Mystery of points charges after C. F. Gauß, J. C. Maxwell and M. Morse

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Main references

[Max] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, **vol. 1**, Republication of the 3rd revised edition, Dover Publ. Inc., 1954.

[MC] M. Morse and S. Cairns, *Critical Point Theory in Global Analysis and Differential Topology*, Acad. Press, 1969.

[Ma] M. Marden, *Geometry of polynomials*, Publ.AMS, 1949.

[Ga] C. F. Gauß, *Gesammelte Verk*, b.7, s. 212., Göttingen.

[GNS] A. Gabrielov, D. Novikov, and B. Shapiro *Mystery of point charges*, Proc. London Math. Soc. (3) vol 95, issue 2 (2007) 443–472.

Newton potential of a point charge ζ located at a point $\bar{x}^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$ is given by

$$V(\bar{x}) = V(x_1, \ldots, x_n) = \begin{cases} \frac{\zeta}{r^{n-2}}, & \text{for } n \geq 3 \\ \zeta \ln r, & \text{for } n = 2, \end{cases}$$

where r is the distance between $\bar{x}$ and $\bar{x}^0$.

The potential of a configuration of point charges is the sum of individual potentials. An **equilibrium point** of a configuration of point charges is a point where the gradient (i.e. the resulting electrostatic force) of its potential vanishes.

Problem

Given a configuration of l positive and negative point charges in $\mathbb{R}^n$ estimate from **above** the maximal possible number of points of equilibrium of this configuration in terms of l and n.

2-dimensional case after C. F. Gauß

In $\mathbb{R}^2 \simeq \mathbb{C}$ the potential of a unit point placed at the origin equals $\ln |r|$. Therefore, the potential of a system of l charges placed at $z_1, \dots, z_l$ with the values $\zeta_1, \dots, \zeta_l$ equals

$$V(z) = \sum_{i=1}^{l} \zeta_i \ln |z - z_i| = \ln \left(\Pi_{i=1}^{l} |z - z_i|^{\zeta_i} \right).$$

Theorem, [Ga]. Points of equilibrium for the potential $\Pi(z)$ coincide with the zeros of the rational function

$$P(z) = \sum_{i=1}^{l} \frac{\zeta_i}{z - z_i}.$$

They are all saddlepoints and their total number is most $l - 1$.

In dimensions 3 or larger

Consider a configuration of $l = \mu + \nu$ fixed point charges in $\mathbb{R}^n$, $n \geq 3$ consisting of μ positive charges with the values $\zeta_1, \ldots, \zeta_\mu$, and ν negative charges with the values $\zeta_{\mu+1}, \ldots, \zeta_l$. They create an electrostatic field whose potential equals

$$V(\bar{x}) = \frac{\zeta_1}{r_1^{n-2}} + \ldots + \frac{\zeta_l}{r_l^{n-2}}, \tag{1}$$

where r_i is the distance between the i-th charge and the point $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ which we assume different from the locations of the charges. Below we consider the problem of **finding effective upper bounds** on the number of critical points of $V(\bar{x})$, i.e. the number of points of equilibrium of the electrostatic force.

In what follows we mostly assume that considered configurations of charges have only nondegenerate critical points. This guarantees that the number of critical points is finite. Such configurations of charges and potentials will be called **nondegenerate**. Surprisingly little is known about this whole topic and the references are very scarce.

In the case of $\mathbb{R}^3$ one of the few known results obtained by direct application of Morse theory to $V(\bar{x})$ is as follows, see [MC], Theorem 32.1.

Theorem. Assume that the total charge $\sum_{i=1}^{l} \zeta_j$ in (1) is negative (resp. positive). Let m_1 be the number of the critical points of index 1 of V, and m_2 be the number of the critical points of index 2 of V. Then $m_2 \geq \mu$ (resp. $m_2 \geq \mu - 1$) and $m_1 \geq \nu - 1$ (resp. $m_1 \geq \nu$). Additionally, $m_1 - m_2 = \nu - \mu - 1$.

Note that the potential $V(\bar{x})$ has no (local) maxima or minima due to its harmonicity.

Remark. The remaining (more difficult) case $\sum_{i=1}^{\mu} \zeta_i + \sum_{j=\mu+1}^{l} \zeta_j = 0$ was treated by Kiang.

Remark. The above theorem has a generalization to any $\mathbb{R}^n$, $n \geq 3$ with m_1 being the number of the critical points of index 1 and m_2 being the number of the critical points of index $n-1$.

Definition. Configurations of charges with all nondegenerate critical points and $m_1 + m_2 = \mu + \nu - 1$ are called **minimal**, see [MC], p. 292.

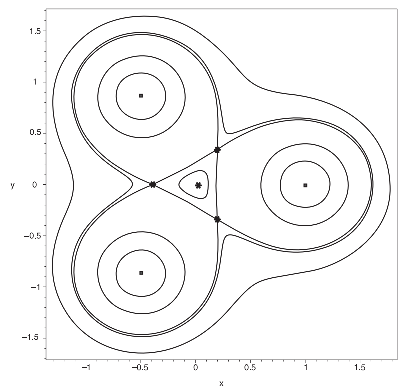
Remark. Minimal configurations occur if one, for example, places all charges of the same sign on a straight line. On the other hand, it is easy to construct generic nonminimal configurations of charges, see [MC].

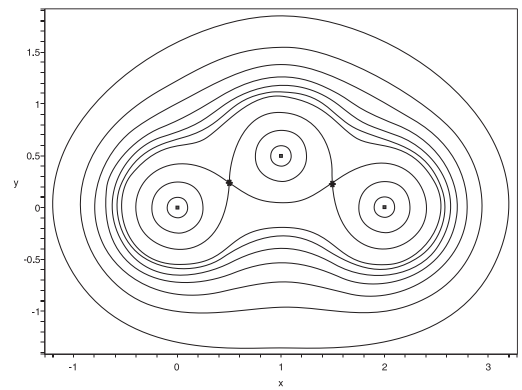
Remark. The major difficulty of this problem is that the **lower bound** on the number of critical points of V_n given by Morse theory is known to be non-exact. Therefore, since we are interested in an effective **upper bound**, the Morse theory arguments do not provide an answer.

The question about the maximum (if it exists) of the number of points of equilibrium of a nondegenerate configuration of charges in $\mathbb{R}^3$ was posed in [MC], p. 293. In fact, J. C. Maxwell in [Max], section 113 made an explicit claim answering exactly this question.

Conjecture, [Max]. The total number of points of equilibrium (all assumed nondegenerate) of any configuration with l charges in $\mathbb{R}^3$ never exceeds $(l-1)^2$.

Remark. In particular, there are at most 4 points of equilibrium for any configuration of 3 point charges according to Maxwell, see Figure 1.





Configurations with two and with four critical points.

Before formulating our results and conjectures let us first generalize the set-up. In the notation of Theorem 1 consider the family of potentials depending on a parameter $\alpha \geq 0$ and given by

$$V_\alpha(\bar{x}) = \frac{\zeta_1}{\rho_1^\alpha} + \ldots + \frac{\zeta_l}{\rho_l^\alpha}, \tag{2}$$

where $\rho_i = r_i^2,\ i = 1, \ldots, l$. (The choice of ρ_i's instead of r_i's is motivated by convenience of algebraic manipulations.)

Notation. Denote by $N_l(n,\alpha)$ the maximal number of the critical points of the potential (2) where the maximum is taken over all nondegenerate configurations with l variable point charges, i.e. over all possible values and locations of l point charges forming a nondegenerate configuration.

Theorem. a) For any $\alpha \geq 0$ and any positive integer n one has

$$N_l(n, \alpha) \leq 4^{l^2} (3l)^{2l}. \tag{3}$$

b) For $l = 3$ one has a significantly improved upper bound

$$N_3(n, \alpha) \leq 12.$$

Remark. Note that the right-hand side of the formula (3) gives even for $l = 3$ the horrible upper bound $139, 314, 069, 504$. On the other hand, computer experiments suggest that Maxwell was right and that for any three charges there are at most 4 (and not 12) critical points of the potential (2), see Figure 1.

The latter Theorem is obtained by a straightforward application of the following important result in the so-called fewnomial theory due to A.Khovanskii.

Consider a system of m quasipolynomial equations

$$P_1(\bar{u}, \bar{w}(\bar{u})) = \ldots = P_m(\bar{u}, \bar{w}(\bar{u})) = 0,$$

where $\bar{u} = (u_1, \ldots, u_m)$, $\bar{w} = (w_1, \ldots, w_k)$ and each $P_i(\bar{u}, \bar{w})$ is a real polynomial of degree d_i in $m + k$ variables $(\bar{u}, \bar{w})$. Additionally,

$$w_j = e^{a_{j,1}u_1 + \ldots + a_{j,m}u_m},\ a_{j,i} \in \mathbb{R}, \quad j = 1, \ldots, k.$$

Theorem. In the above notation the number of real isolated solutions of this system does not exceed

$$d_1 \cdots d_m (d_1 + \cdots + d_m + 1)^k 2^{k(k-1)/2}.$$

Voronoi diagrams and the main conjecture.

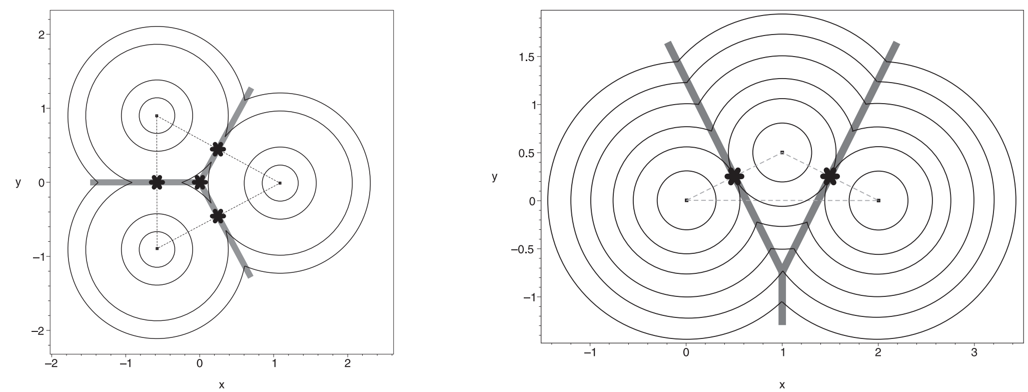
Theorem below determines the number of critical points of the function V_α for large α in terms of the combinatorial properties of the configuration of the charges. To describe it we need to introduce several notions.

Notation. By a (classical) **Voronoi diagram** of a configuration of pairwise distinct points (called sites) in the Euclidean space $\mathbb{R}^n$ we understand the partition of $\mathbb{R}^n$ into convex cells according to the distance to the nearest site.

A **Voronoi cell** S of the Voronoi diagram consists of all points having exactly the same set of nearest sites.

The set of all nearest sites of a given Voronoi cell S is denoted by $\mathcal{NS}(S)$. One can see that each Voronoi cell is a interior of a convex polyhedron, probably of positive codimension. This is a slight generalization of traditional terminology, which considers the Voronoi cells of the highest dimension only. A Voronoi cell of the Voronoi diagram of a configuration of sites is called **effective** if it intersects the convex hull of $\mathcal{NS}(S)$.

Example. Voronoi diagram of three non-collinear points A, B, C on the plane consists of seven Voronoi cells:



1. three two-dimensional cells S_A, S_B, S_C with $\mathcal{NS}(S)$ consisting of one point,

2. three one-dimensional cells S_{AB}, S_{AC}, S_{BC} with $\mathcal{NS}(S)$ consisting of two points. For example, S_{AB} is a part of a perpendicular bisector of the segment $[A, B]$.

3. one zero-dimensional cell S_{ABC} with $\mathcal{NS}(S)$ consisting of all three points. This is a point equidistant from all three points.

There are two types of generic configurations. First type is of an acute triangle ΔABC and then all Voronoi cells are effective. Second type is of an obtuse triangle ΔABC, and then (for the obtuse angle A) the Voronoi cells S_{BC} and S_{ABC} are not effective.

The case of the right triangle ΔABC is non-generic: the cell S_{ABC}, though effective, lies on the boundary of the triangle.

If we have an additional affine subspace $L \subset \mathbb{R}^n$ we call a Voronoi cell S of the Voronoi diagram of a configuration of charges in $\mathbb{R}^n$ **effective with respect to** L if S intersects the convex hull of the orthogonal projection of $\mathcal{NS}(S)$ onto L.

A configuration of points is called **generic** if any Voronoi cell S of its Voronoi diagram of any codimension k has exactly $k+1$ nearest cites and does not intersect the boundary of the convex hull of $\mathcal{NS}(S)$.

A subspace L intersects a Voronoi diagram **generically** if it intersects all its Voronoi cells transversally, any Voronoi cell S of codimension k intersecting L has exactly $k+1$ nearest sites, and S does not intersect the boundary of the convex hull of the orthogonal projection of $\mathcal{NS}(S)$ onto L.

The **combinatorial complexity** (resp. **effective combinatorial complexity**) of a given configuration of points is the total number of cells (resp. effective cells) of all dimensions in its Voronoi diagram.

Theorem.

a) For any generic configuration of point charges of the same sign there exists $\alpha_0 > 0$ such that for any $\alpha \geq \alpha_0$ the critical points of the potential $V_\alpha(\bar{x})$ are in one-to-one correspondence with effective cells of positive codimension in the Voronoi diagram of the considered configuration. The Morse index of each critical point coincides with the dimension of the corresponding Voronoi cell.

b) Suppose that an affine subspace L intersects generically the Voronoi diagram of a given configuration of point charges of the same sign.

Then there exists $\alpha_0 > 0$ (depending on the configuration and L) such that for any $\alpha \geq \alpha_0$ the critical points of the restriction of the potential $V_\alpha(\bar{x})$ to L are in one-to-one correspondence with effective w.r.t. L cells of positive codimension in the Voronoi diagram of the considered configuration. The Morse index of each critical point coincides with the dimension of the intersection of the corresponding Voronoi cell with L.

Conjecture.

a) For any generic configuration of point charges of the same sign and any $\alpha \geq \frac{1}{2}$ one has

$$a_\alpha^j \leq \sharp^j, \tag{4}$$

where a_α^j is the number of the critical points of index j of the potential $V_\alpha(\bar{x})$ and $\sharp^j$ is the number of all effective Voronoi cells of dimension j in the Voronoi diagram of the considered configuration.

b) For any affine subspace L generically intersecting the Voronoi diagram of a given configuration of point charges of the same sign one has

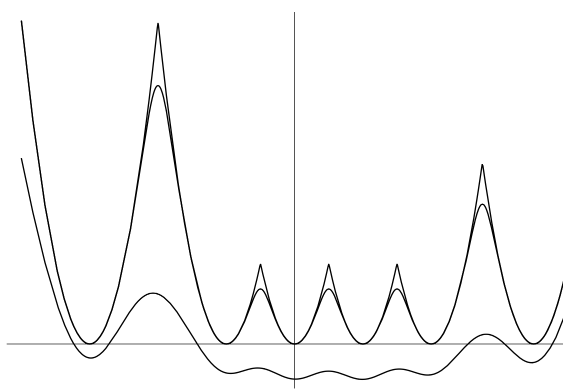
$$a_{\alpha,L}^j \leq \sharp_L^j, \tag{5}$$

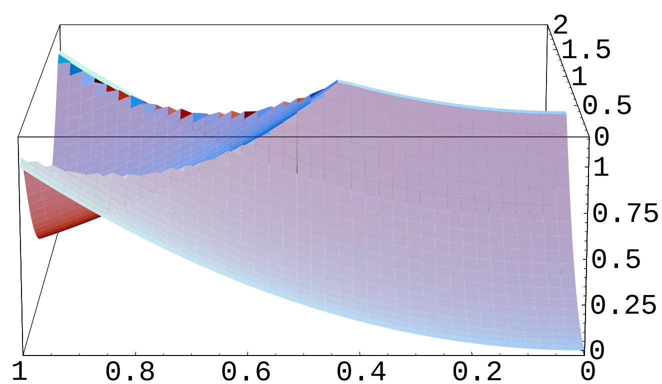
where $a^j_{\alpha,L}$ is the number of the critical points of index j of the potential $V_\alpha(\bar{x})$ restricted to L and $\sharp^j_L$ is the number of all Voronoi cells with dim$(S \cap L) = j$ effective w.r.t L in the Voronoi diagram of the considered configuration.

We will refer to the inequality (4) resp. (5) as **Maxwell** resp. **relative Maxwell** inequality.

Remark. Theorem above and Conjecture were inspired by two observations. On one hand, one can compute the limit of a properly normalized potential $V_\alpha(\bar{x})$ when $\alpha \to \infty$. Namely, one can easily show that

$$\lim_{\alpha\to\infty} V_\alpha{}^{-\frac{1}{\alpha}}(\bar{x}) = V_\infty(\bar{x}) = \min_{i=1,\ldots,l} \rho_i(\bar{x}).$$





This limiting function is only piecewise smooth. However, one can still define critical points of $V_\infty(\bar{x})$ and their Morse indices. Moreover, it turns out that for generic configurations every critical point of $V_\infty(\bar{x})$ lies on a separate effective cell of the Voronoi diagram whose dimension equals the Morse index of that critical point. Our main Theorem above claims that for sufficiently large α the situation is the same, except that the critical point does not lie exactly on the corresponding Voronoi cell (in fact, it lies on $O(\alpha^{-1})$ distance from this Voronoi cell. On the other hand, computer experiments show that the largest number of critical points (if one fixes the positions and values of charges) occurs when $\alpha \to \infty$.

Even the special case of the conjecture when L is one-dimensional is of interest and still open. Its slightly stronger version supported by extensive numerical evidence can be reformulated as follows.

Conjecture. Consider an l-tuple of points $(x_1, y_1), \ldots, (x_l, y_l)$ in $\mathbb{R}^2$. Then for any values of charges $(\zeta_1, \ldots, \zeta_l)$ the function $V_\alpha^*(x)$ in (one real) variable x given by

$$V_\alpha^*(x) = \sum_{i=1}^{l} \frac{\zeta_i}{((x - x_i)^2 + y_i^2)^\alpha} \qquad (6)$$

has at most $(2l-1)$ real critical points, assuming $\alpha \geq \frac{1}{2}$.

Remark. In the simplest possible case $\alpha = 1$ conjecture is equivalent to showing that real polynomials of degree $(4l-3)$ of a certain form have at most $(2l-1)$ real zeros.

Complexity of Voronoi diagram and Maxwell's conjecture

In the classical planar case one can show that the total number of cells of positive codimension of the Voronoi diagram of any l sites on the plane is at most $5l - 11$ and this bound is exact.

Since $(l-1)^2$ is larger than the conjectural exact upper bound $5l - 11$ for all $l > 5$ and coincides with $5l - 11$ for $l = 3, 4$, we conclude that Conjecture implies a stronger form of Maxwell's conjecture for any l positive charges on the plane and any $\alpha \geq \frac{1}{2}$.

For $n > 2$ the worst-case complexity $\Gamma(l, n)$ of the classical Voronoi diagram of an l-tuple of points in $\mathbb{R}^n$ is $\Theta(l^{[n/2+1]})$. Namely, there exist positive constants $A < B$ such that $Al^{[n/2+1]} < \Gamma(l, n) < Bl^{[n/2+1]}$. Moreover, the Upper Bound Conjecture of the convex polytopes theory proved by McMullen implies that the number of Voronoi cells of dimension k of a Voronoi diagram of l charges in $\mathbb{R}^n$ does not exceed the number of $(n-k)$-dimensional faces in the $(n+1)$-dimensional cyclic polytope with l vertices. This bound is exact, i.e. is achieved for some configurations.

In $\mathbb{R}^3$ this means that the number of 0-dimensional Voronoi cells of the Voronoi diagram of l points is at most $\frac{l(l-3)}{2}$, the number of 1-dimensional Voronoi cells is at most $l(l-3)$, and the number of 2-dimensional Voronoi cells is at most $\frac{l(l-1)}{2}$.

We were unable to find a similar result about the number of *effective* cells of Voronoi diagram. However, already for a regular tetrahedron the number of effective cells is 11, which is greater than the Maxwell's bound 9. Thus a stronger version of Maxwell's conjecture in $\mathbb{R}^3$ fails: the number of critical points of V_α could be bigger than $(l-1)^2$ for α sufficiently large.

However Maxwell's original conjecture miraculously agrees with the Maxwell inequalities (4) and we obtain the following conditional statement.

Theorem. The main Conjecture implies the validity of the original Maxwell's conjecture for any configuration of positive charges in $\mathbb{R}^3$ in the standard 3-dimensional Newton potential, i.e. $\alpha = \frac{1}{2}$.

Remarks and problems

Remark 1. What happens in the case of charges of different signs? Note that in a Voronoi cell of highest dimension corresponding to a negative charge the potential of this charge outweighs potentials of all other charges for large α, and $|V_\alpha|^{-1/\alpha}$ converges uniformly on compact subsets of this cell to $V_\infty(\bar{x})$. Therefore it seems that the function defined on the union of Voronoi cells of highest dimension as

$$\tilde{V}_\infty(\bar{x}) = \mathrm{sign}\,\zeta_i \cdot \rho_i(x), \quad \text{if} \quad \rho_i(x) = \min_j \rho_j(x)$$

is responsible for the critical points of V_α as $\alpha \to \infty$.

Remark 2. Conjecturally the number of critical points of $V_\alpha(\bar{x})$ is bounded from above by the number of effective Voronoi cells in the corresponding Voronoi diagram.

The number of all Voronoi cells in Voronoi diagrams in $\mathbb{R}^n$ with l sites has a nice upper bound. What is the upper bound for the number of effective Voronoi cells? Is it the same as for all Voronoi cells?

Remark 3. The initial hope in settling Conjecture was related to the fact that in our numerical experiments for a fixed configuration of charges the number of critical points of $V_\alpha(\bar{x})$ was a **nondecreasing** function of α. Unfortunately this monotonicity turned out to be wrong in the most general formulation: the number of critical points of a restriction of a potential to a line is not a monotonic function of α.

Example.

The potential $V_\alpha(x) = [(x+30)^2+25]^{-\alpha}+[(x+20)^2+49]^{-\alpha}+[(x+2)^2+144]^{-\alpha}+[(x-20)^2+49]^{-\alpha}+[(x-30)^2+25]^{-\alpha}$ has three critical points for $\alpha = 0.1$, seven critical points for $\alpha = 0.2$, again three critical points for $\alpha = 0.3$, and again seven critical points as $\alpha = 1.64$, and nine critical points for $\alpha \geq 1.7$.

Existence of such an example for the potential itself (and not of its restriction) is unknown.

THANK YOU FOR YOUR ATTENTION!

James C. Maxwell on points of equilibrium

In his monumental Treatise [Max[Maxwell has foreseen the development of several mathematical disciplines. In the passage which we have the pleasure to present to the readers his arguments are that of Morse theory developed at least 50 years later. He uses the notions of periphractic number, or, degree of periphraxy which is the rank of H_2 of a domain in $\mathbb{R}^3$ defined as the number of interior surfaces bounding the domain and the notion of cyclomatic number, or, degree of cyclosis which is the rank H_1 of a domain in $\mathbb{R}^3$ defined as the number of cycles in a curve obtained by a homotopy retraction of the domain (none of these notions rigorously existed then). (For definitions of these notions see [Max], section 18.) Then he actually proves Theorem 1 usually attributed to M. Morse. Finally in Section [113] he makes the following claim.

" To determine the number of the points and lines of equilibrium, let us consider the surface or surfaces for which the potential is equal to C, a given quantity. Let us call the regions in which the potential is less than C the negative regions, and those in which it is greater than C the positive regions. Let V_0 be the lowest and V_1 the highest potential existing in the electric field. If we make $C = V_0$ the negative region will include only one point or conductor of the lowest potential, and this is necessarily charged negatively. The positive region consists of the rest of the space, and since it surrounds the negative region it is periphractic.

If we now increase the value of C, the negative region will expand, and new negative regions will be formed round negatively charged bodies. For every negative region thus formed the surrounding positive region acquires one degree of periphraxy.

As the different negative regions expand, two or more of them may meet at a point or a line. If $n+1$ negative regions meet, the positive region loses n degrees of periphraxy, and the point or the line in which they meet is a point or line of equilibrium of the nth degree.

When C becomes equal to V_1 the positive region is reduced to the point or the conductor of highest potential, and has therefore lost all its periphraxy. Hence, if each point or line of equilibrium counts for one, two, or n, according to its degree, the number so made up by the points or lines now considered will be less by one than the number of negatively charged bodies.

There are other points or lines of equilibrium which occur where the positive regions become separated from each other, and the negative region acquires periphraxy. The number of these, reckoned according to their degrees, is less by one than the number of positively charged bodies.

If we call a point or line of equilibrium positive when it is the meeting place of two or more positive regions, and negative when the regions which unite there are negative, then, if there are p bodies positively and n bodies negatively charged, the sum of the degrees of the positive points and lines of equilibrium will be $p-1$, and that of the negative ones $n-1$. The surface which surrounds the electrical system at an infinite distance from it is to be reckoned as a body whose charge is equal and opposite to the sum of the charges of the system.

But, besides this definite number of points and lines of equilibrium arising from the junction of different regions, there may be others, of which we can only affirm that their number must be even. For if, as any one of the negative regions expands, it becomes a cyclic region, and it may acquire, by repeatedly meeting itself, any number of degrees of cyclosis, each of which corresponds to the point or line of equilibrium at which the cyclosis was established. As the negative region continues to expand till it fills all space, it loses every degree of cyclosis it has acquired, a becomes at last acyclic. Hence there is a set of points or lines of equilibrium at which cyclosis is lost, and these are equal in number of degrees to those at which it is acquired.

If the form of the charged bodies or conductors is arbitrary, we can only assert that the number of these additional points or lines is even, but if they are charged points or spherical conductors, the number arising in this way cannot exceed $(n-1)(n-2)$ where n is the number of bodies*.

*{I have not been able to find any place where this result is proved.}.

We finish the paper by mentioning that the last remark was added by J. J. Thomson in 1891 while proofreading the third (and the last) edition of Maxwell's book. Adding the above numbers of obligatory and additional critical points one arrives at the conjecture which was the starting point of our paper.