On level crossing in deterministic and random matrix pencils

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Topics to discuss

1. Basic level crossing problem
2. Two examples of QES-models and their level crossing
3. Level crossing for random matrices
4. Monodromy statistics

Main references

(i) B. Shapiro, M. Tater, On spectral asymptotics of quasi-exactly solvable sextic, Experimental Mathematics, https://doi.org/10.1080/10586458.2017.1325792.

(ii) B. Shapiro, M. Tater, On spectral asymptotics of quasi-exactly solvable quartic and Yablonskii-Vorob'ev polynomials, arXiv:1412.3026, submitted.

(iii) B. Shapiro, K. Zarembo, On level crossing in random matrix pencils. I. Random perturbation of a fixed matrix, Journal of Physics A: Mathematical and Theoretical, Volume 50(4).

(iv) T. Grøsfjeld, B. Shapiro, K. Zarembo, On level crossing in random matrix pencils. II. Random perturbation of a random matrix, in preparation.

Fundamental problem. Given two linear operators A and B with discrete spectrum, consider the pencil $A + tB$, where λ is a complex parameter.

(i) Determine the level crossing set $LC \subset \mathbb{C}$ consisting of all values of λ for which the operator $A + \lambda B$ has a multiple eigenvalue;

(ii) In case when LC is discrete, determine the spectral monodromy of the latter pencil, i.e., the permutations of the eigenvalues when λ traverses different closed loops in $\mathbb{C} \setminus LC$.

The problem seems quite difficult, in general. Especially its monodromy part!

One of the most classical examples of such problems is the quartic oscillator

$$-y'' + (2z^4 + \beta z^2)y = \mu y,$$

with the initial conditions $y(\pm\infty) = 0$ on the real axis. (The perturbation parameter is denoted by β.)

It has been studied by e.g., Bender-Wu (1969), Simon (1970), Voros (1983), Shanley (1988), etc.

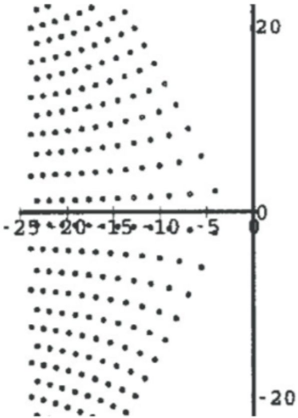


Figure: Numerically obtained level crossing for the classical quartic oscillator, Shanley (1988).

A spectral problem is called *quasi-exactly solvable* if a finite part of its spectrum can be obtained by algebraic methods, i.e., this part of the spectrum satisfies a univariate algebraic equation of some degree. Historically first and the most well-known example of a quasi-exactly solvable problem in quantum mechanics is the *quasi-exactly solvable sextic*. It was originally discovered in [Singh et al. 78] (see also [Turbiner 88, Turbiner and Ushveridze 87, Ushveridze]) and is given by the Schrödinger equation

$$-y'' + \Pi(x)y' = \mu y,$$

with the boundary conditions $y(\pm\infty) = 0$ on $\mathbb{R}$. Here the potential $\Pi(x)$ is given by

$$\Pi_{m,p,b}(x) = x^6 + 2bx^4 + (b^2 - (4m + 2p + 3))x^2,$$

m being a fixed positive integer, $p \in \{0, 1\}$, and $b \in \mathbb{C}$.

It has been shown that for any $b \in \mathbb{C}$, the latter equation has $m+1$ eigenfunctions of the form

$$\phi(x) = Q(x)e^{-\frac{x^4}{4}-\frac{bx^2}{2}},$$

where $Q(x)$ is an even (resp. odd) polynomial of degree $2m$ (resp. $2m+1$) for $p=0$ (resp. $p=1$).
After some straightforward calculations one can show that for $p=0$, the eigenvalues corresponding to the above $m+1$ eigenfunctions form the spectrum of the following $(m+1)\times(m+1)$-matrix

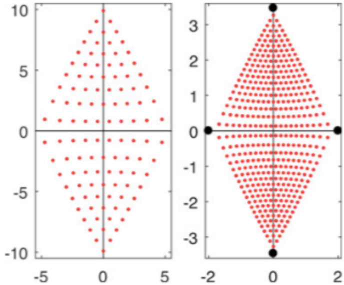
$$M_m(b) = \begin{pmatrix} b & -4m & 0 & 0 & 0 & \dots \\ -1\cdot 2 & 5b & 4-4m & 0 & 0 & \dots \\ 0 & -3\cdot 4 & 9b & 8-4m & 0 & \dots \\ 0 & 0 & -5\cdot 6 & 13b & 12-4m & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \quad (1)$$

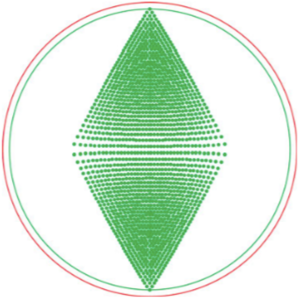
In other words, we need to consider a rather simple matrix pencil $M_m(b) = A_m + bB_m$ with

$$A_m = \begin{pmatrix} 0 & -4m & 0 & 0 & 0 & \dots \\ -1\cdot 2 & 0 & 4-4m & 0 & 0 & \dots \\ 0 & -3\cdot 4 & 0 & 8-4m & 0 & \dots \\ 0 & 0 & -5\cdot 6 & 0 & 12-4m & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

and

$$B_m = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 5 & 0 & 0 & 0 & \dots \\ 0 & 0 & 9 & 0 & 0 & \dots \\ 0 & 0 & 0 & 13 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$





For all real b, the spectrum of $M_m(b)$ is real and simple. On the other hand, there exist $(m+1)m$ non-real values of b for which $M_m(b)$ has a multiple eigenvalue. Calculating the discriminant and solving it numerically, one obtains the above (scaled) pictures.

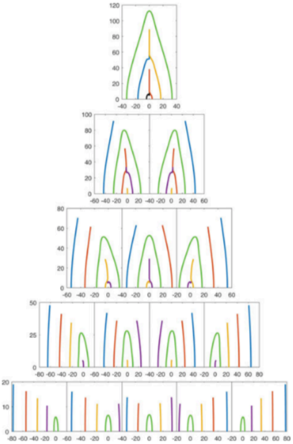


Figure: Monodromy for $m = 5$.

One can approximate the quartic potential by a sequence of QES-sextic potentials. Set $n = 4m + 3$. Then the quasi- exactly solvable equation is related to

$$-y''(z) + [a^2 z^6 + 2a\alpha z^4 + (\alpha^2 - an)z^2]y(z) = \mu y(z)$$

by the scaling $x = a^{1/4}z, \alpha = a^{1/4}z, \lambda = a^{1/2}\mu$. Then $b = \alpha/a^{1/2} = n^{1/2}(1 + (s - t/2)n^{-2/3} + O(n^{-4/3})$. We get the potential

$$n^{-2/3}(1 + O(n^{-2/3}))z^6 + 2(1 + (s + t)n^{-2/3} + stn^{-4/3})z^4$$

$$+((2s - t) + s^2 n^{-2/3})z^2.$$

Next Figure shows location of the level crossing points for the rescaled sextic and for the classical quartic oscillator.

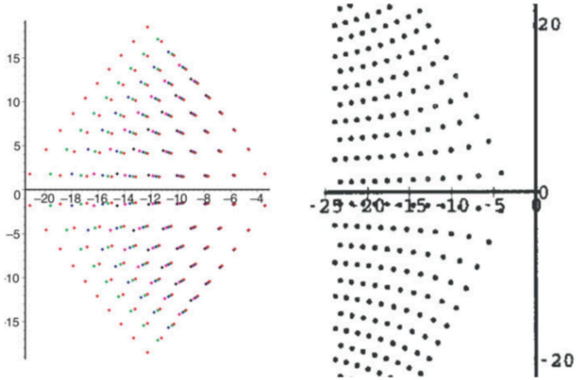


Figure: Level crossing points for the rescaled QES sextics with $m = 6, 7, 8, 9, 10$ and that of the quartic oscillator.

Another well-known quasi-exactly solvable model is the QES-quartic oscillator which was found by Bender and Boetcher in 1998. In its restricted form it is given by is a Schrödinger-type eigenvalue problem of the form

$$L_J(y) = y'' - (x^4/4 - ax^2/2 - Jx)y = \mu y \qquad (2)$$

with the boundary conditions $y(te^{\pm \pi i/3}) \to 0$ as $t \to +\infty$, where $a \in \mathbb{C}$ and J are parameters of the problem.
When $J = n + 1$ is a positive integer, then $L_{n+1}(y)$ maps the space of quasi-polynomials $\{pe^h : \deg p \leq n\}$ to itself where p is a polynomial of degree at most n and $h = -x^3/6 + ax/2$.

Therefore one gets $n+1$ eigenfunctions of the above form. One can check that the eigenvalues of these $n+1$ eigenfunctions form the spectrum of the matrix

$$M_n^{(a)} := \begin{pmatrix} 0 & a & 2 & 0 & 0 & \cdots & 0 \\ n & 0 & 2a & 6 & 0 & \cdots & 0 \\ 0 & n-1 & 0 & 3a & 12 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 & 0 & (n-1)a & n(n-1) \\ 0 & 0 & \cdots & 0 & 2 & 0 & na \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (3)$$

When a is a large positive number the spectrum of $M_n(a)$ is real and distinct.

Thus we need to study a “simple" pencil $M_n^{(a)} = A_n + aB_n$ with

$$A_n = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & \cdots & 0 \\ n & 0 & 0 & 6 & 0 & \cdots & 0 \\ 0 & n-1 & 0 & 0 & 12 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 & 0 & 0 & n(n-1) \\ 0 & 0 & \cdots & 0 & 2 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix};$$

$$B_n = \begin{pmatrix} 0 & 1 & & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & (n-1) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & n \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

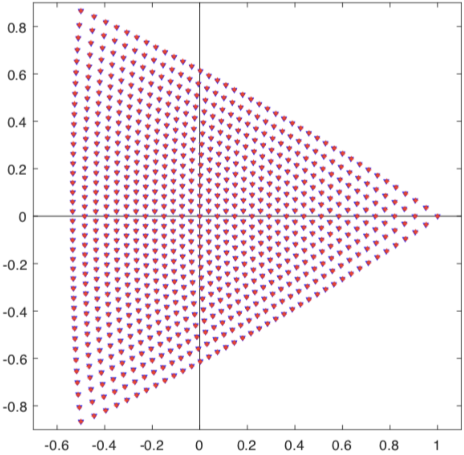


Figure: Level crossing for $J = 40$.

Yablonskii-Vorob'ev polynomials are defined as follows. Set $YV_0 = 1,\ YV_1 = t$. For $n \geq 1$, set

$$YV_{n+1} = \frac{t \cdot YV_n^2 - 4(YV_n \cdot YV_n'' - (YV_n')^2)}{YV_{n-1}}.$$

Although the latter expression apriory determines a rational function, YV_n is in fact a polynomial of degree $\binom{n+1}{2}$. The importance of Yablonskii-Vorob'ev polynomials is explained by the fact that all rational solutions of the second Painlevé equation

$$u_{tt} = tu + 2u^3 + \alpha, \alpha \in \mathbb{C},$$

are presented in the form

$$u(t) = u(t;n) = \frac{d}{dt}\left\{\ln\left[\frac{YV_{n-1}(t)}{YV_n(t)}\right]\right\},\ u(t,0) = 0, u(t;-n) := -u(t;n)$$

Denote by $\mathcal{Z}_n$ the zero locus of YV_n.

Remark

One can show that the maximal absolute value of points in Σ_n grows as $\frac{3}{\sqrt[3]{4}} n^{2/3}$ while the maximal absolute value of points in $\mathcal{Z}_n$ grows as $\left(\frac{9}{2}\right)^{\frac{2}{3}} n^{2/3}$.

Conjecture

(i) *After division by* $\frac{3}{\sqrt[3]{4}} n^{2/3}$ *and* $\left(\frac{9}{2}\right)^{\frac{2}{3}} n^{2/3}$ *respectively, the sequence* $\{\Sigma_n\}$ *tends to the sequence* $\{\widehat{\mathcal{Z}}_n\}$*, where* $\widehat{\mathcal{Z}}_n$ *is the zeros locus of* $YV_n(-t)$*. In particular, the limiting triangular domain* $\mathfrak{F}$ *covered by* $\{\Sigma_n\}$ *and* $\{\widehat{\mathcal{Z}}_n\}$ *when* $n \to \infty$ *is the same after the latter scaling.*

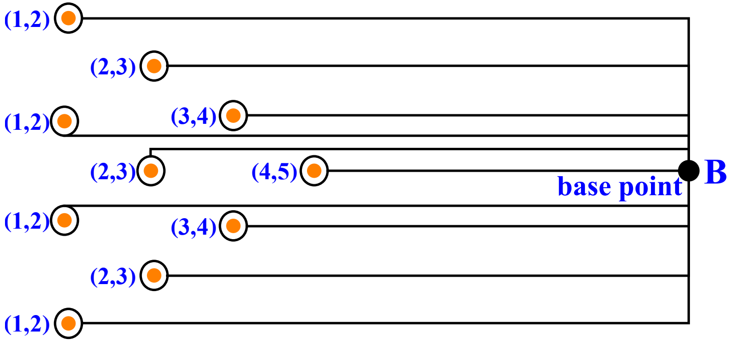


Figure: Monodromy for $J = 5$.

We start with complex Gaussian ensembles. Recall that the complex (non-symmetric) Gaussian ensemble $GE_n^{\mathbb{C}}$ is the distribution on the space $Mat_n^{\mathbb{C}}$ of all complex-valued $n \times n$-matrices, where each entry of a random $n \times n$-matrix is an independent complex Gaussian variable distributed as $N(0, \frac{1}{2}) + iN(0, \frac{1}{2})$. Our initial result is as follows.

Theorem

For any positive integer n, if the matrices A and B are independently chosen from $GE_n^{\mathbb{C}}$, then the distribution of level crossings in $A + \lambda B$ with respect to the affine coordinate $\lambda = x + iy$ of $\mathbb{C}$ is given by

$$\mathcal{P}_{GE_n^{\mathbb{C}}}(\lambda) := \mathcal{P}_{GE_n^{\mathbb{C}}}(x, y)dxdy = \frac{dxdy}{\pi(1 + x^2 + y^2)^2} = \frac{dxdy}{\pi(1 + |\lambda|^2)^2}. \quad (4)$$

Remark

In polar coordinates (r, θ) in the complex plane of parameter λ, the above distribution $\mathcal{P}_{GE_n^{\mathbb{C}}}(\lambda)$ has the form

$$\mathcal{P}_{GE_n^{\mathbb{C}}}(r, \theta) dr d\psi = \frac{r dr d\theta}{\pi(1 + r^2)^2},$$

giving the radial CDF of the form

$$\Psi_{GE_n^{\mathbb{C}}}(r) = \frac{r^2}{1 + r^2}.$$

Let us realize $\mathbb{CP}^1 \simeq S^2$ as the unit sphere in $\mathbb{R}^3$ with coordinates (X, Y, Z) and identify the complex plane of parameter $\lambda = x + iy$ with the horizontal coordinate (X, Y)-plane, where X corresponds to the real axis and Y corresponds to the imaginary axis in $\mathbb{C}$. If we use the standard stereographic projection of the unit sphere in $\mathbb{R}^3$ from its north pole, i.e. from the point $(0, 0, 1)$ onto the (X, Y)-plane, then the usual area element of the sphere induced from the standard Euclidean structure in $\mathbb{R}^3$ is given by

$$dA = \frac{4dxdy}{(1 + x^2 + y^2)^2} = \frac{4dxdy}{(1 + |\lambda|^2)^2}.$$

The latter fact implies that the r.h.s. of (4) presents the constant density $\frac{1}{4\pi}$ with respect to the standard Euclidean area measure on $S^2 \simeq \mathbb{C}P^1$ compactifying the complex plane of parameter λ.

An alternative way to express this fact is as follows. Consider the standard cylindrical coordinate system (ρ, ϕ, Z) in $\mathbb{R}^3$, where $\rho \geq 0, 0 \leq \phi \leq 2\pi, Z \in \mathbb{R}$. Recall that

$$X = \rho \cos \phi, \;\; Y = \rho \sin \phi, \;\; Z = Z.$$

If we consider (ϕ, Z), $0 \leq \phi \leq 2\pi, -1 \leq Z \leq 1$, as coordinates on the unit sphere $S^2 \simeq \mathbb{C}P^1$ (with both poles removed), then in these coordinates the usual area element on the sphere is given by

$$dA = d\phi dZ.$$

Thus, in cylindrical coordinates (ϕ, Z), $0 \leq \phi \leq 2\pi; \; -1 \leq z \leq 1$ parameterising the unit sphere S^2, the measure $\mathcal{P}_{GE_n^{\mathbb{C}}}(x, y)dxdy$ given by (4) transforms into

$$\mathcal{P}_{GE_n^{\mathbb{C}}}(\phi, Z)d\phi dZ = \frac{d\phi dZ}{4\pi}. \tag{5}$$

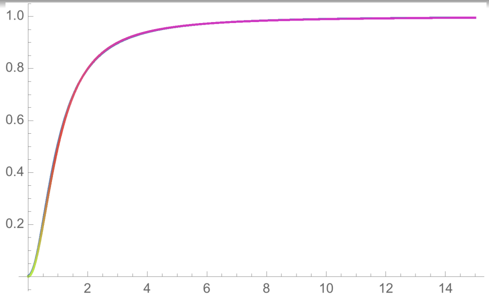


Figure: Radial density of level crossings for $A+\lambda B$, where A and B are independently sampled from $GE_6^{\mathbb{C}}$; (we used 100 random pairs). The above diagram shows a perfect match of the numerical distribution of the absolute values of level crossings obtained in our sampling with the theoretical radial CDF $\frac{r^2}{1+r^2}$.

In fact, we can prove the following generalisation of Theorem 3.

Proposition

Conclusion of Theorem 3 holds, if A and B are independently chosen from the scaled complex Gaussian ensemble $GE^{\mathbb{C}}_{\sigma^2,n}$*, i.e., the ensemble whose off-diagonal entries are i.i.d. standard normal complex variables and whose diagonal entries are i.i.d. normal complex variables with an arbitrary fixed positive variance* σ^2*.*

Moreover consider the following SU_2-action on $Mat_n^{\mathbb{C}} \times Mat_n^{\mathbb{C}}$. A matrix $\mathfrak{U} \in SU_2$ given by $\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}, \ |u|^2 + |v|^2 = 1$ acts on the latter product space by:

$$(A, B) * \mathfrak{U} \mapsto (uA + vB, -\bar{v}A + \bar{u}B). \tag{6}$$

Take any complex linear subspace $W_n \subset Mat_n^{\mathbb{C}}$ such that the product space $W_n \times W_n \subset Mat_n^{\mathbb{C}} \times Mat_n^{\mathbb{C}}$ is preserved by the action (6). Given $\sigma > 0$, denote by $W_{\sigma^2,n}$ the space W_n with the measure induced from the scaled complex Gaussian ensemble $GE_{\sigma^2,n}^{\mathbb{C}}$.

Proposition

In the above notation, level crossings of $A + \lambda B$ *with the random matrices A and B independently chosen from* $W_{\sigma^2,n}$ *are uniformly distributed on* $\mathbb{C}P^1$, *i.e., their probability measure is given by the right-hand side of* (4).

To give an example of such W, recall that $GOE_n^{\mathbb{C}}$ is the distribution on the space $Sym_n^{\mathbb{C}}$ of complex-valued symmetric matrices, where each entry $e_{i,j} = e_{j,i},\ i < j$ of a $n \times n$-matrix has a normal distribution $N(0, 1/2) + iN(0, 1/2)$, and each diagonal entry $e_{i,i}$ is distributed as $\sqrt{2}(N(0, 1/2) + iN(0, 1/2))$.

Further interesting examples of linear subspaces W covered by Proposition 6 include Toeplitz matrices, band matrices, band Toeplitz matrices, diagonal matrices, etc.

Next we consider Gaussian orthogonal and Gaussian unitary ensembles. A very essential feature of all these cases is that their level crossings distribution is invariant under the action of the subgroup $SO_2 \subset SU_2$ given by the same formula (6), but with real u and v satisfying $u^2 + v^2 = 1$.

In the above realization of $\mathbb{C}P^1$ as the unit sphere $S^2 \subset \mathbb{R}^3$, SO_2 acts on it by rotation around the Y-axis. This circumstance implies that the family of orbits of the SO_2-action on the unit sphere $S^2 \simeq \mathbb{C}P^1$ projected to the complex plane of parameter $\lambda = x + iy$ will coincide with the family of circles given by

$$x^2 + (y - t)^2 = t^2 - 1, \quad |t| \geq 1.$$

Let us introduce the cylindrical coordinates (ρ, ψ, Y) where $X = \rho \cos \psi,\ \ Y = Y,\ Z = \rho \sin \psi$. Then (ψ, Y), $0 \leq \psi \leq 2\pi,\ -1 \leq Y \leq 1$ again parameterises the unit sphere $S^2 \simeq \mathbb{C}P^1$. SO_2-action implies that in the cylindrical coordinates (ψ, Y) the distributions of level crossings of the above ensembles on $\mathbb{C}P^1$ are of the form:

$$dens(\psi, Y)d\psi dY = \rho(Y)d\psi dY,$$

for some univariate function ρ, i.e., its density depends only on Y and is independent of the angle variable ψ. (In general, $\rho(Y)dY$ can be a 1-dimensional measure which not necessarily has a smooth density function.) In the affine coordinates,

$$dens(x, y)dxdy = \rho\left(\frac{2y}{x^2 + y^2 + 1}\right)\frac{4dxdy}{(x^2 + y^2 + 1)^2}, \qquad (7)$$

with the same ρ as above.

In the case of $GOE_n^{\mathbb{R}}$ we can show the following.

Theorem

If the matrices A and B are independently chosen from $GOE_2^{\mathbb{R}}$, then the distribution of level crossings in $A + \lambda B$ is uniform on $\mathbb{CP}^1 \supset \mathbb{C}$, i.e., their density is given by the right-hand side of (4)*.*

Extensive numerical experiments strongly support the following guess.

Conjecture

For any size $n > 2$, if the matrices A and B are independently chosen from $GOE_n^{\mathbb{R}}$, then the distribution of level crossings in $A + \lambda B$ is uniform on $\mathbb{CP}^1 \supset \mathbb{C}$.

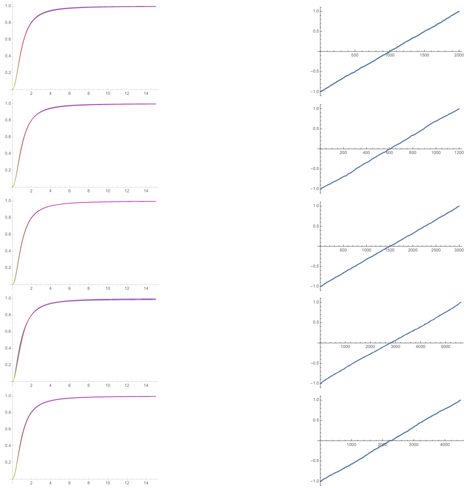


Figure: Numerical and theoretical radial and angle CDF for $A + \lambda B$ with A and B from GOE_n, for $n = 2, 4, 6, 8, 10$.

Our next results deal with the Gaussian unitary ensemble.

Theorem

If the matrices A and B are independently chosen from GUE_2, then the distribution of level crossings in $\mathbb{C}$ is given by

$$\mathcal{P}_{GUE_2}(x,y)dxdy = \frac{4|y|dxdy}{\pi(1+x^2+y^2)^3} = \frac{1}{\pi}\left|\frac{y}{1+x^2+y^2}\right|\frac{4dxdy}{(1+x^2+y^2)^2} \quad (8)$$

In the cylindrical coordinates (ψ, Y) on $\mathbb{C}P^1$, where $0 \leq \psi \leq 2\pi$ and $-1 \leq Y \leq 1$, one has

$$\mathcal{P}_{GUE_2}(\psi, Y)d\psi dY = \frac{|Y|d\psi dY}{2\pi}. \quad (9)$$

Unfortunately, we do not have explicit (conjectural) formulas for the densities $\mathcal{P}_{GUE_n}(x, y)$, for $n \geq 3$ similar to (9). However, we carried out substantial numerical experiments for matrix sizes up to 6. They were conducted as follows. For each $n \in \{2, ..., 6\}$, sampling independently pairs of GUE_n-matrices, we calculated 12,000 level crossing points for every n and plotted the values of $|Y|$ for obtained level crossings in increasing order, see next Figure. These numerical experiments strongly suggest the following.

Conjecture

There exists a limiting distribution $\mathcal{P}_{GUE_\infty}(Y) := \lim_{n\to\infty} \mathcal{P}_{GUE_n}(Y)$.

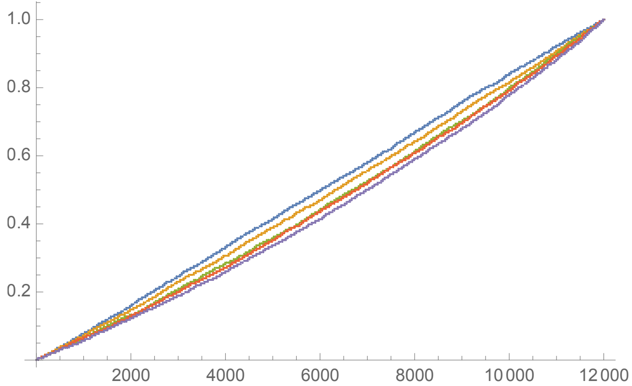


Figure: Empirical distributions of $|Y|$ for $A + \lambda B$ taken from GUE_n with $n = 2, 3, 4, 5, 6$. (Curves corresponding to the increasing values of n lie one below the other; the blue straight line corresponds to $n = 2$, see (9).)

Finally, let us discuss the case of (non-symmetric) real Gaussian distribution of $GE_n^{\mathbb{R}}$. Similarly to the above, we were not able to get the explicit formulas for the distributions of level crossings of $GE_n^{\mathbb{R}}$ with $n \geq 3$, but as in the previous cases, we performed detailed numerical experiments. These experiments strongly suggest the validity of the following guess.

Conjecture

The average of the number real crossing points for A and B independently sampled from $GE_n^{\mathbb{R}}$ equals $\sqrt{n(n-1)}$.

Conjecture

When $n \to \infty$, the level crossing distribution for A and B independently sampled from $GE_n^{\mathbb{R}}$ approaches the uniform distribution on $\mathbb{CP}^1$.

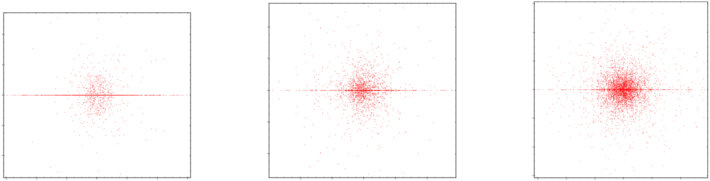


Figure: Distributions of level crossings in the λ-plane when A and B are sampled from $GE_n^{\mathbb{R}}$ for $n = 2, 5, 10$ apparently approaching the uniform distribution on $\mathbb{C}P^1$.

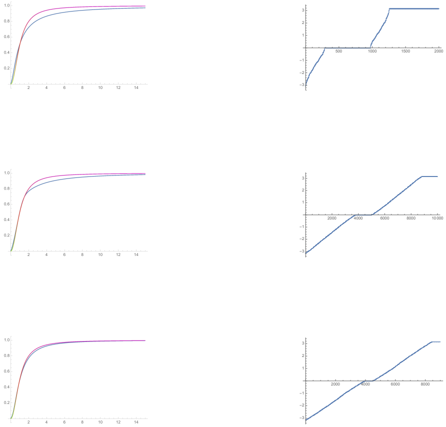


Figure: Radial and angle distributions of level crossings with A and B sampled from $GE_n^{\mathbb{R}}$ with $n = 2, 5, 10$ approaching that of the uniform distribution on $\mathbb{C}P^1$.

GOE_3- and GUE_3-cases

If λ_i is the i-th level crossing point in the upper half-plane in the order of increasing real parts, consider the path in the λ-plane starting on the real axis at $\tau = Re(\lambda_i)$, going straight up to λ_i, making a small loop encircling λ_i counterclockwise, and returning back to τ_i.

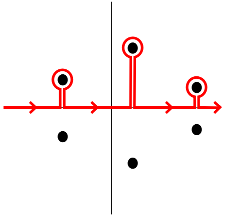
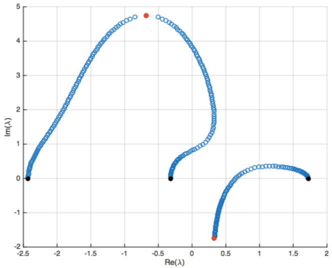


Figure: Creating the monodromy sequence

As a result, one gets a transposition σ_i of two real eigenvalues corresponding to $\tau_i = Re(\lambda_i)$. Doing this for each λ_i, $i = 1, \dots, \binom{n}{2}$, we obtain a sequence of $\binom{n}{2}$ transpositions $\left(\sigma_1, \sigma_2, \dots, \sigma_{\binom{n}{2}}\right)$, $\sigma_i \in S_n$.
One can easily check that the obtained sequence $\left(\sigma_1, \sigma_2, \dots, \sigma_{\binom{n}{2}}\right)$ of transpositions satisfies the following two conditions:

(i) for general A and B, they generate the symmetric group S_n;
(ii) the product $\sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_{\binom{n}{2}}$ coincides with the inverse permutation $(n, n-1, \dots, 1)$.

For $n = 3$, it is easy to check that there are only 8 triples of transpositions in S_3 satisfying conditions (i) and (ii). These triples are: (12)(12)(13); (12)(13)(23); (12)(23)(12); (13)(12)(12);
(13)(23)(23); (23)(12)(23); (23)(13)(12); (23)(23)(13). (For comparison, for $n = 4$, there are already 3840 6-tuples of transpositions in S_4 satisfying (i) and (ii).)



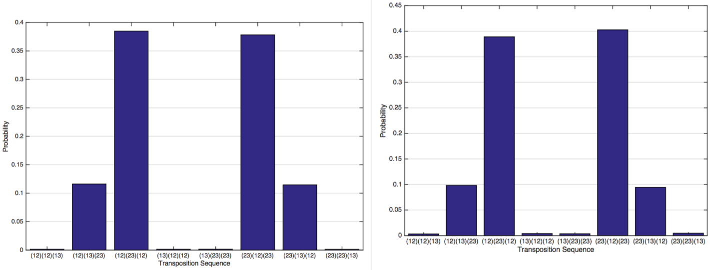


Figure: The probabilities of the monodromy triples of transpositions for GUE_3- and GOE_3-matrices.

$GE_3^{\mathbb{C}}$-case.

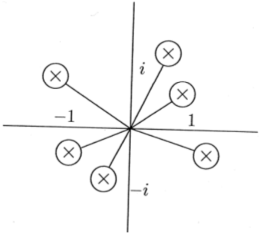


Figure: An example of paths in the λ-plane chosen to determine the monodromy for pairs (A, B) from $GE_3^{\mathbb{C}}$.

For A and B in $GE_3^{\mathbb{C}}$, there are 240 sequences of 6-tuples of transpositions $(\sigma_1, \sigma_2, \ldots, \sigma_6)$ from S_3 satisfying the conditions:
(i) they generate the symmetric group S_3;
(ii) the product $\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_6$ coincides with the identity permutation $(1, 2, 3)$.

We generated 150000 random matrix pairs in $GE_3^{\mathbb{C}}$ and calculated their monodromy sequences. Our numerical results show the following, see Fig. 16.

(i) Of the 240 possible cases, only 209 were realized and only 204 were realized more than once.
(ii) The most common monodromy sequences were of the form (23)(12)(23)(12)(23)(12), which occurred 2.43 % of the time and of the form (12)(13)(13)(23)(23)(12) which occured 2.29 % of the time.
(iii) Monodromy sequences in which one permutation occurs four times in a row followed by two occurrences of another permutation and their cyclic permutations (for example, (12)(12)(12)(12)(13)(13) or (12)(23)(23)(23)(23)(12)) were the most rare, occurring only once or not at all.

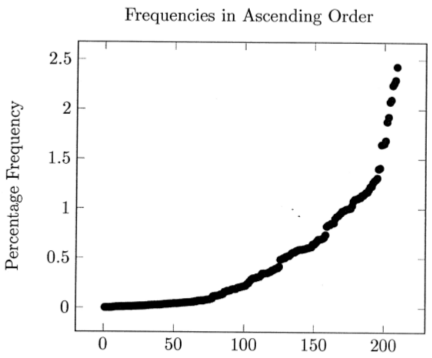


Figure: Frequencies of 240 possible 6-tuples of transpositions from S_3 in the ascending order.

Remark

One particularly strange and interesting result is that the labelling of the eigenvalues seems to affect the frequencies with which certain monodromy sequences appear. In the case of $GE_3^{\mathbb{C}}$-matrices, one can relabel the three preimages of $\lambda = 0$, i.e., the eigenvalues of A, by using the action of S_3. Usually, about half of these six group elements change the frequency by either doubling or halving the original one. The other half of the group tends to keep the frequency the same, but exactly which members of S_3 do what varies from case to case. We have not been able to find a pattern of or an explanation to why relabelling changes the frequencies in this peculiar way.

Open problems

There were many unsolved questions during the talk. In particular, although the simple (conjectural) answer for level crossing distribution in the GOE-case presented in Theorem 2 and Conjecture 1 indicates the possible existence of some extra symmetry complementing the above SO_2-action, we were not able to find such.

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