On moments of a polytope

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jointly with
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To the memory of Mikael Passare

Notation. In what follows we shall always assume that $\mathbb{R}^d$ is endowed with a fixed coordinate system $(x_1, ..., x_d)$ orthonormal with respect to the standard scalar product $\langle \cdot, \cdot \rangle$.

Let μ be a finite complex-valued Borel measure in $\mathbb{R}^d$. Given a multiindex $I = (i_1, ..., i_d)$, let $\mathbf{x}^I$ be the shorthand of the monomial $x_1^{i_1} \dots x_d^{i_d}$ and $|I| = i_1 + \dots + i_d$. For any multiindex I, define the *moment* $m_I(\mu)$ of μ as

$$m_I(\mu) := \int_{\mathbb{R}^d} x_1^{i_1} x_2^{i_2} ... x_d^{i_d} d\mu(x_1, x_2, \dots, x_d) = \int_{\mathbb{R}^d} \mathbf{x}^I d\mu(\mathbf{x}). \tag{1}$$

Define the *normalized moment generating function* of μ by

$$F_\mu(\mathbf{u}) := \sum_{I=(i_1,\dots,i_d),\ |I|\geq 0} \frac{(|I|+d)!}{i_1!\cdots i_d!} m_I(\mu)\mathbf{u}^I,$$

where $\mathbf{u} = (u_1, \dots, u_d)$ and $\mathbf{u}^I = u_1^{i_1} \dots u_d^{i_d}$.

Note that $F_\mu(\mathbf{u})$ admits the integral representation

$$F_\mu(\mathbf{u}) = d! \int_{\mathbb{R}^d} \frac{d\mu(\mathbf{x})}{(1 - \langle \mathbf{x}, \mathbf{u} \rangle)^{d+1}}, \qquad (2)$$

which is a special case of the so-called Fantappiè transformation.

A finite set $S \subset \mathbb{R}^d$ is called *spanning* if it is not contained in any hyperplane in $\mathbb{R}^d$.

As usual, by a *(compact) convex polytope* $\mathcal{P} \subset \mathbb{R}^d$ we mean the closed convex hull of a finite spanning set in $\mathbb{R}^d$.

A *d-simplex* in $\mathbb{R}^d$ is the convex hull of a spanning $(d+1)$-tuple of points.

By an *open polytope (resp. simplex)* we mean the set of interior points of a compact polytope (resp. simplex).

Given a convex polytope $\mathcal{P}$ let $\mathcal{V} = (\mathbf{v}_1, ..., \mathbf{v}_N)$ denote the set of its vertices. Assuming that $\mathcal{P}$ is simple (i.e. every vertex has exactly d incident edges), for each $\mathbf{v} \in \mathcal{V}$ we consider a fixed set of non-zero vectors $w_1(\mathbf{v}), \ldots, w_d(\mathbf{v})$ that are coming from $\mathbf{v}$ parallel to the edges incident to $\mathbf{v}$ in $\mathcal{P}$.

The polyhedral cone $K_{\mathbf{v}}$ coinciding with the non-negative real span of $w_1(\mathbf{v}), \ldots, w_d(\mathbf{v})$ is called *the tangent cone* of $\mathcal{P}$ at $\mathbf{v}$.

For each $K_{\mathbf{v}}$ define $|\det K_{\mathbf{v}}| = |\det(w_1(\mathbf{v}), \ldots, w_d(\mathbf{v}))|$ to be the volume of the parallelepiped formed by $w_1(\mathbf{v})$, $\ldots, w_d(\mathbf{v})$.

Given a bounded domain $\Omega \subset \mathbb{R}^d$ we call the measure $\mu_\Omega = \chi_\Omega dx_1 dx_2 \ldots dx_d$, where χ_Ω is the characteristic function of Ω, the *standard measure* of Ω.

Our first result is as follows.

Theorem 1. For an arbitrary simple convex polytope $\mathcal{P}$,

$$F_{\mathcal{P}}(\mathbf{u}) := F_{\mu_{\mathcal{P}}}(\mathbf{u}) = (-1)^d \sum_{\mathbf{v}\in\mathcal{V}} \frac{\langle \mathbf{v}, \mathbf{u}\rangle^d |\det K_{\mathbf{v}}|}{\prod_{j=1}^{d} \langle w_j(\mathbf{v}), \mathbf{u}\rangle} \cdot \frac{1}{1 - \langle \mathbf{v}, \mathbf{u}\rangle} = \tag{3}$$

$$= (-1)^d \sum_{\mathbf{v}\in\mathcal{V}} \frac{|\det K_{\mathbf{v}}|}{\prod_{j=1}^{d} \langle w_j(\mathbf{v}), \mathbf{u}\rangle} \cdot \frac{1}{1 - \langle \mathbf{v}, \mathbf{u}\rangle}. \tag{4}$$

In particular, if $\mathcal{P} = \Delta \subset \mathbb{R}^d$ is an arbitrary d-simplex then we get the following.

Corollary 2.

$$F_\Delta(\mathbf{u}) = \frac{d!\mathrm{Vol}(\Delta)}{\prod_{\mathbf{v}\in\mathcal{V}} (1 - \langle \mathbf{v}, \mathbf{u}\rangle)}. \tag{5}$$

Example 1. Let Δ be a triangle in $\mathbb{R}^2$ with vertices $v_1 = (1,1)$, $v_2 = (2,5)$ and $v_3 = (3,2)$. Its normalized moment generating function equals

$$F_\Delta(u,v) = \frac{7}{(1-u_1-u_2)(1-2u_1-5u_2)(1-3u_1-2u_2)}.$$

Its Taylor expansion about the origin up to the terms of degree 7 is given by

$$7+42u+56v+175u^2+455uv+329v^2+630u^3+2387u^2v+3367uv^2+1750v^3$$
$$+2107u^4+10318u^3v+21217u^2v^2+21546uv^3+8967v^4+6762u^5$$
$$+40082u^4v+106526u^3v^2+157976u^2v^3+128772uv^4+45276v^5+21175u^6$$
$$+145845u^5v+468895u^4v^2+900123u^3v^3+10744451u^2v^4+741993uv^5+227269v^6,$$

which implies that

$$m_{00}=\frac{7}{2}, m_{10}=7, m_{01}=\frac{28}{3}, m_{20}=\frac{175}{12}, m_{11}=\frac{455}{24}, m_{02}=\frac{329}{12}, m_{30}=\frac{63}{2},$$
$$m_{21}=\frac{2387}{60}, m_{12}=\frac{3591}{20}, m_{03}=\frac{175}{2}, m_{40}=\frac{2107}{30}, m_{31}=\frac{5159}{60}, m_{22}=\frac{21217}{180},$$
$$m_{13}=\frac{3591}{20}, m_{04}=\frac{2989}{10}, m_{50}=161, m_{41}=\frac{2863}{15}, m_{32}=\frac{7609}{30}, m_{23}=\frac{5642}{15},$$
$$m_{14}=\frac{3066}{5}, m_{05}=1078, m_{60}=\frac{3025}{8}, m_{51}=\frac{6945}{16}, m_{42}=\frac{13397}{24},$$
$$m_{33}=\frac{128589}{160}, m_{24}=\frac{153493}{120}, m_{15}=\frac{35333}{16}, m_{06}=\frac{32467}{8}.$$

Our second group of results addresses the problem of distinguishing different polytopes with the same underlying set of vertices from the information about their moments.

First we need to define what we mean by a polytope.

We shall study the following class of polytopal objects.

Definition 1. A subset $\mathcal{P} \subset \mathbb{R}^d$ coinciding with a finite union of arbitrary convex d-dimensional polytopes is called a *generalized polytope*.

We need to introduce the notion of a vertex of a generalized polytope.

Definition 2. Given a generalized polytope $\mathcal{P} \subset \mathbb{R}^d$ we call a finite collection of open disjoint d-dimensional simplices in $\mathbb{R}^d$ a *dissection* of $\mathcal{P}$ if the closure of their union coincides with $\mathcal{P}$.

Definition 3. Given a generalized polytope $\mathcal{P} \subset \mathbb{R}^d$ we call a point $\mathbf{v}$ a *vertex* of $\mathcal{P}$, if $\mathbf{v}$ is a vertex of (the closure of) some open simplex in every dissection of $\mathcal{P}$.

Definition 4. Given a point $p \in \mathcal{P}$ of a generalized polytope $\mathcal{P}$ we denote by the tangent cone $T_p(\mathcal{P})$ of $\mathcal{P}$ at p the set obtained as follows. For a sufficiently small $\epsilon > 0$ set $\mathcal{P}_p(\epsilon) = \mathcal{P} \cap B_p(\epsilon)$ where $B_p(\epsilon)$ is the ϵ-ball centered at p. Define $T_p(\mathcal{P})$ as the set obtained by taking a ray though p and every point of $\mathcal{P}_p(\epsilon)$. In other words, $T_p(\mathcal{P})$ is the cone with the apex at p and the base $B_p(\epsilon)$. (Obviously, $T_p(\mathcal{P})$ is independent of ϵ for a sufficiently small $\epsilon > 0$.)

Lemma 3. A point $\mathbf{v}$ is a *vertex* of $\mathcal{P}$ if and only if $T_{\mathbf{v}}(\mathcal{P})$ does not admit a decomposition in the disjoint union of convex polygonal subcones, such that each subcone in the decomposition has a translation-invariant direction. In particular, if the tangent cone to $\mathcal{P}$ at $\mathbf{v}$ has a connected component with no translation-invariant direction then $\mathbf{v}$ is a vertex.

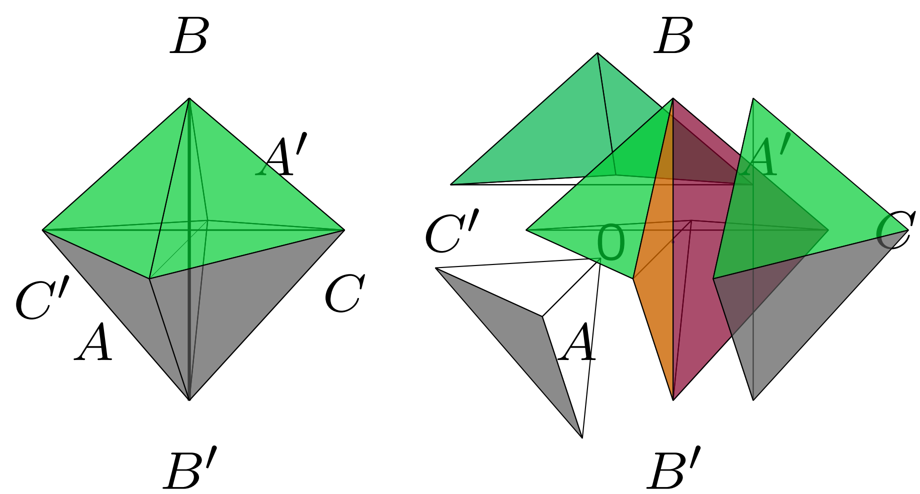
We use Conv(S) to denote the convex hull of arbitrary set S. The above lemma implies that any vertex of Conv$(\mathcal{P})$ is a vertex of $\mathcal{P}$.

Proposition 4. For any generalized polytope $\mathcal{P}$ the denominator $\Omega(\mathbf{u})$ of its normalized moment generating function $F_{\mathcal{P}}(\mathbf{u})$ divides the function

$$\Phi_{\mathcal{P}}(\mathbf{u}) := \prod_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} (1 - \langle \mathbf{v}, \mathbf{u} \rangle),$$

where $\mathcal{V}(\mathcal{P})$ is the set of vertices of $\mathcal{P}$.

Notice that there exist non-convex polytopes which do not admit triangulations with only existing vertices. The simplest example of this kind is the *Schönhardt polyhedron*. Therefore, Proposition 4 is not an immediate consequence of Corollary 2.



Schönhardt polyhedron obtained from an octahedron (left) by removing tetrahedra $[ABB'C]$, $[AA'B'C']$, and $[A'BCC']$.

Absense of a triangulation $\mathcal{T}$ which uses only its 6 vertices can be established by observing that none of the edges AC, $A'B$, and $B'C'$ can appear in a simplex of $\mathcal{T}$, yet any simplex on these 6 vertices must contain one of them.

Let $S \subset \mathbb{R}^d$ be a finite spanning set and $\mathcal{P}(S)$ be the set of all generalized polytopes $\mathcal{P}$ whose set $\mathcal{V}(\mathcal{P})$ of vertices is contained in S.

Given a generalized polytope $\mathcal{P} \in \mathcal{P}(S)$ consider its standard measure $\mu_{\mathcal{P}}$. (Obviously, $\mu_{\mathcal{P}}$ is supported on $\mathcal{P} \subseteq \mathrm{Conv}(S)$.)

Denote by $\mathfrak{M}(S)$ the linear space of all signed measures obtained as the linear span of all standard measures $\mu_{\mathcal{P}}$ where $\mathcal{P} \in \mathcal{P}(S)$.

Let $\mathfrak{M}^{\Delta}(S) \subseteq \mathfrak{M}(S)$ be its subspace spanned by μ_{Δ} where $\Delta \in \mathcal{P}(S)$ runs over the set of all d-dimensional simplices spanned by points in S. We shall refer to elements of $\mathfrak{M}(S)$ as to *polytopal measures* with the vertex set S.

The next conjecture is central in our study.

Conjecture 5. For an arbitrary spanning set S and any generalized polytope $\mathcal{P}$ whose set of vertices is contained in S its standard measure $\mu_{\mathcal{P}}$ belongs to $\mathfrak{M}^{\Delta}(S)$. In other words, $\mathfrak{M}(S) = \mathfrak{M}^{\Delta}(S)$.

The above conjecture is non-trivial as there exist polytopes which do not admit a triangulation using only the set of their vertices.

Notice that at the moment there is no formula or a receipt for how one could calculate the dimension of $\mathfrak{M}^{\Delta}(S)$ in terms of S.

Although we can not prove Conjecture 5 in complete generality we can settle it for sufficiently large class of spanning sets.

Namely, given a finite spanning set $S \subset \mathbb{R}^d$, we say that S is *weakly non-degenerate* if any $(d+2)$-tuple of points from S is spanning. If S satisfies the stronger condition that each $(d+1)$-subset of S is spanning then we call the latter S *strongly non-degenerate*.

Theorem 6. Conjecture 5 holds for any weakly non-degenerate finite set S.

Namely, let $\mathfrak{F}(S)$ (resp. $\mathfrak{F}^{\triangle}(S)$) be the linear space of Fantappiè transformations of signed measures in $\mathfrak{M}(S)$ (resp. $\mathfrak{M}^{\triangle}(S)$). In other words, $\mathfrak{F}(S)$ (resp. $\mathfrak{F}^{\triangle}(S)$) is the space of normalized moment generating functions of signed measures in $\mathfrak{M}(S)$ (resp. $\mathfrak{M}^{\triangle}(S)$).

Notice that since each compactly supported measure is uniquely determined by its complete set of moments the map

$$F_\mu : \mathfrak{M}(S) \to \mathfrak{F}(S), \tag{6}$$

induced by the Fantappiè transformation is a linear isomorphism.

Finally, given a spanning set $S \subset \mathbb{R}^d$, $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_N\}$ denote by $\mathfrak{Rat}(S)$ the linear space of all rational functions whose denominator $\Phi_S(\mathbf{u})$ is given by

$$\Phi_S(\mathbf{u}) = \prod_{i=1}^{N} (1 - \langle \mathbf{v}_i, \mathbf{u} \rangle), \tag{7}$$

and whose numerator is an arbitrary real (inhomogeneous) polynomial of degree at most $N - d - 1$.

Proposition 7. $\mathfrak{F}^{\triangle}(S)$ coincides with $\mathfrak{Rat}(S)$ if and only if S is strongly non-degenerate.

Corollary 8. If S is strongly non-degenerate then

$$\mathfrak{M}^{\triangle}(S) = \mathfrak{M}(S).$$

As a consequence of Corollary 8 we get that for strongly non-degenerate S the dimension of all these linear spaces equals $\binom{N-1}{d}$.

Our final goal is to explicitly solve the following inverse moment problem.

Problem 1. Given a strongly non-degenerate spanning set $S \subset \mathbb{R}^d$, card $S = N$ find the unique polytopal measure in $\mathfrak{M}(S)$ with a given set of all moments up to order $N - d - 1$.

Lemma 9. Given an arbitrary spanning set $S \subset \mathbb{R}^d$, card $S = N$ and an arbitrary polynomial $T(\mathbf{u})$ of degree at most $N-d-1$ there exists a unique rational function $R(\mathbf{u}) = P(\mathbf{u})/\Phi_S(\mathbf{u})$ whose Taylor polynomial of degree $N - d - 1$ at the origin coincides with $T(\mathbf{u})$. Namely, $P(\mathbf{u}) = [T(\mathbf{u})\Phi_S(\mathbf{u})]_{N-d-1}$, where $[\cdot]_{N-d-1}$ stands for the truncated polynomial with all monomials up to degree $N - d - 1$.

Assuming that $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_N\} \subset \mathbb{R}^d$ is strongly non-degenerate we present an explicit inversion formula determining the densities of an unknown polytopal measure having a given set of moments up to order $N-d-1$ on each simplex from a natural basis of $\mathfrak{M}^\Delta(S)$.

In fact, using Lemma 9 we can assume that we are already given an arbitrary rational function $R(\mathbf{u}) = P(\mathbf{u})/\Phi_S(\mathbf{u})$ where $\deg P(\mathbf{u}) \leq N - d - 1$ and we want to determine the densities of the required signed measure from $\mathfrak{M}(S)$ in terms of numerator $P(\mathbf{u})$.

From now on we shall choose the basis of $\mathfrak{M}^\Delta(S)$ consisting of the standard measures of all simplices containing the last vertex $\mathbf{v}_N$.

Let $\mathfrak{L} = \{l_1, l_2, ..., l_{N-1}\}$ be the $(N-1)$-tuple of linear forms corresponding to vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{N-1}$, where $l_i(\mathbf{u}) = 1 - \langle \mathbf{v}_i, \mathbf{u} \rangle$.

Consider the linear span $V_{\mathfrak{L}}$ of all possible products of the form $l_{j_1} \cdot l_{j_2} \cdot ... \cdot l_{j_{N-d-1}}$, $1 \leq j_1 < j_2 < ... < j_{N-d-1}$. Notice that the number of such products equals to $\binom{N-1}{d}$ and each such product is a polynomial of degree at most $N-d-1$.

On the other hand, the dimension of the space $Pol(N-d-1, d)$ of all (inhomogeneous) polynomials of degree at most $N-d-1$ in d variables also equals $\binom{N-1}{d}$.

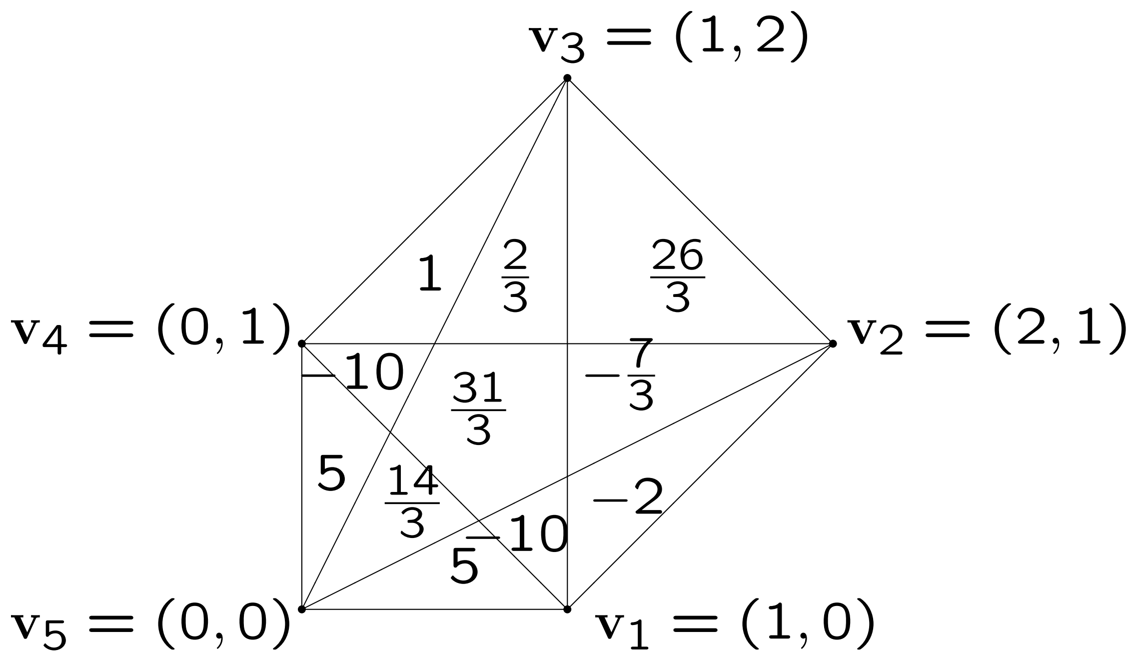
Define the square matrix Mat_S of size $\binom{N-1}{d}$ whose elements are coefficients of the above products of linear forms with respect to the standard monomial basis in $Pol(N-d-1, d)$. We assume that Mat_S acts on the space $V_{\mathfrak{L}}$ of *column* vectors.

Theorem 10. For an arbitrary strongly non-degenerate spanning set $S \subset \mathbb{R}^d$, card $S = N$ the matrix Mat_S is invertible. Moreover, for a rational function $R(\mathbf{u}) = P(\mathbf{u})/\Phi_S(\mathbf{u})$ where $P(\mathbf{u})$ is an arbitrary polynomial of degree $N-d-1$ there exists a unique measure $\mu_R \in \mathfrak{M}(S)$ whose Fantappiè transform equals $R(\mathbf{u})$. It is given by:

$$\mu_R = Mat_S^{-1}(P(\mathbf{u})). \tag{8}$$

We finish by explicitly solving the above inverse problem for a concrete 5-tuple of points in $\mathbb{R}^2$.

Example 2. Set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ where $\mathbf{v}_1 = (1,0), \mathbf{v}_2 = (2,1), \mathbf{v}_3 = (1,2), \mathbf{v}_4 = (0,1), \mathbf{v}_5 = (0,0)$.



Final measure in Example 2.

The corresponding set $\mathfrak{L} = \{l_1, l_2, l_3, l_4\}$ of linear forms is given by $l_1 = 1 - u_1$, $l_2 = 1 - 2u_1 - u_2$, $l_3 = 1 - u_1 - 2u_2$, $l_4 = 1 - u_2$. Additionally, $l_5 = 1$.

We are considering the basis of $\mathfrak{M}^{\Delta}(S)$ consisting of (the standard measures of) 6 triangles containing $\mathbf{v}_5$.

Therefore we need 6 quadratic forms obtained as pairwise products $l_i l_j$, $1 \leq i < j \leq 4$. We get

$$\begin{cases} l_1 l_2 = 1 - 3u_1 - u_2 + 2u_1^2 + u_1 u_2 \\ l_1 l_3 = 1 - 2u_1 - 2u_2 + u_1^2 + 2u_1 u_2 \\ l_1 l_4 = 1 - u_1 - u_2 + u_1 u_2 \\ l_2 l_3 = 1 - 3u_1 - 3u_2 + 2u_1^2 + 5u_1 u_2 + 2u_2 \\ l_2 l_4 = 1 - 2u_1 - 2u_2 + 2u_1 u_2 + u_2^2 \\ l_3 l_4 = 1 - u_1 - 3u_2 + u_1 u_2 + 2u_2^2. \end{cases}$$

Notice that l_1l_2 corresponds to triangle Δ_{345}, l_1l_3 to Δ_{245}, l_1l_3 to Δ_{245}, l_1l_4 to Δ_{234}, l_2l_3 to Δ_{145}, l_2l_4 to Δ_{135}, and l_3l_4 to Δ_{125}. Ordering monomials spanning the space $Pol(2,2)$ as $(1, u_1, u_2, u_1^2, u_1u_2, u_2^2)$ we get the inverse 6×6-matrix Mat_S^{-1} as follows

$$4Mat_S^{-1} = \begin{array}{c} \\ l_1l_2 \\ l_1l_3 \\ l_1l_4 \\ l_2l_3 \\ l_2l_4 \\ l_3l_4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & u_1 & u_2 & u_1^2 & u_1u_2 & u_2^2 \end{array} \\ \begin{pmatrix} 1 & -1 & 1 & 1 & 1 & -1 \\ -4 & 0 & -4 & 0 & 0 & -4 \\ 9 & 3 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -4 & 0 & -4 & 0 & 0 \\ 1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array}.$$

Given an arbitrary rational function $R(u_1, u_2) = \frac{P(u_1,u_2)}{\Phi_S(u_1,u_2)}$ where

$$P(u_1, u_2) = a_{00} + a_{1,0}u_1 + a_{0,1}u_2 + a_{2,0}u_1^2 + a_{11}u_1u_2 + a_{02}u_2^2$$

and $\Phi_S(u_1, u_2) = l_1 l_2 l_3 l_4 l_5$ we get

$$\begin{cases} w_{345} = \frac{1}{4}(a_{00} - a_{10} + a_{01} + a_{20} - a_{11} + a_{02}) \\ w_{245} = -a_{00} - a_{01} - a_{02} \\ w_{235} = \frac{1}{4}(9a_{00} + 3a_{01} + 3a_{10} + a_{20} + a_{11} + a_{02}) \\ w_{145} = \frac{1}{4}(a_{00} + a_{01} + a_{10} + a_{20} + a_{11} + a_{02}) \\ w_{135} = -a_{00} - a_{10} - a_{20} \\ w_{125} = \frac{1}{4}(a_{00} + a_{10} - a_{01} + a_{20} - a_{11} + a_{02}), \end{cases}$$

where w_{ijk} is the weight of the signed measure in question which should be placed at triangle Δ_{ijk}.

Here by the weight of a simplex we mean the following.

Definition 5. Given a signed measure μ in $\mathbb{R}^d$ and a d-dimensional simplex $\Delta \subset \mathbb{R}^d$ we define the *weight* w_Δ of Δ by the formula:

$$w_\Delta = d! \int_\Delta d\mu. \tag{9}$$

In other words, the density d_Δ of the measure in question which should be placed at Δ equals

$$d_\Delta = \frac{w_\Delta}{d! Vol(\Delta)}.$$

To illustrate all steps of solution of our inverse moment problem assume in the latter example that we are looking for a polygonal measure with the above vertex set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ and having the (ad hoc chosen) moments

$$m_{00} = 1, m_{10} = 2, m_{01} = 3, m_{20} = 4, m_{11} = 5, m_{02} = 6.$$

Then its normalized moment generating function $F_\mu(\mathbf{u})$ satisfies the relation

$$F_\mu(\mathbf{u}) = 1\frac{2!}{0!0!} + 2\frac{3!}{1!0!}u_1 + 3\frac{3!}{0!1!}u_2 + 4\frac{4!}{2!0!}u_1^2 + 5\frac{4!}{1!1!}u_1u_2 +$$

$$+6\frac{4!}{0!2!}u_2^2 + \cdots = \frac{P(u_1, u_2)}{l_1 l_2 l_3 l_4 l_5},$$

where $P(u_1, u_2)$ is a (non-homogeneous) polynomial of at most second degree. Thus truncating the product of the left-hand side and $l_1 l_2 l_3 l_4 l_5$ up to the second degree we obtain

$$P(u_1, u_2) = 2 + 4u_1 + 10u_2 + 10u_1^2 + 24u_1u_2 + 10u_2^2,$$

i.e. $a_{00} = 2, a_{10} = 4, a_{01} = 10, a_{20} = 10, a_{11} = 24, a_{02} = 10$. Thus $w_{345} = 1, w_{245} = -22, w_{235} = 26, w_{145} = 15, w_{135} = -16, w_{125} = -2$, see (9).

The areas of the corresponding triangles are equal to: $Area(\Delta_{345}) = \frac{1}{2}; Area(\Delta_{245}) = 1; Area(\Delta_{235}) = \frac{3}{2}; Area(\Delta_{145}) = \frac{1}{2}; Area(\Delta_{135}) = 1; Area(\Delta_{125}) = \frac{1}{2}$.

This implies that the densities of the measure of the corresponding triangles are equal to

$$d_{345} = 1, d_{245} = -11, d_{235} = \frac{26}{3}, d_{145} = 15, d_{135} = -8, d_{125} = -2.$$

To obtain the final densities in the convex hull $Conv(S)$ of S one has to decompose $Conv(S)$ into domains obtained by removing from $Conv(S)$ the set of all hyperplanes spanned by vertices in S. For each such domain we should add up the densities of all basic simplices containing this domain.

Choosing an arbitrary basis of $\mathfrak{M}^{\Delta}(S)$ consisting of the standard measures of simplices let us introduce the integer lattice in $\mathfrak{M}^{\Delta}(S)$ using the latter basis. (One can easily see that this lattice is invariantly defined independently of the choice of a basis of standard measures of simplices.) Denote by $\mathfrak{M}^{\Delta}_{\mathbb{Z}}(S)$ the space $\mathfrak{M}^{\Delta}(S)$ with the latter lattice. We can prove the following conditional statement valid if Conjecture 5 holds.

Proposition 11. For any generalized polytope $\mathcal{P} \in \mathcal{P}(S)$ whose standard measure $\mu_{\mathcal{P}}$ belongs to $\mathfrak{M}^{\Delta}(S)$ it is a rational point in $\mathfrak{M}^{\Delta}_{\mathbb{Z}}(S)$.

Let us pose the following tantalizing question.

Problem 2. Does there exist a generalized polytope $\mathcal{P}$ whose coordinates in $\mathfrak{M}_{\mathbb{Z}}(S(\mathcal{P}))$ are not integer where $S(\mathcal{P})$ is the set of its own vertices.

Problem 3. One can also define an important rational convex cone $\mathfrak{Pos}(S) \subset \mathfrak{M}_{\mathbb{Z}}(S)$ by taking non-negative linear combinations of all $\mu_{\mathcal{P}}$ where $\mathcal{P}$ runs over the set of all generalized polytopes in $\mathcal{P}(S)$.

Conjecture 12. The rational cone $\mathfrak{Pos}(S)$ is uniquely determined by the oriented matroid associated to S.

THANK YOU!