Short presentation of research

Boris Shapiro

About myself

Boris Shapiro, shapiro@math.su.se

Doctoral student by correspondence at Moscow State University under the advisorship of Vl. Arnold, 1984–1989. Defense failed in December 1989.

PhD from SU defended in June 1990.

Professor at SU since 2001.

About 110 papers, earliest publication – 1979. Principal advisor for 8 PhD students, coadvisor for 7 students. Currently 2 students.

For papers see: https://staff.math.su.se/shapiro/

Research interests

To avoid dullness I was trying to regularly change the fields of study. Have papers in the following areas:

- singularity theory;
- Schubert calculus and hyperplane arrangements;
- characteristic classes and distributions of manifolds;
- complex and real linear ordinary differential equations;
- Hamiltonian systems;
- Kazhdan-Lusztig theory;
- general combinatorics;
- real and complex analysis, Polya-Schur theory;
- potential theory;
- orthogonal polynomials and random matrices;
- asymptotic of polynomial sequences of different origins;
- Heine-Stieltjes theory;
- mathematical physics (Schrödinger equations etc);
- commutative algebra including the Waring problem;
- algebraic geometry and some other stuff . . .

Is it a good idea to change fields often?

I am not sure.

Probably not, since people (including in the first place those who administrate science) do not know in what pocket to place you . . .

On the other hand, some famous mathematicians were doing this with great success . . . . (I do not belong to this group.)

After all, the majority of us do math research just because we like it (and probably can not do anything else meaningful in life). Some do mathematics as a kind of sport (not my style though). Mathematics very seldom is about money, prices, or career.

If you do not like mathematics per se you can make MUCH MORE money by doing other things.

First project, joint with Vl. Kostov

Descartes rule of signs, Sur la construction de problémes solides ou plus que solide, La Géométrie, 1635.

Theorem. Given a real univariate polynomial with all non-vanishing coefficients, the number of its negative roots does not exceed the number of sign changes in the sequence of its coefficients.

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There are still basic unsolved questions related to the above rule. How could it be?

Fix a positive degree d and consider polynomials of degree d with all coefficients non-vanishing. An arbitrary ordered sequence $\overline{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_d)$ of $\pm$-signs is called a *sign pattern*. When working with monic polynomials we will use their *shortened sign patterns* $\widehat{\sigma}$ representing the signs of all coefficients except the leading term which equals 1. For the actual sign pattern $\overline{\sigma}$, we write $\overline{\sigma} = (1, \widehat{\sigma})$ to emphasise that we consider monic polynomials.

Given a shortened sign pattern $\widehat{\sigma}$, we say that its *Descartes pair* $(p_{\widehat{\sigma}}, n_{\widehat{\sigma}})$ is the pair of non-negative integers counting sign changes and sign preservations of $\overline{\sigma} = (1, \widehat{\sigma})$. By Descartes' rule of signs, $p_{\widehat{\sigma}}$ (resp. $n_{\widehat{\sigma}}$) gives the upper bound on the number of positive (resp. negative) roots of any monic polynomial from of degree d. (Observe that, for any $\widehat{\sigma}$, $p_{\widehat{\sigma}} + n_{\widehat{\sigma}} = d$.)

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To any monic polynomial $q(x)$ with the sign pattern $\overline{\sigma} = (1, \widehat{\sigma})$, we associate the pair (pos_q, neg_q) giving the numbers of its positive and negative roots counted with multiplicities. Obviously the pair (pos_q, neg_q) satisfies the standard restrictions

$$pos_q \leq p_{\overline{\sigma}},\ pos_q \equiv p_{\overline{\sigma}} (\bmod 2),\ neg_q \leq n_{\overline{\sigma}},\ neg_q \equiv n_{\overline{\sigma}} (\bmod 2). \tag{1}$$

We call pairs (pos, neg) satisfying (1) *admissible* for $\overline{\sigma}$. Conversely, for a given pair (pos, neg), we call a sign pattern $\overline{\sigma}$ such that (1) is satisfied *admitting* the latter pair. It turns out that there exist couples ($\overline{\sigma}$, (pos, neg)), where $\overline{\sigma}$ is a sign pattern and (pos, neg) is a pair admissible for $\overline{\sigma}$, which are not realizable by polynomials.

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D. J. Grabiner found the first example of non-realizable combination for polynomials of degree 4. He has shown that the sign pattern $(+, -, -, -, +)$ does not allow to realize the pair $(0, 2)$ and the sign pattern $(+, +, -, +, +)$ does not allow to realize $(2, 0)$. Observe that their Descartes pairs equal $(2, 2)$.

His argument is very simple. (Due to symmetry induced by $x \mapsto -x$ it suffices to consider only the first case.) Observe that a fourth-degree polynomial with only two negative roots for which the sum of roots is positive could be factored as $a(x^2 + bx + c)(x^2 - sx + t)$ with $a, b, c, s, t > 0,\ s^2 < 4t$, and $b^2 \geq 4c$. The product of these factors equals $a(x^4 + (b - s)x^3 + (t + c - bs)x^2 + (bt - cs)x + ct)$. To get the correct sign pattern, we need $b < s$ and $bt < cs$, which gives $b^2 t < s^2 c$ and thus $b^2/c < s^2/t$. But we have $b^2/c \geq 4 > s^2/t$. Contradiction!

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Unsolved problem. Describe which $(\overline{\sigma}, (pos, neg))$ are realizable.

Second project, Evolutes, joint with R.Piene and C.Riener

As we usually tell our students in calculus classes, the evolute of a curve in the Euclidean plane is the locus of its centers of curvature. The following intriguing information about evolutes can be found on Wikipedia :

"Apollonius (c. 200 BC) discussed evolutes in Book V of his treatise Conics. However, it is Huygens who is often credited for being the first to study them. Huygens formulated his theory of evolutes sometime around 1659 to help solve the problem of finding the tautochrone curve, which in turn helped him construct an isochronous pendulum. This was because the tautochrone curve is a cycloid, and cycloids have the unique property that their evolute is a cycloid of the same type. The theory of evolutes, in fact, allowed Huygens to achieve many results that would later be found using calculus.

Among several dozens of books on (plane) algebraic curves available now only very few mention evolutes at all, the best of them being Salmon's treatise of higher plane curves first published more than one and half century ago.

Some properties of evolutes have been studied in connection with the so-called 4-vertex theorem of Mukhopadhyaya–Kneser as well as its generalizations. Their definition has been generalized from the case of plane curves to that of plane fronts and also from the case of Euclidean plane to that of the Poincaré disk. Singularities of evolutes and involutes have been discussed in details by V. Arnold and his school.

Questions

We consider the evolutes of plane real-algebraic curves and discuss some of their complex and real algebraic properties. In particular, for a given degree $d \geq 2$, we provide lower bounds for the following 4 numerical invariants:

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– the maximal number of real cusps which can occur on the evolute of a real algebraic curve of degree d;

– the maximal number of (cru)nodes which can occur on the evolute of a real algebraic curve of degree d;

– the maximal number of (cru)nodes which can occur on the dual curve to the evolute of a real algebraic curve of degree d.

Illustrations

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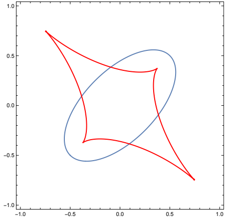

Figur: A rotated ellipse in blue and its evolute in red

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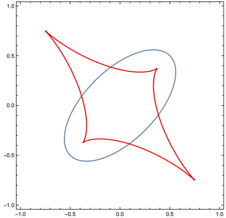

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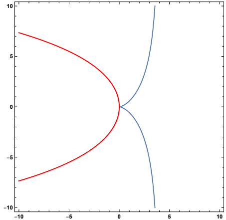

Figur: A Cissoid $(x^2 + y^2)x = 4y^2$ in blue and its evolute in red.

Illustrations, 2

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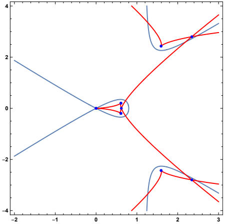

Figur: The nodal cubic in blue and its evolute in red with marked singular points.

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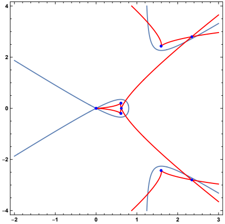

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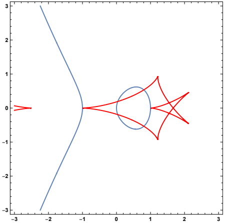

Figur: The Weierstrass cubic in blue and its evolute in red.

Illustrations, 3

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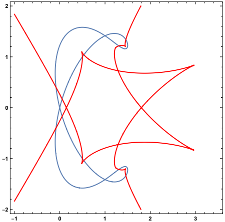

Figur: The ampersand curve in blue with its evolute in red

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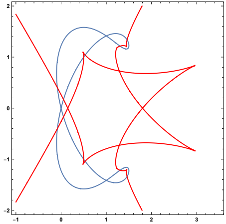

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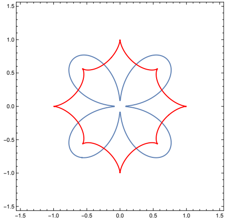

Figur: The quadfolium in blue with its evolute in red.