Root asymptotics for the eigenpolynomials of Bochner-Krall operators

joint with G. Masson, H. Rullgård, R. Bøgvad, J. Borcea, T. Bergkvist

Boris Shapiro

March 3, 2016

Topics to discuss

Main references

G. Masson and B. Shapiro, **On polynomial eigenfunctions of a hypergeometric-type operator**, Exper. Math., vol. 10, 609-618, (2001).

T. Berqkvist and H. Rullgård, **On polynomial eigenfunctions for a class of differential operators**, Math. Res. Lett., vol.9, 153-171, (2002).

T. Berqkvist, H. Rullgård and B. Shapiro, **On Bochner-Krall orthogonal polynomial systems**, Math. Scand., vol 94, 148–154, (2004).

Main references continued

J. Borcea, B. Shapiro, **Root asymptotics of spectral polynomials for the Lamé operator**, Comm.Math.Phys, vol 282 (2008) 323–337.

J. Borcea, R. Bøgvad and B. Shapiro, **Homogenized spectral pencils for exactly solvable operators: asymptotics of polynomial eigenfunctions**, Publ. RIMS, vol 45 (2009) 525–568.

T. Bergkvist, **On asymptotics of polynomial eigenfunctions for exactly-solvable differential operators**, J. Approx. Theory 149 (2007), no. 2, 151–187.

Bochner-Krall problem

Definition 1. An operator $T = \sum_{i=1}^{k} Q_i(z)\frac{d^i}{dz^i}$ is called **Bochner-Krall** if $\deg Q_i(z) \leq i$ and there exists a value i such that $\deg Q_i(z) = i$.

Obviously, $T(z^j) = a_j z^j +$ lower order terms, i.e. T acts diagonally in the monomial basis.

Lemma. For any Bochner-Krall operator T and sufficiently large n there exists a unique eigenpolynomial $p_n(z)$ of degree n.

In fact, S.Bochner has shown in 1929 that any linear differential univariate operator having an infinite sequence of polynomial eigenfunctions is a Bochner-Krall operator.

Bochner-Krall problem, cont.

Problem (Bochner (1929), Krall (1938))

Which Bochner-Krall operators have orthogonal polynomials (with respect to a positive or a signed measure supported on $\mathbb{R}$) as their sequences of eigenpolynomials?

Lemma (Bochner)

A Bochner-Krall operator having a sequence of orthogonal polynomials (BKOPS) must be of even order and formally self-adjoint.

Bochner-Krall problem, cont.

Theorem (Bochner)

The only second order BKOPS correspond to four classical orthogonal polynomial families: Jacobi, Bessel, Laguerre and Hermite polynomials.

Theorem (Krall)

The fourth order BKOPS additionally contain three more families: the Legendre-type, the Laguerre-type, the Jacobi-type polynomials.

Bochner-Krall problem, cont.

For orders exceeding 4 the complete answer is still unknown!!!

All known BKOPS have distributional weights of the form $w = u + v$, where u is a classical orthogonal weight and v consists of some point masses supported on the boundary of the support of u.

It is conjectured that this is true for all BKOPS.

It is also conjectured that the leading coefficient of a BKOPS is a power of either linear or quadratic polynomial.

Bochner-Krall problem, cont.

Theorem (Kwon-Lee)

The only BKOPS with compactly supported positive measure on $\mathbb{R}$ as the Jacobi-type polynomials, i.e. after an appropriate linear real change of variables they are orthogonal w.r.t

$$w = (1-x)^{\alpha}(1+x)^{\beta}H(1-x^2) + c\delta(x-1) + d\delta(x+1),$$

where $\alpha, \beta > -1$ and $c, d \geq 0$, and $H(x)$ is the Heaviside step function.

Main problem and examples

Our main problem. Given an arbitrary Bochner-Krall operator T describe the root asymptotics for the polynomial sequence $\{p_n(z)\}$.

Examples

$$T_1 = z(z-1)(z-I)\frac{d^3}{dz^3}$$

$$T_2 = (z-I)(z+I)(z-2+3I)(z-3-2I)\frac{d^4}{dz^4}$$

$$T_3 = (z-I)(z+I)(z-2+3I)(z-3-2I)(z+3)\frac{d^5}{dz^5}$$

$$T_4 = (z^2+1)(z-2+3I)(z-3-2I)(z+3)(z+1+I)\frac{d^6}{dz^6}$$

Roots of eigenpolynomials

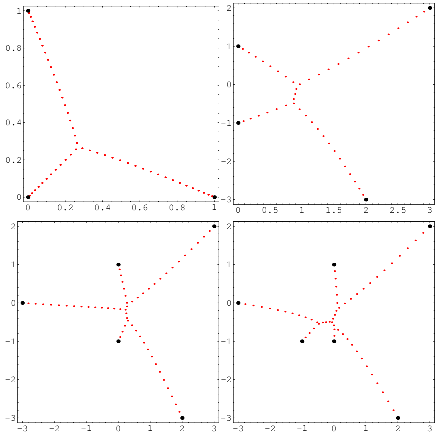


Figure: Roots of $p_{55}(z)$ for the above T's.

Basic Definitions

Definition 3. Cauchy transform of a (complex-valued) measure ρ satisfying $\int_{\mathbb{C}} d\rho(\xi) < \infty$ is given by

$$C_\rho(z) = \int_{\mathbb{C}} \frac{d\rho(\xi)}{z - \xi}.$$

Example. If $d\rho(z) = \frac{1}{\pi\sqrt{1-x^2}}$, $x \in [-1, 1]$ then $C_\rho = \frac{1}{\sqrt{z^2-1}}$ in $\mathbb{C} \setminus [-1, 1]$

Definition 1.1. A Bochner-Krall operator $T = \sum_{i=1}^{k} Q_i(z) \frac{d^i}{dz^i}$ is called of **non-degenerate** if $\deg Q_k(z) = k$.

First results

Proposition 1.2. Assuming that each $p_n(z)$ is monic and that $\Psi(z) = \lim_{n\to\infty} \frac{p_n'(z)}{np_n(z)}$ exists in some open neighborhood Ω of $\mathbb{C}$ one gets that $\Psi(z)$ satisfies in Ω the algebraic equation

$$Q_k(z)\Psi^k(z) = 1.$$

Theorem 1.1, H. Rullgård. Let $Q_k(z)$ be a monic degree k polynomial. $\exists!$ probability measure μ_Q such that
a) supp μ_Q is compact;
b) its Cauchy transform C_μ satisfies the equation $Q_k(z)C_\mu^k(z) = 1$ almost everywhere in $\mathbb{C}$.

Main Theorem

Main theorem. In the above notation
1) supp μ_Q is a curvilinear tree which is straightened out by the analytic mapping

$$\xi(z) = \int_a^z \frac{dz}{\sqrt[k]{Q_k(z)}}.$$

2) supp μ_Q contains all the zeros of $Q_k(z)$ and is contained in the convex hull of those.
3) There is a natural formula for the angles between the branches and the masses of the branches satisfy Kirchhoff law.

Illustration

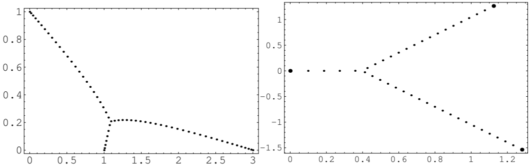


Figure: The measure μ_Q before and after the transformation.

Here $Q(z) = (z - 1)(z - 3)(z - I)$

Illustration cont.

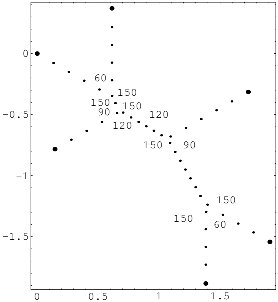


Figure: Example of a μ_Q with angles.

Speculation. Distribution should be related to the algebraic curve $y^k = Q_k(z)$ and Stokes lines for the differential operator.

Degenerate case

A Bochner-Krall operator T of order k is called **degenerate** iff $\deg Q_k < k$.

Examples: $T = z\frac{d^2}{dz^2} + (az+b)\frac{d}{dz}$, $T = \frac{d^2}{dz^2} + (az+b)\frac{d}{dz}$ leading to Laguerre resp. Hermite polynomials.

Proposition. The union of all roots of all polynomial eigenfunctions of a Bochner-Krall operator T is unbounded if and only if T is degenerate.

Question. Given a degenerate T with the family of eigenpolynomials $\{p_n(z)\}$ how fast does the maximum modulus among the roots of $p_n(z)$ grow?

Main Conjecture

Conjecture. (T.Bergkvist) Given a degenerate $T = \sum_{j=1}^{k} Q_j(z)\frac{d^j}{dz^j}$ denote by j_0 the largest j for which $\deg Q_j(z) = j$. Then

$$\lim_{n\to\infty} \frac{r_n}{n^d} = c_T$$

where $c_T > 0$ is a positive constant and

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j} \right).$$

Main Conjectural Corollary

The Cauchy transform $C(z)$ of the asymptotic root measure μ of the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$ of a degenerate T satisfies the following algebraic equation for almost all complex z:

$$z^{j_0} C^{j_0}(z) + \sum_{j \in A} \alpha_{j,\deg Q_j} z^{\deg Q_j} C^j(z) = 1,$$

where A is the set consisting of all j for which the maximum $d := \max_{j \in [j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j} \right)$ is attained, i.e. $A = \{ j : (j - j_0)/(j - \deg Q_j) = d \}$.

More Pictures

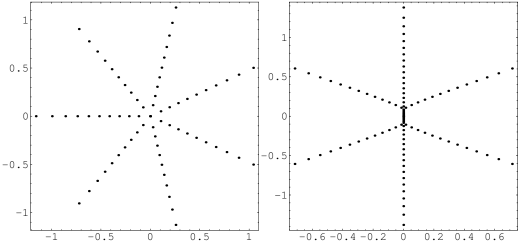


Figure: Degenerate case.

What do we need to solve Bochner-Krall problem?

Problem (Seems doable)

Assuming that Bergkvist's conjecture is settled, deduce the information about the support and the density of the measure from the above algebraic equation.

In particular,

Conjecture

The support of the measure satisfying a.e. the algebriac equation of T.Bergkvist is always a tree.

What do we need to solve Bochner-Krall problem? cont.

Problem

Assuming that $\{p_n(x)\}$ is an OPS with respect to a positive measure with unbounded (connected?) support on $\mathbb{R}$, is there an asymptotic root-counting measure for the scaled roots of $\{p_n(x)\}$?

Problem

If such a measure exists, can one get it explicitly for some natural classes of OPS?

(Such results are known if the support is compact but in the unbounded case there seems to be no information!)

Homogenized stuff (non-degenerate case so far)

Only the leading coefficient was important in the previous set-up which is unfare! To improve the situation we use (following Wasow, Fedoryuk etc) the *homogenized spectral problem* of the form

$$T_\lambda = \sum_{i=0}^{k} Q_i(z)\lambda^{k-i}\frac{d^i}{dz^i},$$

where each $Q_i(x) = a_{ii}z^i + a_{i,i-1}z^{i-1} + \ldots$ is a polynomial of degree i.

basic facts

Definition. A non-degenerate T is called **of general type** iff $\sum_{i=0}^{k} a_{ii}\lambda^{k-i} = 0$ has k distinct zeros.

Proposition 2.1. If T is of general type,

1) for all sufficiently large n there exist exactly k distinct values $\lambda_{n,j},\ j = 1, \dots, k$ of the spectral parameter λ such that the operator T_λ has a polynomial eigenfunction $p_{n,j}(z)$ of degree n.

2) Asymptotically $\lambda_{n,j} \sim n\lambda_j$ where $\lambda_1, \dots, \lambda_k$ is the set of roots of the algebraic equation $\sum_{i=0}^{k} a_{i,i}x^{k-i} = 0$.

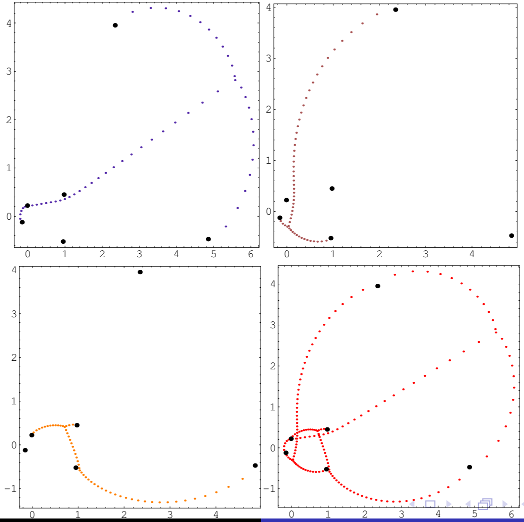
main conjectures

Conjecture 1. If T is of general type and all $\lambda_1, \dots, \lambda_k$ have distinct arguments then for each $j = 1, \dots, k$ $\exists!$ probability measure μ_j with compact support whose Cauchy transform $C_j(z)$ satisfies almost everywhere in $\mathbb{C}$

$$\sum_{i=1}^{k} Q_i(z)(\lambda_j C_j(z))^i = 0.$$

Conjecture 2. $C_j(z) = \lim_{n\to\infty} \frac{p'_{n,j}(z)}{\lambda_{n,j} p_{n,j}(z)}$ outside the support of μ_j which is the union of finitely many segments of analytic curves.

Even more nice pictures



Remarks

Observation. Near $\infty \in \mathbb{CP}^1$ the Cauchy transforms $\lambda_1 C_1(z), \dots, \lambda_k C_k(z)$ are independent sections of the symbol equation of T_λ considered as a branched cover over $\mathbb{CP}^1$.

Open Problems. Find "explicit" description of the measures μ_i and their supports. What is their relation to the plane curve $\sum_{i=1}^k Q_i(z)y^i = 0$?

Heine-Stieltjes theory

Take a general linear operator $T = \sum_{i=0}^{k} Q_i(z)\frac{d^i}{dz^i}$ with polynomial coefficients and set

$$r = \max_i(\deg Q_i(z) - i).$$

If $r \geq 0$, $\deg Q_k(z) = k + r$ and $Q_k(z)$ has at least two distinct roots we call T **general Lame-type** operator.
Consider the generalized spectral problem

$$T(p(z)) + V(z)p(z) = 0,$$

where $p(z)$ is an eigenpolynomial and $V(z)$ is a spectral polynomial. (Classically, $p(z)$ is called a **Stieltjes** polynomial and $V(z)$ is called a **Van Vleck** polynomial.) Note that $\deg V(x) \leq r$.

Results

Proposition 3.1. Under the above assumptions for any sufficiently large n there exist exactly $\binom{n+r}{r}$ degree n Stieltjes polynomials $p_{n,j}(z)$ and corresponding Van Vleck polynomials $V_{n,j}(z)$.

Results, cont.

Proposition 3.2. If a sequence $\{\widetilde{V}_{n,j_n}(z)\},\ n = 1, \dots,$ of scaled Van Vleck polynomials converge to some polynomial $\widetilde{V}(z)$ then the sequence of finite measures $\mu_{n,j}$ of the corresponding family of eigenpolynomials $\{p_{n,j_n}(z)\}$ converge to a measure $\mu_{\widetilde{V}}$ satisfying the properties:
a) supp $\mu_{\widetilde{V}}$ is a forest of curvilinear trees;
b) the union of the leaves of supp $\mu_{\widetilde{V}}$ coincides with the union of all zeros of $Q_k(z)$ and that of $\widetilde{V}(z)$.
c) supp $\mu_{\widetilde{V}}$ is straightened out by the transformation given by

$$\int_a^z \frac{\widetilde{V}(z)dz}{Q_k(z)}.$$

Example

$$T = (z^2 + 1)(z + 2I - 3)(z - 3I - 2)\frac{d^3}{dz^3}$$

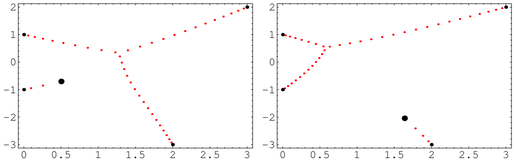


Figure: Examples of μ_Q's for the above T.

Example, cont.

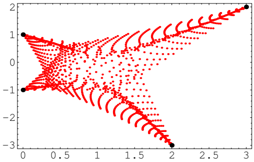


Figure: Union of μ_Q's for the above T.

Example, cont.

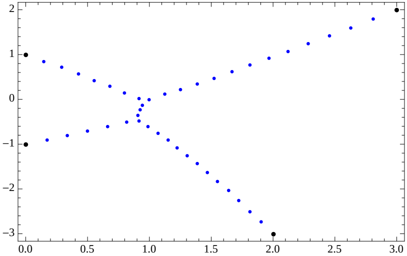


Figure: The union of roots for the Van Vleck polynomials for the above T.

Problems

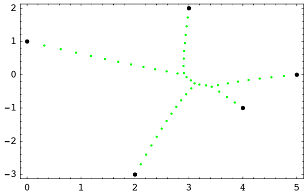


Figure: The union of roots for the Van Vleck polynomials for $T = Q_5(z)\frac{d^4}{dz^4}$.

The asymptotic distributions for the roots of Van Vleck polynomials are completely unknown for operators of order > 2!

Thanks for your patience!