from some point on $\mathbb{CP}^2$.

The critical points on $\mathcal{C} \subset \mathbb{CP}^2$ for the projection from some point $pt \in \mathbb{CP}^2$ are the points where the pencil of lines through pt is tangent to $\mathcal{C}$. Thus in order to prove that a given triple (P_0, P_1, P_2) of real points on a given real elliptic curve $\mathcal{C} \subset \mathbb{CP}^2$ can not serve as the set of critical points of a real degree 3 function we have to show that for any choice of a fourth real point $P \in \mathcal{C}$ the (real) tangent lines to $\mathcal{C}$ at the points $P_0 + P, P_1 + P, P_2 + P$ never meet at the same (real) point on $\mathbb{CP}^2$. Recall that (up to a sign change) the addition of two point A and B on a real elliptic $\mathcal{C} \subset \mathbb{CP}^2$ can be interpreted as the third intersection point of the line $\overline{AB}$ with $\mathcal{C}$.

Consider the real elliptic curve $(\mathcal{C}, \sigma)$ given by the equation $y^2 = x^3 + x$ with 3 real points P_0, P_1, P_2, where P_0 is "almost" infinite point $(100.35, 1005.3)$, P_1 is the point slightly above the origin $(0.05, 0.224)$, and, finally, P_2 is the inflection point $(0.4, -0.67)$.

Then we claim that for any point P on the real part $\mathcal{C}_\sigma$ there is no real function whose inflection points coincide with $P_0 + P, P_1 + P, P_2 + P$. Indeed as we explained above the existence of such a function would mean that tangents to $\mathcal{C}_\sigma$ would intersects at the same point of $\mathbb{RP}^2$, or the corresponding dual points in $(\mathbb{RP}^2)^*$ are collinear. However, computing the determinant of these three dual points we see that the obtained function of P is non-vanishing, see the above Figure.

Remarks and problems

I. Analogously to the conjecture on total reality for rational curves one can ask a similar question for projective curves of any genus, namely

Problem 1. Given a real algebraic curve $(\mathcal{C}, \sigma)$ with compact $\mathcal{C}$ and nonempty real part $\mathcal{C}_\sigma$ and a complex algebraic map $\Psi : \mathcal{C} \to \mathbb{CP}^n$ such that the inverse images of all the flattening points of $\Psi(\mathcal{C})$ lie on the real part $\mathcal{C}_\sigma \subset \mathcal{C}$ is it true that Ψ is a real algebraic up to a Möbius transformation of the image $\mathbb{CP}^n$?

(Recent private communication from Prof. Gabrielov claims that this is now proven for the plane rational curves of degree 5.)

II.

Problem 2. Describe the properties of the stratification of the space of all meromorphic functions of a given degree on a fixed real curva according to their real-factorizability.