Non-unitary analogs of zonotopal algebras for undirected graphs

Boris Shapiro, Stockholm University, joint with A.Kirillov, G.Nenashev, A.Vaintrob

January 24, 2024

Topics to discuss

Abstract. About two decades ago three types of zonotopal algebras (external, central, and internal) have been associated to an arbitrary undirected graph G. They contain an abundance of information about G encoded in its Tutte polynomial. In particular, external algebras distinguish graphical matroids of graphs. Below we introduce their analogs in which we double each edge of G. The resulting algebras have nice combinatorial properties and, in particular, are monomial.

The *circulation algebra* is defined as follows. Let $G = (V, E)$ be a undirected graph (multiple edges and loops are allowed) with the vertex set V and the edge set E. For a field K of zero characteristic, consider the square-free algebra defined as the quotient

$$B(E) = K[E]/(x_e^2),\ e \in E,$$

of the polynomial algebra in the *edge variables* $x_e,\ e \in E$, by the ideal generated by their squares.

Given an orientation σ of G, we define the standard directed *incidence matrix* $A(G,\sigma)$ of G whose entries are given by

$$a_{v,e} = \begin{cases} -1, & \text{if the edge } e \text{ begins at } v; \\ 1, & \text{if } e \text{ ends at } v; \\ 0, & \text{if } e \text{ is a loop or it is not incident to } v. \end{cases} \quad (1.1)$$

(The rows of $A(G,\sigma)$ are labeled by the vertices and its columns by the edges of G). Define the algebra

$$\mathcal{C}(G,\sigma) := K[y_v] \subset B(E) \quad (1.2)$$

generated by the elements

$$y_v = \sum_{e \in E} a_{v,e} x_e \in B(G), \text{ for } v \in V. \quad (1.3)$$

Remark

Observe that reversing the orientation of any edge $e \in E$ we simply change the sign of the corresponding generator x_e. So the isomorphism class of $\mathcal{C}(G, \sigma)$ as a graded algebra does not depend on the choice of σ. Therefore we will denote this graded algebra by $\mathcal{C}(G)$ skipping σ.

If we view each edge variable x_e as representing the flow along the directed edge e in G, then the variable y_v can be interpreted as the total flow through the vertex v. Thus $\mathcal{C}(G)$ can be thought of as the algebra generated by the flows through the set of all vertices and, following D. Wagner, we call it the *circulation algebra* of G.

One of the main results about $\mathcal{C}(G)$ is the formula for its Hilbert function

$$h_{\mathcal{C}(G)}(t) := \sum_{k\geq 0} \dim \mathcal{C}^{(k)}(G) \cdot t^k$$

obtained by Postnikov-Shapiro and independently by D. Wagner which contains important information about the spanning subgraphs of G.

Definition

Let us fix a linear ordering of all edges of the graph G. For a spanning forest $F \subset G$, an edge $e \in G \setminus T$ is called *externally active* for F if there exists a cycle $C \subseteq G$ such that e is the minimal edge of C in the chosen ordering and $(C \setminus \{e\}) \subset T$. The *external activity* of F is the number of its externally active edges.

Let N_G^k denote the number of spanning forests $F \subset G$ of external activity k. Even though the notion of external activity depends on a particular choice of ordering of edges, the numbers N_G^k are known to be independent of the latter choice.

Theorem

The dimension $\dim_K \mathcal{C}(G)$ of the algebra $\mathcal{C}(G)$ is equal to the number of spanning forests in G. Additionally, the dimension $\dim_K \mathcal{C}^k(G)$ of the k-th graded component of $\mathcal{C}(G)$ is equal to the number of spanning forests $S \subset E$ of G with *external activity* $|E| - |S| - k$. Here $|\Omega|$ stands for the number of edges in a subgraph Ω.

Later G. Nenashev proved that the circulation algebras $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ of two graphs are isomorphic if and only if the usual graphical matroids of G_1 and G_2 are isomorphic.

A natural algebra similar to $\mathcal{C}(G)$ and whose definition does not require any orientation of edges of G can be introduced as follows. Denoting it by $\mathcal{C}_+(G)$ we define it exactly like $\mathcal{C}(G)$ except for the formula for the coefficients $a_{v,e}$ which now will be given by

$$a_{v,e} = \begin{cases} 1, & \text{if the edge } e \text{ contains the vertex } v; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\mathcal{C}_+(G) = \mathcal{C}(A_+(G))$, where $A_+(G)$ is the standard *undirected* incidence matrix of G.
Similarly to $\mathcal{C}(G)$, the graded algebra $\mathcal{C}_+(G)$ contains interesting information about G.

Definition

(a) Given a graph $G = (V, E)$, we say that its (edge-induced) subgraph $F \subset E$ is an *odd-circle pseudoforest* if every connected component of F is either a tree or a unicycle whose cycle has odd length. (A *unicycle* is a graph containing a unique cycle, i.e. a connected graph obtained from a tree by adding exactly one edge connecting some of its vertices.)
(b) A connected graph $G = (V, E)$ is called an *even circuit* if it either is an even cycle or is a union of two odd cycles sharing exactly one vertex or is a union of two disjoint odd cycles exactly two vertices of which are connected by a bridge. (The three types of even circuits are shown in Figure 1 a,b,c respectively; the second type can be thought as a degenerate case of the third type when the bridge contracts of a single vertex).

Definition

Denote by $\mathcal{P}_+(G)$ the set of all odd-circle pseudoforests in G. Given a linear ordering of the set E of edges in G, we call an edge e *evenly active* for a pseudoforest $F \in \mathcal{P}_+(G)$ if $F \cup \{e\}$ contains an even circuit in which e is the smallest edge with respect to the chosen ordering.

Theorem

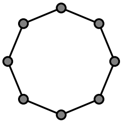
Given an undirected graph $G = (V, E)$, one has the following.

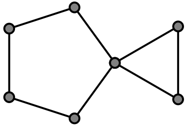
1. *The dimension* $\dim_{\mathbb{K}} \mathcal{C}_+(G)$ *of the algebra* $\mathcal{C}_+(G)$ *is equal to the number of spanning odd-circle pseudoforests in G, i.e.*

$$\dim_{\mathbb{K}} \mathcal{C}_+(G) = |\mathcal{P}_+(G)|.$$

2. *The dimension* $\dim_{\mathbb{K}} \mathcal{C}^k_+(G)$ *of the k-th graded component of* $\mathcal{C}_+(G)$ *is equal to the number of odd-circle pseudoforests* $F \subseteq G$ *whose even activity equals* $|E \setminus F| - k$*, i.e.*

$$\dim_{\mathbb{K}} \mathcal{C}^k_+(G) = \left|\{F \in \mathcal{P}_+(G) \,|\, \mathrm{act}_+(F) = |E \setminus F| - k\}\right| .$$


(a) even cycle


(b) odd figure-eight

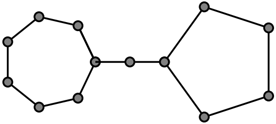

(c) odd handcuff

Figure: Even circuits

Theorem 5 is a direct consequence of the following lemma and Theorem B.

Lemma

A set of columns S in the undirected incidence matrix $A_+(G)$ is dependent if and only if the edge-induced subgraph of G whose edges correspond to the columns in S contains an even circuit.

Proof. One can easily check that the set of columns corresponding to any odd-circle pseudoforest is linear independent. Indeed, an edge incident to a hanging vertex can not be a part of a linear dependence. Similarly, a single odd cycle gives no linear dependence of its edges/columns. On the other hand, all three types of even circuits provide linearly dependent sets of edges. These dependences are straightforward. For an even cycle, one has to assign the coefficients ± 1 alternatingly. For the second type, see Figure 1 b, one traverses this graph as an Eulerian assigning the coefficients ± 1 alternatingly along the path.

Finally, for the third type, see Figure 1 c, let us first choose one of two odd cycles. Then we traverse the even circuit starting at the vertex of the chosen cycle attached to the bridge first going around the chosen cycle and assigning ± 1 alternatingly on the way, then we follow the bridge assigning the coefficients ± 2 alternatingly, and, finally we traverse the second odd cycle again assigning the coefficients ± 1. By construction, the sum of the coefficients assigned to all edges incident to any given vertex vanishes.

To finish the proof, observe that adding an edge to an odd-circle pseudoforest, we either obtain a new pseudoforest (in which case the set of edges/columns of the incidence matrix is still independent) or a graph containing an even circuit (in which case the set of edges is dependent). Indeed, if the new graph is not a pseudoforest, then it either contains an even cycle of two odd cycles connected by a bridge. The fact that removing any edge from a even circuit one obtains an odd-circle pseudoforest finishes the proof. □

Remark

Algebra $\mathcal{C}_+(G)$ counts integer points in the zonotope generated by the set of vectors $x_i + x_j, i < j$.

Remark

The matroid appearing in Theorem 5 is called the *even-circle matroid* by M. Doob and J.M.S. Simões-Pereira. It also coincides with a *signed graphic matroid* of T. Zaslavsky, which corresponds to the case when all the edges of G are given the negative sign. Additionally, the even-circle matroid of G is the same object as the *factor matroid*. It is also worth mentioning that the matrix $A_+(G)^T A_+(G) - 2I$ equals the adjacency matrix of the line graph of G which explains the appearance of the even-circle matroid in Doob's paper.

NEW ALGEBRA

The following definition was introduced by A. Kirillov.

Given an undirected graph $G = (V, E)$, let us label the vertices of V as $\{v_1, \dots, v_n\}$ where $n = |V|$. Now consider the double edge set $DE = E \cup \tilde{E}$ where for each edge $e \in E$, we introduce an edge $\tilde{e} \in \tilde{E}$ connecting the same pair of vertices. Next, consider the double edge algebra

$$\mathcal{A}DE_G = \frac{\mathbb{K}[DE]}{< x_e^2, x_{\tilde{e}}^2, x_e x_{\tilde{e}} >}$$

which is the quotient of the polynomial algebra over the field $\mathbb{K}$ generated by the variables x_e and $x_{\tilde{e}}$ corresponding to all edges in DE by the ideal generated by the squares of all edges in DE plus the relations $x_e x_{\tilde{e}} = 0$ for all edges in E.

To each vertex $v_\ell \in V, \ \ell = 1, \dots, n$, we associate the linear form in $\mathcal{A}DE_G$:

$$y_\ell = \sum_{i<\ell, e=(i,\ell)\in E} x_e + \sum_{j>\ell, \tilde{e}=(j,\ell)\in \tilde{E}} x_{\tilde{e}} = \sum_{k\neq\ell} x_{(k,\ell)}. \qquad (2.1)$$

Finally, associate to the graph $G = (V, E)$ with labelled vertices $(v_1, \dots, v_n)$ the subalgebra $\mathcal{N}\mathcal{U}_G \subset \mathcal{A}DE_G$ generated by the linear forms $y_1, \dots, y_n$. Obviously, $\mathcal{N}\mathcal{U}_G$ is a graded algebra a priori depending on the choice of vertex labelling.

Remark

The last formula in (2.1) implies that the algebra $\mathcal{N}\mathcal{U}_G$ is independent of the labelling of vertices.

Remark

For any choice of vertex labeling, there exists a natural surjective homomorphism of $\mathcal{NU}_G$ onto $\mathcal{C}_+(G)$ if one adds additional relations $x_{\tilde{e}} = x_e$ for all $e \in E$. Similarly, for any choice of vertex labeling, there exists a natural surjective homomorphism of $\mathcal{NU}_G$ onto $\mathcal{C}(G)$ if one adds additional relations $x_{\tilde{e}} = -x_e$ for all $e \in E$.

The main motivation of its consideration is extension of known results valid for simple graphs (i.e. without loops and multiple edges) by Orlik-Solomon and Postnikov-Shapiro-Shapiro. These results have been obtained by studying of some elements in anti-commutative algebras related to simple graphs in question. A natural even analog of these constructions is missing at the moment.

A equally natural question is an extension of the known results from the case of simple graphs to the case of graphs with multiple edges (and loops) which might be related to K-theory analogs of Orlik-Solomon and similar algebras. The Double edge (non-unitary) forest algebra is directly connected to the double edge graph without loops.

The next result describes $\mathcal{NU}_G$ in terms of generators and relations. Let's call $(a_1, a_2, \ldots, a_n)$ a *partial score vector* if there is a subset of edges $E' \subset E(G)$ and its orientation such that outgoing degrees of vertices are $a_1, a_2, \ldots, a_n$ respectively.

We define the *G-parking function polytope* (denoted by $\mathcal{P}_G$) as the convex hull of all partial score vectors. Observe that the classical parking function polytope which corresponds to the case of complex graphs and some generalizations different from ours have been studied in several recent papers.

Lemma

For any undirected graph $G = (V, R)$ on n vertices, the algebra $\mathcal{NU}_G$ is monomial.
Furthermore, $y_1^{a_1} y_2^{a_2} \dots y_n^{a_n} \neq 0$ if and only if $(a_1, a_2, \dots, a_n)$ is a partial score vector.

Proof.

We first observe the following: take an edge $e = (i, j)$ where $i < j$, then the variable x_e only occurs in the linear form y_j while the variable $x_{\tilde{e}}$ occurs only in y_i.
Therefore any monomial $\mathfrak{m}$ in the variables $x_e, x_{\tilde{e}}$ appears only in the expansion of the unique monomial in the variables $y_1, \ldots, y_n$. Therefore $\mathcal{NU}_G$ is a monomial algebra, furthermore, $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} \neq 0$ if and only if there is a subset of edges $E' \subset E(G)$ and its orientation such that outgoing degrees of vertices are $a_1, a_2, \ldots, a_n$ respectively. □

Lemma 11 shows that $\mathcal{NU}_G$ is the monomial algebra and one can extract all relations, however this set of relations is huge. Below we construct a better interpretation.

Notation. For any undirected graph $G = (V; E)$, and any subset $V_I = (v_{i_1}, v_{i_2}, \dots, v_{i_\ell})$ of its vertices, denote by κ_I the total number of edges of G at least one of whose vertices belong to V_I. Further, associate to V_I the set of monomials $\mathfrak{M}_I$ in the variables $v_{i_1}, v_{i_2}, \dots, v_{i_\ell}$ of the form

$$v_{i_1}^{k_1} v_{i_2}^{k_2} \dots v_{i_\ell}^{k_\ell} \text{ with } k_1 + k_2 + \dots + k_\ell = \kappa_I + 1.$$

Theorem

For any undirected graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$, one has

$$\mathcal{NU}_G \simeq \frac{\mathbb{K}[v_1, \dots, v_n]}{< \cup_{I \subseteq \{v_1, \dots, v_n\}} \mathfrak{M}_I >}.$$

Here the denominator is the monomial ideal generated by the union of all monomials occurring in $\mathfrak{M}_I$ where I runs over the set of all non-empty subsets of the set of vertices V.

Proof. We first show that for any subset $V_I = (v_{i_1}, v_{i_2}, \dots, v_{i_\ell})$ of vertices of G and any multiindex $(k_1, k_2, \dots, k_\ell)$ satisfying the equality $k_1 + k_2 + \cdots + k_\ell = \kappa_I + 1$, one has

$$y_{i_1}^{k_1} y_{i_2}^{k_2} \dots y_{i_\ell}^{k_\ell} = 0, \qquad (2.2)$$

where $y_1, y_2, \dots, y_n$ are given by (2.1). Indeed, if we expand $y_{i_1}^{k_1} y_{i_2}^{k_2} \dots y_{i_\ell}^{k_\ell}$ in x_e and $x_{\tilde{e}}$ only edges at least one end of which belongs to V_I will be involved. Since the total number of such edges is κ_I in each monomial of the expansion, then either x_e^2, $x_{\tilde{e}}^2$ or $x_e x_{\tilde{e}}$ for some e will appear in every such monomial. Thus (2.2) follows.

In order to prove the converse, we use Lemma 11. It remains to show that if $v_1^{a_1} v_2^{a_2} \cdots v_n^{a_n} \neq 0$, then there is a subset E' and its orientation such that all outgoing degrees are $a_1, a_2, \ldots, a_n$ respectively. Consider the bipartite graph $\mathcal{B}$ with two sets of vertices $B_1 = \{p_{1,1}, \ldots, p_{1,a_1}, p_{2,1}, \ldots, p_{2,a_2}, p_{3,1}, \ldots, p_{n,a_n}\}$ and $B_2 = E$ with the set of edges given by $(p_{i,k}, e)$ if and only if $v_i \in e$. Note that the condition $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} \neq 0$ is equivalent to the condition from Hall's marriage theorem. Therefore there exists a perfect matching M in $\mathcal{B}$.
We construct a E' and its orientation in the following way: if $(p_{i,k}, e) \in M$, then we choose orientation of e away from the vertex i. It is easy to check that the corresponding score vector is $a_1, a_2, \ldots, a_n$.

Theorem

The Hilbert function

$$h_{\mathcal{NU}_G}(t) := \sum_{k \geq 0} \dim \mathcal{NU}_G^{(k)} \cdot t^k$$

has the following properties.

(1) *It is a polynomial of degree $|E|$ which is the total number of edges in G;*

(2) $\dim \mathcal{NU}_G^{(|E|)}$ *equals the number of spanning forests of G;*

(3) $\dim \mathcal{NU}_G^{(\ell)}, \ell = 1, 2, \dots, |E|$ *equals the number of partial score vectors of G with the sum coordinates ℓ;*

(4) *the total dimension* $\dim \mathcal{NU}_G$ *equals the number of integer points in the G-parking function polytope.*

Proof of (1)-(3) Theorem 13.

Items (1) and (3) immediately follow from Lemma 11. Since the number of score vectors for any graph is equal to the number of spanning forests, the item (2) holds. □

We do not have other integer points in our polytope (it is (4)).

Proposition

Given G and $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$*, then* $(a_1, a_2, \ldots, a_n) \in \mathcal{P}_G$ *if and only if* $(a_1, a_2, \ldots, a_n)$ *is a partial score vector.*

Proof of Proposition 14 and (4) Theorem 13
We know that $(a_1, a_2, \ldots, a_n)$ is a partial score vectors if and only if $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \notin \langle \cup_{I \subseteq \{v_1,\ldots,v_n\}} \mathfrak{M}_I \rangle$ by Theorem 12. Hence, $(a_1, a_2, \ldots, a_n)$ is a partial score vector if and only if $\sum_{i \in I} a_i \leq \kappa_I$ for all $I \subset V$. Therefore, the set of partial score vectors is exactly the set of integer points of polytopes described by the above equations.

We can also describe all vertices of $\mathcal{P}_G$.

Theorem

$(a_1, a_2, \ldots, a_n)$ is a vertex of $\mathcal{P}_G$ if and only there is a pair $\pi \in S_n, k \leq n$ such that

$$a_{\pi_i} = \begin{cases} \textit{the number of edges between } v_{\pi_i} \textit{and } \{v_{\pi_1}, \ldots, v_{\pi_{i-1}}\}, & \textit{if } i > k. \\ 0, & \textit{otherwise.} \end{cases}$$

Lemma

For each graph G, the set function κ_X is submodular.

Proof.

We need to show that $\kappa_I + \kappa_J \geq \kappa_{I \cap J} + \kappa_{I \cup J}$. Remember that κ_X counts edges that incident to X. Let's count occurences of edge (a, b) in LHS and in RHS.

If a or b belongs to $I \cap J$, then we counted edge twice in LHS and twice in RHS. If $a, b \notin I \cap J$ and (a, b) is incident to at least one set, then we counted it at least once in LHS and exactly once in RHS.

Since the occurence of each edge is a submodular function, then κ is also submodular. □

Proof of Theorem 15
We use induction on the size of the graph G. For the graph everything is clear.
Let $\bar{a} = (a_1, a_2, \ldots, a_n)$ be a vertex of $\mathcal{P}_G$. We have two cases:
For any $\emptyset \neq I \subset V$, we have $\sum_{i \in I} a_i < \kappa_I$. Note that if $a_{k_1}, a_{k_2} > 0$, then $\bar{a} + e_{k_1} - e_{k_2}, \bar{a} - e_{k_1} + e_{k_2} \in \mathcal{P}_G$. Since $\bar{a}$ is a vertex, then there is k such that $a_i = 0$ for $i \neq k$. Hence $\bar{a}$ is a linear combination of 0 and $deg(v_k)e_k$. Hence $\bar{a} = 0$, because $\bar{a}$ is a vertex with the condition $a_k < deg(v_k)$.

There is $\emptyset \neq I \subset V$ such that $\sum_{i \in I} a_i = \kappa_I$. Let I be a minimal such set (by inclusion). If $I = \{v_k\}$, then we consider $\pi_n = k$ and $G' = G - v_k$. Note that $(a_1, a_2, \dots a_{k-1}, a_{k+1}, \dots a_n)$ is a vertex $\mathcal{P}_{G'}$, therefore by induction we can construct a permutation $\pi_1, \pi_2, \dots \pi_{k-1}, \pi_{k+1}, \dots \pi_n$ of $\{1, 2, \dots, k-1, k+1, \dots n\}$.

It remains the case when $|I| > 1$. Consider $k_1, k_2 \in I$. Since I is a minimal by inclusion set, $a_{k_1} > 0$ and $a_{k_2} > 0$. Since κ is a submodular function on subsets of vertices, there is no $J \subset V$ such that $\sum_{j \in J} a_j = \kappa_J$ and exactly one vertex of k_1, k_2 belongs to J (otherwise we can consider $I \subset J$, which is smaller than I). Hence $\bar{a} + e_{k_1} - e_{k_2}, \bar{a} - e_{k_1} + e_{k_2}$ both belong to $\mathcal{P}_G$. Indeed let's check for $\bar{a}' = \bar{a} + e_{k_1} - e_{k_2}$. For any $J \subset V$, we have $\sum_{j \in J} a'_j = \sum_{j \in J} a_j + \mathbb{1}_{k_1 \in J} - \mathbb{1}_{k_2 \in J}$. If $\{k_1, k_2\} \cap J = \{k_1, k_2\}$ or $\emptyset$, then

$$\sum_{j \in J} a_j + \mathbb{1}_{k_1 \in J} - \mathbb{1}_{k_2 \in J} = \sum_{j \in J} a_j \leq \kappa_J;$$

if $\{k_1, k_2\} \cap J = \{k_1\}$ or $\{k_2\}$, then

$$\sum_{j \in J} a_j + \mathbb{1}_{k_1 \in J} - \mathbb{1}_{k_2 \in J} < \kappa_J + \mathbb{1}_{k_1 \in J} - \mathbb{1}_{k_2 \in J} \leq \kappa_J + 1.$$

Hence, $\bar{a} + e_{k_1} - e_{k_2} \in \mathcal{P}(G)$, similarly $\bar{a} - e_{k_1} + e_{k_2} \in \mathcal{P}(G)$.

Note that for some graphs vertices can be described in a few different ways in Theorem 15, however we have a good bound.

Proposition

Given a graph G with n vertices, the number of vertices of $\mathcal{P}_G$ is at most $[(e-1)n!]$ with equality only for the complete graph K_n.

Proof.

Fixing $k \leq n$, it is easy to see that the vector from Theorem 15 depends only on $\pi_{k+1}, \ldots, \pi_n$. Moreover if there is edge between π_j and π_{j+1}, then we can swap them. Hence, the number of vertices for k is at most $(n-k)!\binom{n}{n-k} = \frac{n!}{k!}$ with equality only for complete graph.
Then the total number of vertices is at most

$$\sum^{n} \frac{n!}{k!} = n! \sum^{n} \frac{1}{k!} = n!(e-1) - n! \sum^{\infty} \frac{1}{j!} = [(e-1)n!]$$

Theorem

Given two graphs G_1 and G_2 without isolated vertices. Then $\mathcal{NU}_{G_1}$ and $\mathcal{NU}_{G_2}$ are isomorphic if and only if G_1 and G_2 are isomorphic.

Proof. It is obvious that for isomorphic graphs algebras are also isomorphic. Let's proof converse. Given two graphs G_1 and G_2 such that $\mathcal{NU}_{G_1} \simeq \mathcal{NU}_{G_2}$.
Define function $deg(u)$ as the smallest k such that $u^{k+1} = 0$ for $u \in \mathcal{NU}_{G_1}$. Note that, for non-zero $u \in \mathcal{NU}^{(1)}_{G_1}$, $deg(u)$ is exactly the number of edges appearing in u. We know that $dim(\mathcal{NU}_{G_1})$ is equal to number of vertices. Consider basis $u_1, u_2, \dots, u_n$ of $\mathcal{NU}^{(1)}_{G_1}$ such that $deg(u_1) + deg(u_2) + \dots + deg(u_n)$ is the smallest possible.

We have

$$deg(u_1)+deg(u_2)+\ldots+deg(u_n) = 2*|E(G_1)|-\#\{e: \ e \text{ is a loop}\},$$

because each non-loop edge appears at least in two different u_i and u_j and we have equality for basis corresponding to vertices. Let $y_1, \ldots, y_n \in \mathcal{NU}_{G_1}$ correspond to vertices of G_1. Since $y_1, \ldots, y_n$ and $u_1, \ldots, u_n$ are bases of $\mathcal{NU}_{G_1}$, there is a permutation $\pi \in S_n$ such that $u_i = c_{i,1}y_1 + \ldots + c_{i,n}y_n$ has non-zero coefficient c_{i,π_i}. Clearly $deg(u_i) \geq deg(y_{\pi_i})$, hence $deg(u_i) = deg(y_{\pi_i})$. Therefore the support of u_i has only edges corresponding to edges incident to v_{π_i}.

Hence, the number of edges between v_{π_i} and v_{π_j} is equal to $deg(u_i) + deg(u_j) - deg(u_i + \lambda u_j)$, where $\lambda \in \mathbb{K}$ is generic. Since we know how many edges incident to π_i and how many edges between π_i and π_j for each j, we can find the number of loops for vertex π_i. We reconstructed G_1 from $\mathcal{NU}_{G_1}$ (Note that u_i is not necessary $c_{i,\pi_i} v_{\pi_i}$). Similarly we can reconstruct G_2, therefore graphs are isomorphic. □

Definition

For a given undirected multigraph G and its edge e (which is not a loop), define
(i) $G - e$ as the graph G with the edge e deleted;
(ii) G/e as the graph G in which the end vertices of e were glued together and the edge e is transformed into a loop.

Theorem

For any undirected multigraph G and its edge e, one has:
(a) the deletion-contraction property:

$$h^e_G(t) = h^e_{G/e}(t) + t \cdot h^e_{G-e}(t)$$

(b) the multiplicativity property:

$$h^e_{G_1 \sqcup G_2}(t) = h^e_{G_1}(t) \cdot h^e_{G_2}(t)$$

with the initial conditions, for $n \geq 1$

$$h^e_{B_n}(t) = 1 + t + \cdots + t^n, \; h^e_{\circ}(t) = 1$$

Here B_n is the wedge of n loops.

Hilbert functions and dimensions for complete graphs

External algebras (K_2–K_9)

1, 2;
1, 3, 6, 7;
1, 4, 10, 20, 31, 40, 38;
1, 5, 15, 35, 70, 121, 185, 255, 310, 335, 291;
1, 6, 21, 56, 126, 252, 456, 756, 1161, 1666, 2232, 2796, 3281, 3546, 3516, 2932;
1, 7, 28, 84, 210, 462, 924, 1709, 2954, 4809, 7420, 10906, 15309, 20559, 26454, 32655, 38591, 43589, 46984, 47649, 45150, 36961;
1, 8, 36, 120, 330, 792, 1716, 3432, 6427, 11376, 19160, 30864, 47748, 71184, 102524, 142920, 193117, 253240, 322596, 399344, 480390, 561472, 637400, 701296, 746089, 765640, 748532, 691720, 561948;

1, 9, 45, 165, 495, 1287, 3003, 6435, 12870, 24301, 43677, 75177, 124485, 199035, 308187, 463287, 677520, 965493, 1342513, 1823553, 2421927, 3147723, 4005819, 4993839, 6100350, 7303545, 8570601, 9855829, 11101599, 12241305, 13203705, 13902291, 14254524, 14195199, 13575951, 12369033, 10026505.

Dimensions:
$3, 17, 144, 1623, 22804, 383415, 7501422, 167341283.$

Central algebras (K_2–K_{10})

1;
1, 3, 3;
1, 4, 10, 16, 19, 16;

1, 5, 15, 35, 65, 101, 135, 155, 155, 125;
1, 6, 21, 56, 126, 246, 426, 666, 951, 1246, 1506, 1686, 1731, 1626, 1296;
1, 7, 28, 84, 210, 462, 917, 1667, 2807, 4417, 6538, 9142, 12117, 15267, 18327, 20958, 22827, 23667, 23107, 21112, 16807;
1, 8, 36, 120, 330, 792, 1716, 3424, 6371, 11152, 18488, 29184, 44052, 63792, 88852, 119288, 154645, 193880, 235292, 276592, 315078, 347880, 371820, 384112, 382817, 364232, 328392, 262144;
1, 9, 45, 165, 495, 1287, 3003, 6435, 12861, 24229, 43353, 74097, 121515, 191907, 292743, 432399, 619677, 863109, 1170073, 1545777, 1992195, 2506983, 3082599, 3705795, 4357593, 5013801, 5645313, 6219649, 6703245, 7064073, 7267815, 7285959, 7100739, 6660495, 5966613, 4782969;

1, 10, 55, 220, 715, 2002, 5005, 11440, 24310, 48610, 92278, 167410, 291730, 490270, 797170, 1257454, 1928575, 2881450, 4200670, 5983570, 8337880, 11377750, 15218050, 19966990, 25717165, 32535466, 40452550, 49452730, 59465230, 70357750, 81931942, 93922750, 106002685, 117791350, 128869900, 138786766, 147077890, 153294730, 157034680, 157852210, 155381665, 149411470, 139011220, 124170310, 100000000.

Dimensions: 1, 7, 66, 792, 11590, 200469, 90759016, 2301604074

Internal algebras (K_2–K_{10}):

0;
1;
1, 4, 6, 4, 1;

1, 5, 15, 30, 45, 51, 45, 30, 15;
1, 6, 21, 56, 120, 216, 336, 456, 546, 580, 546, 456, 336, 216;
1, 7, 28, 84, 210, 455, 875, 1520, 2415, 3535, 4795, 6055, 7140, 7875, 8135, 7875, 7140, 6055, 4795, 3430;
1, 8, 36, 120, 330, 792, 1708, 3368, 6147, 10480, 16808, 25488, 36688, 50288, 65808, 82384, 98813, 113688, 125588, 133288, 135954, 133288, 125588, 113688, 98533, 81488, 61440;
1, 9, 45, 165, 495, 1287, 3003, 6426, 12789, 23905, 42273, 71127, 114387, 176463, 261891, 374808, 518301, 693693, 899857, 1132677, 1384803, 1645791, 1902663, 2140866, 2345553, 2503053, 2602341, 2636263, 2602341, 2502423, 2342907, 2134062, 1881243, 1596861, 1240029;

1, 10, 55, 220, 715, 2002, 5005, 11440, 24300, 48520, 91828, 165760, 286780, 477400, 767140, 1193104, 1799920, 2638800, 3765520, 5237200, 7107880, 9423040, 12213400, 15488560, 19231180, 23392456, 27889620, 32606080, 37394620, 42083800, 46487332, 50415760, 53689450, 56151700, 57679450, 58192656, 57660550, 56103820, 53573530, 50159560, 45988330, 41027500, 35399710, 28000000.

Dimensions: 1, 16, 237, 3892, 72425, 1521810, 35794801, 933875704.

Many thanks for your patience!