Some problems by Anatol Kirillov

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Main references

a) A. Kirillov, On some quadratic algebras $1\frac{1}{2}$, arXiv:1502.00426.
b) A. Kirillov, On some quadratic algebras II, unpublished draft with many typos, available upon request, but not for distribution.
c) J. Blasiak, R. Liu, K. Meszaros, Subalgebras of the Fomin-Kirillov algebras, arXiv:1310.4112.
d) S. Homayouni, Topics in Fomin-Kirillov Algebra, PhD thesis 2021.

Three main non-commutative algebras

The following definitions can be found in the above references.

Definition

The two–term relations algebra $2T_n$ is an associative algebra over $\mathbb{Q}$ with the set of generators $\{u_{ij} | 1 \leq i < j \leq n\}$ subject to the set of relations:
a) Locality relations: $u_{ij}u_{k\ell} = u_{k\ell}u_{ij}$, if i, j, k, ℓ are pairwise distinct;
b) Two–term relations: $u_{ij}u_{jk} = u_{ik}u_{ij} = u_{jk}u_{ik}, u_{jk}u_{ij} = u_{ij}u_{ik} = u_{ik}u_{jk}, \quad 1 \leq i < j < k \leq n$.

The algebra $2T_n$ is given by $\binom{n}{2}$ generators and $6\binom{n}{3} + 3\binom{n}{4}$ relations.

Definition

The three–term relations algebra $3T_n$ is an associative algebra over $\mathbb{Q}$ with the set of generators $\{u_{ij} | 1 \leq i < j \leq n\}$ subject to the set of relations:

a) Locality relations: $u_{ij}u_{k\ell} = u_{k\ell}u_{ij}$, if i, j, k, ℓ are pairwise distinct;

b) Three–term relations:

$$\begin{cases} u_{ij}u_{jk} = u_{jk}u_{ik} + u_{ik}u_{ij}, \\ u_{jk}u_{ij} = u_{ik}u_{jk} + u_{ij}u_{ik}, \quad 1 \leq i < j < k \leq n \end{cases}$$

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The algebra $3T_n$ is given by $\binom{n}{2}$ generators and $2\binom{n}{3} + 3\binom{n}{4}$ relations.

Definition

The six–term relations algebra $6T_n$ is an associative algebra over $\mathbb{Q}$ with the set of generators $\{u_{ij} | 1 \leq i < j \leq n\}$ subject to the set of relations:

a) Locality relations: $u_{ij}u_{k\ell} = u_{k\ell}u_{ij}$, if i, j, k, ℓ are pairwise distinct;

b) six–term relations (i.e classical Yang-Baxter relations):

$$[u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] = 0, \quad 1 \leq i < j < k \leq n.$$

The algebra $6T_n$ is given by $\binom{n}{2}$ generators and $\binom{n}{3} + 3\binom{n}{4}$ relations.

Project1: Kirchhoff algebras

We denote by $2T_n^{(0)}$ the quotient of the algebra $2T_n$ by the two-sided ideal generated by the elements $\{u_{ij}^2, 1 \leq i < j \leq n\}$. Finally we denote by $2T_n^{ab}$ the abelinization of the algebra $2T_n^{(0)}$.

Proposition (CHECK!)

$$Hilb(2T_n^{ab}, t) = \sum_k \left\{ \binom{n}{n-k+1} \right\} t^k, \ \dim_{\mathbb{Q}} 2T_n^{ab} = Bell_n,$$

where $\left\{ \binom{n}{n-k+1} \right\}$ *stands for the Stirling number of the second type, and* $Bell_n$ *denotes the n-th Bell number.*

Recall that $\left\{ \binom{n}{k} \right\}$ is the number of ways to partition n objects into k non-empty subsets.
B_n is the number of all partitions of n objects.

Example (CHECK!)

$$Hilb(2T_4^{ab}, t) = (1, 6, 7, 1), Betti(2T_4^{ab}, t) = (1, 14, 32, 50, 50, 21, 1).$$

$$Hilb(2T_5^{ab}, t) = (1, 10, 25, 15, 1), Hilb(2T_6^{ab}, t) = (1, 15, 65, 90, 31, 1).$$

Definition

The Kirchhoff algebra Krh_n is the quotient of the algebra $2T_n^{ab}$ by the ideal generated by the elements (classical Yang-Baxter)

$$u_{i\ell}u_{jk}+u_{ik}u_{j\ell}+u_{ij}u_{k\ell}-u_{ik}u_{jk}-u_{i\ell}u_{j\ell}-u_{j\ell}u_{k\ell}, \quad 1 \le i < j < k < \ell \le n.$$

Proposition (CHECK!)

$$Hilb(Krh_n, t) = Narayana(n, k)t^k, \quad \dim Krh_n = Cat_n.$$

Recall that $Narayana(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ and $Cat_n = \frac{1}{n+1}\binom{2n}{n}$.

Problem

Generalize the definitions and claims about the Kirchhoff algebra from the case of complete graph K_n to some other class of graphs.

Remark

Per Alexandersson and Darij Grinberg have started to work on this, but they need a younger person to finish the project! Any volunteers in the audience?

Extra information

Let $\{x_{ij}\}_{1\leq i\neq j\leq n}$ be a symmetric matrix of mutually commuting variables, i.e $x_{ij}=x_{ji}$. Set $\partial_{ij}:=\partial/\partial_{x_{ij}}$. Define the *spanning tree polynomial* $F_n(x_{ij})$ as

$$F_n(x_{ij}):=\sum_T \prod_{(ij)\in Edge(T)} x_{ij},$$

where the sum is taken over the set of spanning trees of the complete graph K_n.

By the classical Kirchhoff theorem $F_n(x_{ij})$ is equal to any of the principal $(n-1)\times(n-1)$ minor of the Laplace matrix $L_n:=(\ell_{ij})$, where

$$\ell_{ij}:=\begin{cases}\sum_{k\neq i} x_{ik} & \text{if } i=j,\\ -x_{ij} & \text{if } i\neq j.\end{cases}$$

Next consider the ring of differential operators $\mathbb{Q}[\partial_{ij}]_{1\leq i<j\leq n}$ and the following collection I_n of differential operators

$$\partial_i\partial_{jk}-\partial_{ik}\partial_{ij},\partial_i\partial_{jk}-\partial_{jk}\partial_{ik},\partial_{jk}\partial_{ij}-\partial_{ij}\partial_{ik},\partial_{jk}\partial_{ij}-\partial_{ik}\partial_{jk},\quad 1\leq i<j<k\leq n$$

$$\partial_{i\ell}\partial_{jk}+\partial_{ik}\partial_{j\ell}+\partial_{ij}\partial_{k\ell}-\partial_{ik}\partial_{jk}-\partial_{i\ell}\partial_{j\ell}-\partial_{j\ell}\partial_{k\ell},\quad 1\leq i<j<k<\ell\leq n.$$

WHY DO WE NEED I_n ARE ALL INDICES CORRECT???

Finally, denote by J_n the ideal

$$J_n := Ann(F_n) := \{D\in\mathbb{Q}[\partial_{ij}]_{1\leq i<j\leq n} | D\circ F = 0\},$$

and denote the quotient ring by $R(K_n) := \mathbb{Q}[\partial_{ij}]_{1\leq i<j\leq n}/J_n$.

Theorem (CHECK!)

In the above notation, one has

$$R(K_n) \simeq Krh_n.$$

Conjecture

The algebra Krh_n *is Koszul and* $Krh'_n \simeq NACYB_n$ *where* $NACYB_n$ *denotes the non-local associative classical Yang-Baxter algebra.*

Project 2: Fomin-Kirillov algebras

Definition

The Fomin-Kirillov algebra FK_n (alias $3T_n^{(0)}$) is the quotient of the three-term algebra $3T_n$ modulo the two-sided ideal generated by $u_{ij}^2, \quad 1 \leq i < j \leq n$.

Remark

Known that FK_n are finite-dimensional for $n \leq 5$. Conjectured that FK_n are infinite-dimensional for $n \geq 6$.

Example

FK_6 has 15 generators and 100 relations.

Problem

Compute $Hilb(FK_6(t))_d$, *for* $d > 15$.

Problem

Describe the subalgebra $A \subset FK_6$ *generated by* $\{u_{13}u_{14}u_{15}; u_{24}u_{25}u_{26}; u_{35}u_{36}, u_{46}\}$. *Is A infinite-dimensional?*

For any subgraph $G \subseteq K_n$, one can define FK_G by killing the generators of FK_n for non-edges.

Problem (Very fundamental!)

For which graphs G, the algebra FK_G is finite-dimensional?

In the article "Subalgebras of the Fomin-Kirillov algebras", arXiv:1310.4112, one finds results of experiments in Bergman for graphs with up to 6 vertices. Some conjectures and results are presented for Dynkin diagrams of simple Lie algebras and affine Lie algebras.
Even a conjecture about the latter problem would be an important first step.

Further directions

There exists a natural action of FK_n on the group ring $\mathbb{Z}[S_n]$ given by:

$$u_{ij} \cdot w = \begin{cases} w \cdot s_{ij}, \text{ if } \ell(w \cdot s_{ij}) = \ell(w) + 1 \\ 0, \text{otherwise}, \end{cases}$$

where $w \in S_n$.

Problem

What is the kernel of this representation?

Conjecture

Let A be finitely generated nil-quadratic algebra over $\mathbb{Q}$ (i.e. $e_i^2 = 0$ for any generator of A). Let d be the number of generators of A and r be the number of defining relations.

$$\text{If } \binom{d}{2} > r \text{ then } A \text{ is infinite-dimensional.}$$

Corollary (Conjectural)

FK_6 has 15 generators and 100 relations. Since $\binom{15}{2} = 105 > 100$ then FK_6 is infinite-dimensional.

There are natural actions of S_n on FK_n and on $3T_n$. Define the **Dunkl elements**

$$\Theta_i = \sum_{j \neq i} u_{ij}, \ i = 1, \ldots, n.$$

One can easily check that Θ_i commute with each other in FK_n.

Problem

Show that for any $\lambda \vdash n$, the Schur polynomial $s_\lambda(\Theta_1, \ldots, \Theta_n)$ can be written in the algebra FK_n^{ab} as a non-negative linear combination of monomials.

Denote by $C^{+}(FK_n) \subset FK_n$ the cone of non-negative combinations of monomials in FK_n.

Problem

(a) describe $C^{+}(FK_n)$;
(b) check whether for $w \in S_n$,

$$\sigma_w(\Theta_1, \ldots, \Theta_n) \in C^{+}(FK_n),$$

where σ_w (apparently) is the Schubert polynomial corresponding to w.

Project 3: Algebra of chord diagrams

Definition

Define the algebra of chord diagrams CD_n as the quotient of $6T_n$ by the two-sided ideal generated by the elements $u_{ij}^2, 1 \leq i < j \leq n$.
Denote by $\widetilde{CD}_n$ the subalgebra of CD_n generated by the additive Dunkl elements

$$\theta_i = -\sum_{j<i} u_{ji} + \sum_{j>i} u_{ij}, \quad 1 \leq i \leq n.$$

Set $NL_n(t) := Hilb(CD_{2^n}, t)$. (SEEMS TO BE SEVERAL TYPOS IN THE CLAIM BELOW)

Proposition (CHECK!)

$$Hilb(CD_n, t) = \sum_{2k \le n} \frac{n!}{k!(n-2k)!2^k} t^k,$$

$$Hilb(CD(K_{n,n}), t) = \sum_{k \ge 0} k! \binom{n}{k}^2 t^k,$$

$$\deg_t NL_n(t) = n, \; NL_n(1) = [A293073(n)],$$

$$Coeff_{t^n} NL_n(t) = [A053871(n)],$$

In other words, $Coeff_{t^n} NL_n(t)$ is the number of perfect matchings in the complete multipartite graph K_{2^n} and it is equal to $\sum_{k \ge 0} (-1)^k \binom{n}{k} (2(n-k)-1)!!$.

Proposition (CHECK!)

$$Hilb(\widetilde{CD},t) = \sum_{2k \le n} \frac{n-2k+1}{n-k+1} \binom{n}{k} t^k;$$

$$Hilb(\widetilde{CD(K_{2^n})}, t) = Hilb(\widetilde{CD}_{2n}, t).$$

Examples:

$$Hilb(CD(K_{2^5}), t) = (1, 40, 500, 2240, 3020, 544), \quad \dim CD(K_{2^5}) = 634$$

$$Hilb(\widetilde{CD(K_{2^5})}, t) = (1, 9, 35, 75, 90, 42), \quad \dim \widetilde{CD(K_{2^5})} = 252 = \binom{10}{5}$$

Comments

Comment 1. It is known that the k-th coefficient of the polynomial $Hilb(CD_n, t)$ given by $\frac{n!}{k!(n-2k)!2^k}$ equals the number of k-matchings of the complete graph K_n, see A100861. The number $Hilb(CD_n, 1)$ counts the number of standard tableaux with n cells which also equals the number of involutions in the symmetric group, see A000085. Further $Coeff_{t^{[n/2]}} Hilb(CD_n, t) = \left(2\left[\frac{n+1}{2}\right] - 1\right)!!$.

Comment 2. The k-th coefficient of the polynomial $Hilb(\widetilde{CD}_n, t)$ given by $\frac{n-2k+1}{n-k+1}\binom{n}{k} = \binom{n}{k} - \binom{n}{k-1}, 2k \leq n$ equals the number of standard tableaux with n cells, see A008315.

Problem

Generalize chord algebras CD_n and $\widetilde{CD}_n$ to other graphs and understand the combinatorics of their Hilbert series.

More information

For a subset $I \subset [1, n]$, define the monomial

$$\eta_I := \prod_{i \in I, j \in [1,n] \setminus I} u_{ij}.$$

Consider the ideal J_n in the Postnikov-Shapiro algebra $\mathcal{A}_n$ generated by the elements $\eta_I, \emptyset \neq I \subsetneq [1, n]$. Denote by $\mathcal{A}_n^\sharp$ the quotient algebra $\mathcal{A}_n / J_n$.

Lemma (CHECK!)

$$Hilb(\mathcal{A}_n^\sharp, t) = t^{n-1} Tutte(K_n, t^{-1}, 1).$$

Problem

Generalize chord algebras $\mathcal{A}_n^\sharp$ to other graphs and understand the combinatorics of their Hilbert series.