On isodynamic points of binary forms and their ratios

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Topics to discuss

Main references

(i) Ch. Hägg, B. Shapiro, and M. Shapiro, Introducing isodynamic points for binary forms and their ratios, Complex Analysis and its Synergies, volume 9, Article number: 2 (2023).

(ii) https://en.wikipedia.org/wiki/Isodynamic_point.

(iii) S. Rabinowitz, Catalog of Properties of the First Isodynamic Point of a Triangle, International Journal of Computer Discovered Mathematics (IJCDM), Volume 6, 2021, pp. 108–136.

In geometry, a **triangle center** is a point in the triangle's plane that is in some sense in the middle of the triangle. For example, the centroid, circumcenter, incenter and orthocenter were familiar to the ancient Greeks, and can be obtained by simple constructions.

Each of these classical centers has the property that it is invariant (more precisely equivariant) under similarity transformations. In other words, for any triangle and any similarity transformation (rotations, reflections, dilations, or translations), the center of the transformed triangle is the same point as the transformed center of the original triangle.

This invariance is the defining property of a general **triangle center**. Encyclopedia of triangle centers lists more than a hundred such points.

Among all centers there exists a unique pair of points which are invariant under a larger group, namely under the action of the Möbius group $\mathcal{M}$ on the set of triangles. These points are called *first and second isodynamic points*.

There are typically two such points called of the triangle under consideration. (Every equilateral triangle however has just one isodynamic point; the second one can be thought as lying at infinity.)

The classical geometric problem about triangles in the Euclidean plane which isodynamic points solve is that for them the distances to the vertices are inversely proportional to the lengths of the opposite sides. In other words, they solve the equation

$$|u - z_1||z_3 - z_2| = |u - z_2||z_3 - z_1| = |u - z_3||z_2 - z_1| \qquad (1)$$

An elementary construction of the isodynamic points using Apollonian circles of a triangle can be found in Figure 1 below.

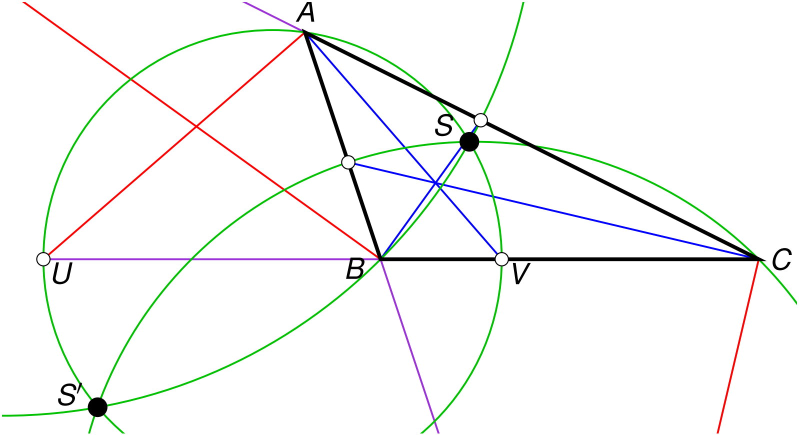


Figure: Let U and V be the points on BC met by the exterior and interior bisectors of angle A. The circle with diameter UV is called the *A-Apollonian*. The B- and C-circles are constructed similarly. Each circle passes through a vertex and both isodynamic points S and S'.

We introduce below the *isodynamic* map associating to a univariate polynomial of degree $d \geq 3$ with at most double roots a polynomial of degree (at most) $2d - 4$ such that this map commutes with the action of the Möbius group $\mathcal{M}$ on the zero loci of the initial polynomial and its image.

The roots of the image polynomial will be called the *isodynamic points* of the preimage polynomial. Our construction naturally extends from univariate polynomials to binary forms and further to their ratios.

In particular, the above classical geometric recipe associating to a plane triangle its isodynamic point(s) can be substituted by a pleasant explicit formula (15) associating to the unique monic cubic polynomial whose roots are the vertices of the triangle an appropriate polynomial of degree at most two vanishing at its isodynamic point(s). This formula seems to be missing in the existing literature.

Inspired by the latter observations, we consider the following question.

Problem

Find a (natural) map from an open and dense subset of complex-valued monic polynomials of a given degree $d \geq 3$ to some space of univariate polynomials which commutes with the action of the Möbius group $\mathcal{M} \simeq PGL_2(\mathbb{C})$ on the respective zero loci of the preimages and of the images.

Remark

Observe that the Möbius group $\mathcal{M}$ does not quite act on the space of monic polynomials or on their zero loci since a Möbius transformation typically has a pole in $\mathbb{C}$, i.e., it sends one point in $\mathbb{C}$ to ∞. The same difficulty already occurs in case of isodynamic points of triangles.

Therefore, to ensure a well-defined action of the Möbius group $\mathcal{M}$ and for our Problem 1 to be correctly stated, we will later instead of the space of monic polynomials of a given degree d consider the space of homogeneous binary forms of the same degree d.

In what follows, we present one non-trivial solution to Problem 1 which for $d = 3$ sends cubic monic polynomials to polynomials of degree at most two and which, on the level of their zero loci, associates to a triple of points in the Euclidean plane the above classical isodynamic point(s) of their convex hull.

We will call this map *isodynamic*.

More exactly, denote by Pol_d the affine space of monic complex-valued polynomials of degree d and by $Pol_d^\star \subset Pol_d$ its subset consisting of monic polynomials with roots of multiplicity at most 2.

Further denote by $\mathcal{P}_n$ the linear space of all complex-valued univariate polynomials of degree at most n. (Observe that $\mathcal{P}_n$ is isomorphic to the linear space $\mathcal{B}_n$ of binary forms of degree n.)

Denote by $\mathcal{B}_n^\star \subset \mathcal{B}_n$ the subset of homogeneous binary forms with roots of multiplicity at most 2 (considered as points in $\mathbb{C}P^1$).

Given a monic polynomial $P(z)$ of a given degree d, consider its *polar derivative*[1]

$$\mathcal{D}(z,u) = d\,P(z) + (u-z)P'(z). \tag{2}$$

Note that $\mathcal{D}(z,u)$ is a bivariate polynomial in (z,u) which is linear non-homogeneous in the variable u and has degree at most $d-1$ in z. The *(univariate) isodynamic map*

$$ID_d : Pol_d^{\star} \to \mathcal{P}_{2d-4} \tag{3}$$

sends a monic polynomial $P(z)$ of degree d with roots of multiplicity at most 2 to the polynomial $ID_d(P)$ in the variable u given by

$$ID_d(P) := \mathrm{Discr}(\mathcal{D}(z,u),z). \tag{4}$$

[1] Polar derivative has been already considered by E. Laguerre and G. Pólya jointly with G. Szegö, who used different terminology. A nice survey of its properties is Ch. 3 of the famous treatise by M. Marden.

In other words, ID_d sends $P(z)$ to the discriminant of its polar derivative $\mathcal{D}(z,u)$ with respect to the variable z.

The roots of $ID_d(P)$ are, by definition, the *isodynamic points* of $P(z)$.

We can also formulate a slightly different recipe of obtaining the isodynamic divisor $\mathcal{I}_P$.

Namely, given a polynomial $P(z) = z^d + a_1 z^{d-1} + a_2 z^{d-2} + \cdots + a_d$ of degree d as above, consider the rational function

$$R_P(z) = z - \frac{d\,P(z)}{P'(z)} = -\frac{a_1 z^{d-1} + 2a_2 z^{d-2} + \cdots + da_d}{dz^{d-1} + (d-1)a_1 z^{d-2} + \cdots + a_{d-1}} \quad (5)$$

which we call the **associated rational function** of P and take the divisor of the critical values of $R_P(z)$.

One can show that the latter divisor coincides with $\mathcal{I}_P$, i.e. the critical values of $R_P(z)$ are the isodynamic points of $P(z)$.

Thus the map (3) can be thought as a version of the *Lyashko-Looijenga map* sending a monic polynomial of degree d with simple roots to the univariate polynomial whose zero locus is the set of all critical values of its associated rational function.

Let us now modify the above construction to accommodate the case of binary forms and to ensure a well-defined action of the Möbius group $\mathcal{M}$ on the preimage and the image spaces.

Given $P(z)$ as above, define its homogenization $P(x,y) := y^d P(\frac{x}{y})$, where $z = \frac{x}{y}$. Using this notation, in the homogeneous coordinates $(x:y)$ on $\mathbb{C}P^1$ one has the following alternative expression for the homogenization of the associated rational function

$$R_P(x,y) = -\frac{P'_y(x,y)}{P'_x(x,y)}, \tag{6}$$

where $R_P(x,y)$ is obtained by the homogenization of the enumerator and denominator of the above $R_P(z)$.

Noticing that $R_P(x, y)$ has the same degrees of its numerator and denominator and therefore defines a rational function on $\mathbb{C}P^1$, we can introduce the *(bivariate) isodynamic map*

$$ID_d : Proj(\mathcal{B}_d^\star) \to Proj(\mathcal{B}_{2d-4}) \quad (7)$$

sending a binary form $P(x, y) \in \mathcal{B}_d^\star$ (considered up to a constant factor) to the binary form ID_d^P (considered up to a constant factor) whose projectivized zero locus is the set of critical values of the rational function $-\frac{P'_y(x,y)}{P'_x(x,y)}$.

The latter set of critical values is *(projective) isodynamic divisor* and its points are *(projective) isodynamic points* of $P(x, y)$. As above $\mathcal{B}_d^\star$ is the space of binary forms of degree d with roots of multiplicity at most 2 on $\mathbb{C}P^1$.

Our first result is as follows.

Theorem

The natural action of the Möbius group $\mathcal{M}$ on Proj($\mathcal{B}_d^\star$) and Proj($\mathcal{B}_{2d-4}$) commutes with the bivariate isodynamic map ID_d.

Note that binary forms of degree d can be thought of as holomorphic sections of the sheaf $\mathcal{O}(d)$ on $\mathbb{C}P^1$.

This circumstance sparkles the idea that the above construction of isodynamic map might extend to meromorphic sections of $\mathcal{O}(d)$ as well, i.e. to the ratios of binary forms $\frac{P(x,y)}{Q(x,y)}$, where $\deg P - \deg Q = d$. And indeed, such an extension exists.

Namely, consider the space $B^{\star}_{d+\partial,\partial}$ of pairs of binary forms $(P(x,y),Q(x,y))$ where P and Q are coprime, both P and Q have roots of multiplicity at most 2, $\deg P = d+\partial,\ \deg Q = \partial$, $d>0$, and $\partial \geq 0$.

We can interpret the pair $(P(x,y),Q(x,y))$ as the bivariate rational function $W(x,y)=\frac{P(x,y)}{Q(x,y)}$. (Note that $W(x,y)$ is not a rational function on $\mathbb{C}P^1$ in the usual sense, but is naturally a meromorphic section of $\mathcal{O}(d)$.) We can associate to $W(x,y)$ its divisor $D(W)$ of degree d whose positive part, i.e. divisor of zeros $D^0(W)$ has degree $d+\partial$ and whose negative part, i.e. divisor of poles $D_\infty(W)$ has degree $-\partial$.

Next, let us define the associated rational function of W as

$$R_W(x,y) = -\frac{W'_y(x,y)}{W'_x(x,y)}. \tag{8}$$

We will show that $R_W(x,y)$ has the same degree of its denominator $W'_x(x,y)$ as of its numerator $W'_y(x,y)$ implying that it is a well-defined rational function on $\mathbb{C}P^1$ or, equivalently, a meromorphic section of $\mathcal{O}(0)$. Namely, for generic binary forms $P(x,y)$ and $Q(x,y)$ as above the numerator and denominator both have degrees equal to $2d + 4\partial - 4$.

This circumstance explains why although formula (8) looks the same for all meromorphic sections of $\mathcal{O}(d)$ it makes sense not only to fix d, but also the degree ∂ of the pole divisor.
Let us finally associate to the divisor $D(W)$ the positive divisor $D(R_W)$ of all critical values of the associated rational function R_W.

Formula (8) can also be rewritten in a way similar to (5). Indeed, restriction of the above section $W(x, y)$ of $\mathcal{O}(d)$ to the local chart $y = 1$ on $\mathbb{C}P^1$ can be identified with the univariate rational function

$$w(z) = \frac{p(z)}{q(z)}$$

where $p(t) = P(t, 1)$ and $q(t) = Q(t, 1)$. Typically, $\deg p(t) = \deg P(x, y) = d + \partial$ and $\deg q(t) = \deg Q(x, y) = \partial$.

Moreover we can define the associated (univariate) rational function

$$R_w(z) := z - \frac{d\, w(z)}{w'(z)} = z - \frac{d\, p(z)q(z)}{p'(z)q(z) - p(z)q'(z)}.$$

One can easily see that $R_w(z) = R_W(x, y)|_{y=1}$ and its set of critical values can be defined similarly to (2).

Namely, for $w(z)=\frac{p(z)}{q(z)}$ with coprime $p(z)$ and $q(z)$ having roots of multiplicity at most 2 such that the degree $\deg p = d+\partial$ and $\deg q=\partial$ where $d\geq 1$, define its polar derivative as

$$\mathcal{D}(z,u)=d\,w(z)+(u-z)w'(z). \tag{9}$$

The numerator of $\mathcal{D}(z,u)$ is given by the expression

$$N\mathcal{D}(z,u)=d\,p(z)q(z)+(u-z)(p'(z)q(z)-p(z)q'(z)). \tag{10}$$

We define the *(univariate) rational isodynamic map*

$$ID_{d,\partial} : Rat^{\star}_{d+\partial,\partial} \to \mathcal{P}_{2d+4\partial-4} \tag{11}$$

as sending a rational function $w(z)$ to the polynomial given by

$$ID_{d,\partial}(u) := \mathrm{Discr}(N\mathcal{D}(z,u), z). \tag{12}$$

Here $Rat^{\star}_{d+\partial,\partial}$ is the space of rational functions $w = \frac{p}{q}$ with coprime pairs of polynomials (p, q) having only roots of multiplicity at most 2 , $\deg p = d + \partial$ and $\deg q = \partial$ for $\partial \geq 1$. If $\partial = 0$ then $Rat^{\star}_{d,0} = Pol^{\star}_{d}$.

Now define the *rational isodynamic divisor* $\mathcal{I}_w$ of the rational function w as the zero divisor of $ID_{d,\partial}(w)$. Similarly to the above, we can show that $\mathcal{I}_w$ is the divisor of the criticial values of $R_w(z)$.

Theorem

In the above notation, the action of the Möbius group $\mathcal{M} \simeq PGL_2(\mathbb{C})$ *on* $Rat^{\star}_{d+\partial,\partial}$ *and* $\mathcal{P}_{2d+4\partial-4}$ *commutes with the operation of taking the divisor of isodynamic points, i.e. with the map* $ID_{d,\partial}$.

Proofs. We start with the following statement. As above set $w(z) = \frac{p(z)}{q(z)}$ with coprime p and q having roots of multiplicity at most 2 and $\deg p = d + \partial$, $\deg q = \partial$, $d \geq 1$. Further, let $R_w(z) = z - \frac{d\,w(z)}{w'(z)}$ and $W(x,y) = \frac{P(x,y)}{Q(x,y)}$, where $P(x,y) = y^{d+\partial} p\left(\frac{x}{y}\right)$ and $Q(x,y) = y^{\partial} q\left(\frac{x}{y}\right)$. Finally, set $\mathcal{D}(z,u) = d\,w(z) + (u-z)w'(z)$.

Lemma

In the above notation,
(i) u^* *is a value of the variable* u *such that the polar derivative* $\mathcal{D}(z, u^*)$ *has a multiple root in* z *if and only if* u^* *is a critical value of* $R_w(z)$*;*
(ii) *the associated rational function* $R_W(x, y)$ *obtained by homogenizing the numerator and denominator of* $R_w(z)$ *coincides with* $-\frac{W'_y(x,y)}{W'_x(x,y)}$.

Proof.

To settle (i), note that if u^* is a value of the variable u for which the polar derivative $\mathcal{D}(z,u)$ has a multiple root in the variable z then denoting this root by z^* we get

$$0 = d\,w(z^*) + (u^* - z^*)w'(z^*) \Leftrightarrow u^* = z^* - \frac{d\,w(z^*)}{w'(z^*)} = R_P(z^*).$$

Note that the latter expression is well-defined if $w'(z^*) \neq 0$. Since z^* is a multiple root of $R_w(z)$, u^* has to be a critical value of $R_w(z)$ at z^*.

To settle (ii), let us rewrite the ratio $-\frac{W'_y}{W'_x}$. By definition, $W(x,y) = y^d \frac{p\left(\frac{x}{y}\right)}{q\left(\frac{x}{y}\right)}$. Thus

$$W'_x = y^{d-1} \frac{\left(p'\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right) q'\left(\frac{x}{y}\right)\right)}{q^2\left(\frac{x}{y}\right)}.$$

Similarly,

$$W'_y = d \cdot y^{d-1} \frac{p\left(\frac{x}{y}\right)}{q\left(\frac{x}{y}\right)} + \frac{y^d\left(-\frac{x}{y^2}\right)\left(p'\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right) q'\left(\frac{x}{y}\right)\right)}{q^2\left(\frac{x}{y}\right)}.$$

The above implies

$$-\frac{W'_y}{W'_x} = -\frac{d\,p\left(\frac{x}{y}\right)q\left(\frac{x}{y}\right) - \frac{x}{y}\left(p'\left(\frac{x}{y}\right)q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right)q'\left(\frac{x}{y}\right)\right)}{p'\left(\frac{x}{y}\right)q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right)q'\left(\frac{x}{y}\right)}$$

$$= \frac{x}{y} - \frac{d\,p\left(\frac{x}{y}\right)q\left(\frac{x}{y}\right)}{p'\left(\frac{x}{y}\right)q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right)q'\left(\frac{x}{y}\right)} = z - \frac{d\,p(z)q(z)}{p'(z)q(z) - p(z)q'(z)},$$

where $z = \frac{x}{y}$. □

Since Theorem 3 is a special case of Theorem 4 we present below only the proof of the latter result.

Proof. Using the above notation, set $R_W(x,y) = -\frac{W'_y}{W'_x}$ and make a change of coordinates $u = \alpha x + \beta y;\ v = \gamma x + \delta y$. Using the chain rule we obtain

$$\begin{cases} W'_x = \alpha W'_u + \gamma W'_v \\ W'_y = \beta W'_u + \delta W'_v \end{cases} \quad \Leftrightarrow \quad \begin{cases} W'_u = \frac{1}{\mathfrak{D}}\left(\delta W'_x - \gamma W'_y\right) \\ W'_v = \frac{1}{\mathfrak{D}}\left(-\beta W'_x + \alpha \frac{\partial}{\partial y}\right) \end{cases}, \tag{13}$$

where $\mathfrak{D} = \alpha\delta - \beta\gamma$. Thus introducing $R_W(u,v) = -\frac{W'_v}{W'_u}$, we obtain

$$R_W(u,v) = -\frac{-\beta W'_x + \alpha W'_y}{\delta W'_x - \gamma W'_y} = \frac{\alpha\left(\frac{-W'_y}{W'_x}\right) + \beta}{\gamma\left(\frac{-W'_y}{W'_x}\right) + \delta} = \frac{\alpha R_W(x,y) + \beta}{\gamma R_W(x,y) + \delta}.$$

Thus the action of the Möbius group $\mathcal{M}$ in the homogeneous coordinates $(x : y)$ results in the same action of $\mathcal{M}$ on the associated rational functions which implies that the locus of critical values experiences the same action. $\square$

Discriminant of the isodynamic map

Below we discuss possible situations when the image of the isodynamic map $ID_{d,\partial}$ is degenerate. Namely, we describe

(i) for which pairs $(p(z), q(z))$ the map $ID_{d,\partial}$ is well-defined (respectively, for which polynomials $P(z)$ the map ID_d is well-defined);

(ii) for which pairs $(p(z), q(z))$ the corresponding divisor $\mathcal{I}_{p,q}$ has degree less than $2d + 4\partial - 4$ in $\mathbb{C}$ (respectively, for which polynomials $P(z)$ of degree d the divisor $\mathcal{I}_P$ has degree less than $2d - 4$);

(iii) for which pairs $(p(z), q(z))$, the corresponding divisor $\mathcal{I}_{p,q}$ has multiple roots (respectively, for which polynomials $P(z)$ of degree d, the corresponding divisor $\mathcal{I}_P$ has multiple roots).

In order to answer the latter questions we need an additional statement.

Lemma

A polynomial pencil $f(z,u) = A(z) + uB(z)$ has a multiple root w.r.t the variable z for each value of the variable u if and only if $A(z)$ and $B(z)$ have a common root which has multiplicity at least 2 for both $A(z)$ and $B(z)$.

The following claim now answers the above question (i).

Proposition

The map $ID_{d+\partial,\partial}$ *applied to a rational function* $w(z) = \frac{p(z)}{q(z)}$, $\deg p = d + \partial$, $\deg q = \partial$ *given by* (12) *is well-defined if and only if*
either
(*) $\partial > 0$, *and the polynomials* p, q *are coprime and have roots of multiplicity at most* 2*;*
or
(**) $\partial = 0$ (i.e. $q \equiv 1$), *and all roots of the polynomial* P *have multiplicity at most* 2.

Proof. To settle (*), let us first show that in case $\partial > 0$, if $p(z)$ and $q(z)$ have a common root z^* then $N\mathcal{D}(z,u)$ and $N\mathcal{D}'_z(z,u)$ have a common root w.r.t z for any choice of u. Thus for such $(p(z), q(z))$, the map $ID_{d+\partial,\partial}$ is not defined. Indeed, we have

$$\begin{cases} N\mathcal{D}(z^*,u) = d\cdot p(z^*)q(z^*) + (u-z^*)(p'(z^*)q(z^*) - p(z^*)q'(z^*)) \equiv \\ N\mathcal{D}'_z(z^*,u) = (d-1)p'(z^*)q(z^*) + (d+1)p(z^*)q(z^*) + (u-z^*)(p'' \\ -p(z^*)q''(z^*)) \equiv 0, \end{cases}$$

i.e., both $N\mathcal{D}(z^*,u)$ and $N\mathcal{D}'_z(z^*,u)$ vanish identically in the variable u. Exactly the same argument holds if either p or q has a root z^* of multiplicity exceeding 2.

Let us now show the converse implication, i.e., that $ID_{d+\partial,\partial}$ is well-defined if $p(z)$ and $q(z)$ are coprime and have no roots of multiplicity bigger than 2. Assume that $ID_{d+\partial,\partial}$ is not defined for the pair (p,q) which is equivalent to the fact that $N\mathcal{D}(z,u)$ has a multiple root w.r.t z for all values of u. Observe that $N\mathcal{D}(z,u)$ is a polynomial pencil of the form $F(z)+u\Phi(z)$ where $F(z)=d\,pq-z(p'q-pq')$ and $\Phi(z)=(p'q-pq')$. Thus by Lemma 6, the univariate polynomials $F(z)$ and $\Phi(z)$ must have a common root of multiplicity at least 2 at some point z^*. But if $F(z)$ and $\Phi(z)$ have a multiple common root at z^* then pq and $p'q-pq'$ have a multiple common root at z^* as well. But the latter claim is impossible since p and q are coprime and therefore pq has no multiple roots.

To settle (**) let us first show that if $P(z)$ has a root z^* of multiplicity at least 3, then $\mathcal{D}(z,u) = d\,P(z) + (u-z)P'(z)$ has a multiple root w.r.t. z for any choice of u. Indeed,

$$\begin{cases} \mathcal{D}(z^*,u) = d\,P(z^*) + (u-z^*)P'(z^*) \equiv 0, \\ \mathcal{D}'_z(z^*,u) = (d-1)\,P'(z^*) + (u-z^*)P''(z^*) \equiv 0. \end{cases}$$

since P has a least a triple root at z^*. Now assume the converse, i.e., that $\mathcal{D}(z,u)$ has a multiple root w.r.t z for any value of u. Since $\mathcal{D}(z,u)$ is a polynomial pencil of the form $dP(z) - zP'(z) + uP'(z)$, Lemma 6 implies that $P(z)$ and $P'(z)$ must at least have a common double root z^*, i.e., $P(z)$ has a least a triple root. □

The following Lemma settles question (ii).

Define the polynomial families
$p(z) = z^{d+\partial} + a_1 z^{d+\partial-1} + \cdots + a_{d+\partial}$ and
$q(z) = z^{\partial} + b_1 z^{\partial-1} + \cdots + b_{\partial}$.

Lemma

(i) *For* $\partial > 0$, *up to a constant factor, the expression for the map* $ID_{d,\partial}$ *has a factor* $\mathrm{Resultant}(p,q)$, *where* $\mathrm{Resultant}(p,q)$ *stands for the resultant of* p *and* q. *The leading coefficient of* $ID_{d,\partial}$, *i.e. the coefficient of* $u^{2d+4\partial-4}$ *equals* $\mathrm{Discr}(p'q - pq')$, *where* $\mathrm{Discr}(p'q - pq')$ *stands for the discriminant of* $\Phi(z) = p'q - pq'$.

(ii) *For* $\partial = 0$ *the leading coefficient of* ID_d *equals the discriminant* $\mathrm{Discr}(P')$ *of* $P'(z)$. *In other words, the polynomial* $ID_d(P)$ *has degree less than* $2d-4$ *if and only if the derivative of the initial polynomial* P *has multiple zeros.*

Proof. To settle (i), observe that by Proposition 7, if $p(z)$ and $q(z)$ have a common root z^* then the bivariate polynomial $N\mathcal{D}(z, u)$ given by (11) has a multiple root at z^* in the variable z for all values of the second variable u which implies that the map $ID_{d,\partial}$ vanishes identically. To show that the leading coefficient of $ID_{d,\partial}$ equals Discr$(p'q - pq')$ let us calculate $ID_{d,\partial}$ using the standard determinantal formula for the resultant of the Sylvester matrix.

In our situation each non-vanishing entry of the Sylvester matrix will be a linear non-homogeneous polynomial in u with the leading coefficient and the constant term being polynomials in z. If we drop all the constant terms and just keep the terms containing u in the Sylvester matrix, then the determinant of this matrix will be equal to $u^{2d+4\partial-4}$ times the discriminant of the coefficient of u in the original expression $N\mathcal{D}(z, U)$. Since this coefficient equals $p'q - q'p$, the claim follows.

Remark

Observe that Lemma 8 explains why in the classical situation there exists only one isodynamic point for a triple of non-collinear points in the plane if and only if they form an equilateral triangle.

To solve question (iii), let us first discuss a more general set-up. Assume that we consider a family $\Phi_\lambda(z) = \frac{U_\lambda(z)}{V_\lambda(z)}$ of univariate rational functions depending on some multi-dimensional parameter λ taking values in some connected complex algebraic variety Λ. We assume that
(a) univariate polynomials $U_\lambda(z)$ and $V_\lambda(z)$ are coprime for generic values of $\lambda \in \Lambda$, but they can have a common factor for special values of λ belonging to some complex algebraic subvariety of Λ;
(b) for generic λ, $\Phi_\lambda(z)$ has distinct and simple critical values.

By our assumptions, the number of distinct critical values of $\Phi_\lambda(z)$ is constant for almost all $\lambda \in \Lambda$. Denoting this number by κ, let us define the Lyashko-Looijenga map $\mathcal{L} : \Lambda \to \mathbb{C}^\kappa$ sending every point $\lambda \in \Lambda$ to the divisor of the critical values of $\Phi_\lambda(z)$. We define the *Hurwitz discriminant* $\mathcal{H}D \subset \Lambda$ as the set of all $\lambda \in \Lambda$ for which the divisor $\mathcal{L}(\lambda)$ does not consist of κ simple points.

Set-theoretically, the Hurwitz discriminant $\mathcal{H}D$ typically contains three irreducible components $\mathcal{H}D^0 \cup \mathcal{H}D^W \cup \mathcal{H}D^M$, where

(i) $\mathcal{H}D^0 = \{\lambda \in \Lambda | \exists z^* \text{ such that } U_\lambda(z^*) = V_\lambda(z^*) = 0\};$

(ii) $\mathcal{H}D^W$ is the closure of $\mathcal{H}D_o^W$ where

$$\mathcal{H}D_o^W = \{\lambda \in \Lambda \setminus \mathcal{H}D^0 | \exists z^* \text{ such that } W_\Lambda(U;V) \text{ has a double root at } z^*\}$$

and $W_\Lambda(U;V) = U'_\lambda(z)V_\lambda(z) - U_\lambda(z)V'_\lambda(z)$ is the Wronskian of $U_\lambda(z)$ and $V_\lambda(z)$;

(iii) $\mathcal{H}D^M$ is the closure of $\mathcal{H}D_o^M$ where

$$\mathcal{H}D_o^M = \{\lambda \in \Lambda | \exists z_1 \neq z_2 \text{ with } \Phi'_\lambda(z_1) = \Phi'_\lambda(z_2) = 0;\ \Phi_\lambda(z_1) = \Phi_\lambda(z_2)\}$$

In our specific situation the family of rational functions $R_P(z)$ under consideration is given by (5) where $P(z)$ runs over the space Pol_d. Set

$$\begin{cases} \mathcal{U}_P(z) = a_1 z^{d-1} + 2a_2 z^{d-2} + \cdots + (d-1)a_{d-1}z + da_d = zP'(z) - d\,P(z); \\ \mathcal{V}_P(z) = dz^{d-1} + (d-1)a_1 z^{d-2} + \cdots + 2a_{d-2}z + a_{d-1} = P'(z) \end{cases} \tag{14}$$

i.e., $\mathcal{U}_P(z)$ is the numerator and $\mathcal{V}_P(z)$ is the denominator of the associated rational function $R_P(z)$.

Set $\mathcal{D}_d := \mathrm{Resultant}\left(ID_d, \frac{\partial ID'_d}{\partial u}, u\right)$. Numerical experiments with Mathematica strongly support the following guess.

Conjecture

(a) *In the above notation, for any $d \geq 3$,* $\mathcal{D}_d = (\mathcal{D}_d^0)^{j_d^0}(\mathcal{D}_d^W)^{j_d^W}(\mathcal{D}_d^M)^{j_d^M}$, *where* $\mathcal{D}_d^0, \mathcal{D}_d^W, \mathcal{D}_d^M$ *are irreducible polynomials whose zero loci satisfy the above conditions* (i), (ii), (iii) *respectively and* j_d^0, j_d^W, j_d^M *are some non-negative multiplicities.*

(b) *The non-negative multiplicities* j_d^0, j_d^W, j_d^M *are all positive for* $d \geq 5$.

Lemma

In the above notation, $\mathcal{U}_P(z)$ and $\mathcal{V}_P(z)$ have a common zero if and only $P(z)$ has a multiple root, i.e., the discriminant $\mathcal{D}_d^0$ coincides with the discriminant of $P(z)$.

Proof.

Indeed, if $\mathcal{U}_P(z^*) = \mathcal{V}_P(z^*) = 0$ then $P(z^*) = P'(z^*) = 0$ which means that $P(z)$ has a multiple root at z^*. □

Examples.

Cubic polys. In the classical case $d = 3$, direct calculations give the following explicit formula for the isodynamic map

$$ID_3 : z^3 + az^2 + bz + c \ \mapsto \ (a^2 - 3b)u^2 + (ab - 9c)u + b^2 - 3ac. \tag{15}$$

Equation (15) implies that if the image $ID_3(P)$ of a cubic polynomial P is linear, i.e., $a^2 = 3b$ then $P(z) = (z + a/3)^3 + c - a^3/27$ which means that its zero locus is necessarily an equilateral triangle.

In the above notation, the discriminant $\mathcal{D}_3$ of the family $ID_3(P)$ w.r.t. the variable u equals

$$\mathcal{D}_3 = 27c^2 + (4a^3 - 18ab)c + 4b^3 - a^2b^2$$

which is exactly the discriminant $\mathcal{D}_3^0$ of the original polynomial family $P(z) = z^3 + az^2 + bz + c$ w.r.t. the variable z. In other words, two isodynamic points of a triple of points in the plane coincide if and only if at least two of the points in the triple are equal. Thus for $d = 3$, $j_3^0 = 1$ and the discriminants $\mathcal{D}_3^W$ and $\mathcal{D}_3^M$ are empty, see Conjecture 10.

Quartic polys. To simplify our formulas, note that the affine shift $z \to z + t$ acts trivially on the isodynamic map ID_4. Using this fact, let us restrict our considerations to the standard reduced polynomial family

$$P_4(z) = z^4 + az^2 + bz + c$$

which is the classical versal deformation of the singularity z^4. The isodynamic map restricted to the latter family is explicitly given by

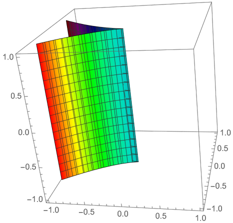
$$ID_4 : z^4 + az^2 + bz + c \mapsto (32a^3 + 108b^2)u^4 + (72a^2b + 864bc)u^3 + (-4a^4 + 108ab^2 - 288a^2c + 1728c^2)u^2$$
$$+(-4a^3b + 108b^3 - 432abc)u - 9a^2b^2 + 32a^3c. \qquad (16)$$

Up to a constant factor, the leading coefficient $(32a^3 + 108b^2)$ in the right-hand side of (16) is the discriminant of the derivative of $P_4(z)$ with respect of z.

The discriminant of the right-hand side of (16) w.r.t. u is given by

$$\mathcal{D}_4 = (2a^3 + 27b^2 - 72ac)^6(-4a^3b^2 - 27b^4 + 16a^4c + 144ab^2c - 128a^2c^2 + 256c^3). \quad (17)$$

The second factor $-4a^3b^2 - 27b^4 + 16a^4c + 144ab^2c - 128a^2c^2 + 256c^3$ of the right-hand side of (17) is $\mathcal{D}_4^0$ which is the standard discriminant of the above family $P_4(z)$.
The first factor $2a^3 + 27b^2 - 72ac$ is the Wronski discriminant $\mathcal{D}_4^W$. (Both discriminants are quasi-homogeneous with weights $w(a) = 2$, $w(b) = 3$, $w(c) = 4$ of the variables. The total weight of $\mathcal{D}_4$ equals 12; the total weights of $\mathcal{D}_4^0$ and of $\mathcal{D}_4^W$ equal 6.)
The Maxwell discriminant $\mathcal{D}_4^M$ is empty and $j_4^0 = 1,\ j_4^W = 6$.
Figure 2 shows these discriminants separately and together.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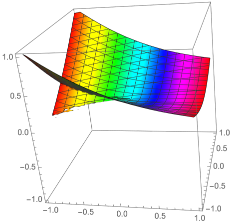
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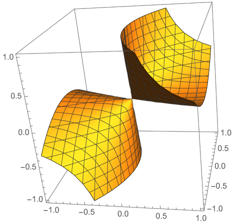
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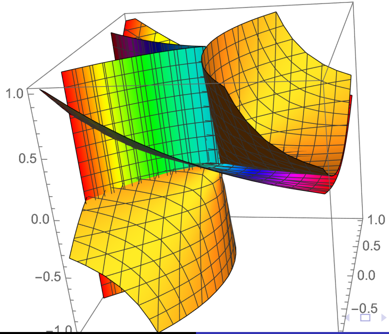
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Rational functions

Case 1-1. Let us consider the isodynamic map for the simplest nontrivial family of rational functions $w(z) = \frac{p(z)}{q(z)}$ where $p(z) = z^2 + az + b,\ q(z) = z + c$. (In this case $d = \partial = 1$.) Then

$$ID_{1,1} : \frac{z^2 + az + b}{z + c} \mapsto 4(b - ac + c^2)(u^2 + au + b)$$

and its discriminant equals

$$\mathcal{D}_{1,1} = 16(a^2 - 4b)(b - ac + c^2)^2.$$

Case 2-1. Consider $w(z) = \frac{p(z)}{q(z)}$ where $p(z) = z^3 + az^2 + bz + c,\ q(z) = z + e$. (In this case $d = 2$ and $\partial = 1$.) Then

$$ID_{2,1} : \frac{z^3 + az^2 + bz + c}{z + e} \mapsto 4(e^3 - c + be - ae^2)((-a^3 + 27c + 9a^2e - 27be)u^4 + (-6a^2b + 54ac + 8a^3e - 18abe - 54ce)u^3$$

$$+(18a^2c - 12ab^2 + 54bc + 12a^2be - 18b^2e - 54ace)u^2 + (18abc - 8b^3 + 54c^2 + 6ab^2e - 54bce)u$$

$$-9b^2c + 27ac^2 + b^3e - 27c^2e).$$

The discriminant of $ID_{2,1}$ is given by

$$\mathcal{D}_{2,1} = -1289945088(-a^2b^2 + 4b^3 + 4a^3c - 18abc + 27c^2)^4(c - be + ae^2 - e^3)^8.$$

Case 1-2. Consider $w(z) = \frac{p(z)}{q(z)}$ where $p(z) = z^3 + az^2 + bz + c,\ q(z) = z^2 + ez + f$. (In this case $d = 1$ and $\partial = 2$.) Then

$$ID_{1,2} : \frac{z^3 + az^2 + bz + c}{z^2 + ez + f} \mapsto -16(c+bu+au^2+u^3)(c^2-bce+ace^2-ce^3+b^2f-2acf-abef+3cef+be^2f+a^2f^2-2bf^2-ae$$

$$((-b^3+27c^2+3ab^2e-27bce-3a^2be^2+27ace^2+a^3e^3-27ce^3+18b^2f-54acf-9abef+81cef-9a^2e^2f+27be^2f+27a^2f^2-8$$

$$+(-9b^2c+27ac^2+3b^3e-9abce-6ab^2e^2+18a^2ce^2+3a^2be^3-27ace^3+24ab^2f-54a^2cf-54bcf-21a^2bef+18b^2ef+135ac$$

$$+9abe^2f+27a^3f^2-72abf^2-81cf^2-9a^2ef^2+27bef^2)u^2+(-9b^2ce+27ac^2e-3b^3e^2+9abce^2+3ab^2e^3-27bce^3+27b^3f-7$$

$$-21ab^2ef+18a^2cef+135bcef-6a^2be^2f+18b^2e^2f+24a^2bf^2-54b^2f^2-54acf^2+3a^3ef^2-9abef^2-9a^2f^3+27bf^3)u-9b^2ce$$

$$+b^3e^3-27c^2e^3+27b^2cf-81ac^2f-9abcef+81c^2ef-3ab^2e^2f+27bce^2f+18a^2cf^2-54bcf^2+3a^2bef^2-27acef^2-a^3f^3+27c$$

The discriminant of $ID_{1,2}$ equals

$$\mathcal{D}_{1,2} = 21641687369515008(-a^2b^2 + 4b^3 + 4a^3c - 18abc + 27c^2)(e^2 - 4f)\times$$

$$(b^4-6ab^2c+9a^2c^2-ab^3e+3a^2bce+9b^2ce-27ac^2e+b^3e^2+a^3ce^2-9abce^2+27c^2e^2+3a^2b^2f-10b^3f-10a^3cf+ 36abcf-27c^2f-a^3bef+3ab^2ef+9a^2cef-27bcef+a^4f^2-6a^2bf^2+9b^2f^2)^2(c^2-bce+ace^2-ce^3+b^2f-2acf-abef +3cef+be^2f+a^2f^2-2bf^2-aef^2+f^3)^12(-2b^3c+9abc^2-27c^3+b^4e-3ab^2ce-9a^2c^2e+27bc^2e-ab^3e^2+6a^2bce^2 -9b^2ce^2+b^3e^3-a^3ce^3-ab^3f+6a^2bcf-9b^2cf+6b^3ef-6a^3cef+a^3be^2f-6ab^2e^2f+9a^2ce^2f+a^3bf^2-6ab^2f^2 +9a^2cf^2 - a^4ef^2 + 3a^2bef^2 + 9b^2ef^2 - 27acef^2 + 2a^3f^3 - 9abf^3 + 27cf^3)^6.$$

Appendix I. Classical isodynamic points for plane triangles in our context

Classical isodynamic points for plane triangles have been studied for more than a century. The main result of this appendix is the following statement which relates the construction of the present paper to one of the definitions of the classical isodynamic points.

Proposition

The zeros of $ID_3(P)$ given by (15) are the first and second isodynamic points of the triangle $T \subset \mathbb{C}$ whose vertices are the (noncollinear) zeros of $P(z) = z^3 + az^2 + bz + c$.

Proof. Let $z_1,\, z_2$ and z_3 denote the zeros of $P(z)$. We will show that the zeros $u_1,\, u_2$ of $ID_3(P)$ satisfy the equations

$$|u - z_1||z_3 - z_2| = |u - z_2||z_3 - z_1| = |u - z_3||z_2 - z_1| \qquad (18)$$

which is one of the definitions for the isodynamic points of T. Due to symmetry of the vertices of T, it is sufficient to show that the first of these equations is satisfied. Furthermore, we can restrict ourselves to the case $z_1 = 0,\, z_2 = 1$ and $z_3 = \rho \in \{\zeta \in \mathbb{C} \mid \mathrm{Im}(\zeta) > 0\}$ since these points are the vertices of any triangle $T \subset \mathbb{C}$ under the action of affine transformations.

Under these conditions we have that

$$u_1 = \frac{\rho(\rho+1) - \sqrt{-3(\rho(\rho-1))^2}}{2+2\rho(\rho-1)}$$

and

$$u_2 = \frac{\rho(\rho+1) + \sqrt{-3(\rho(\rho-1))^2}}{2+2\rho(\rho-1)}.$$

Thus, for u_1 we need to show that

$$|u_1 - z_1||z_3 - z_2| = |u_1 - z_2||z_3 - z_1|, \tag{19}$$

or, equivalently,

$$\left|\frac{(u_1 - z_1)(z_3 - z_2)}{(u_1 - z_2)(z_3 - z_1)}\right| = 1 \iff \left|\frac{\left(\frac{\rho(\rho+1)-\sqrt{-3(\rho(\rho-1))^2}}{2+2\rho(\rho-1)} - 0\right)(\rho - 1)}{\left(\frac{\rho(\rho+1)-\sqrt{-3(\rho(\rho-1))^2}}{2+2\rho(\rho-1)} - 1\right)(\rho - 0)}\right| = 1. \tag{20}$$

Equation (20) simplifies to

$$\left| \frac{\sqrt{3}\rho(\rho-1)}{\sqrt{-(\rho(\rho-1))^2}} - 1 \right| = 2 \tag{21}$$

which is equivalent to $| \pm \sqrt{-3} - 1| = 2$. Since $\pm\sqrt{-3} = \pm i\sqrt{3}$ the later fact is trivial. Calculations for u_2 are completely similar. □

Final remarks.

1. The following analog of the famous theorem of E. Laguerre, about the location of the roots of the polar derivative in our current setting is supported by our numerical experiments with polynomials of degrees ≥ 3 with randomly distributed roots in various rectangles.

Conjecture (10)

For a univariate polynomial P of degree $d \geq 3$ with at most double roots, no circle or line in $\mathbb{C}$ separates the zero locus of $P(z)$ from $\mathcal{I}_P$.

2. Our numerical experiments with the isodynamic points for the Legendre and the Laguerre polynomials resulted in the following intriguing pictures which need to be explained, see Figure 3 below.

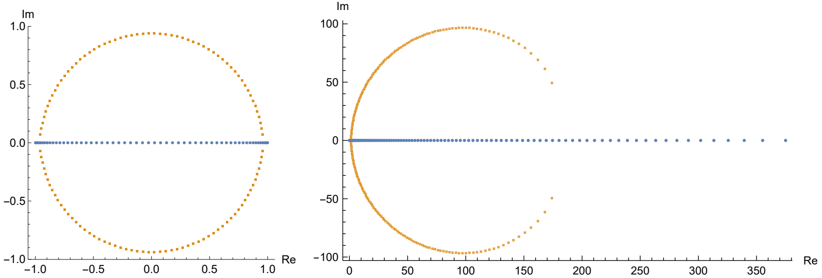

Figure: The zeros of the 60th Legendre polynomial (left, blue) and the 100th Laguerre polynomial (right, blue), shown along with the isodynamic points of these polynomials (brown).

3. Is our construction of the isodynamic map unique in some appropriate sense?

4. Is it possible to find a natural algebraic – geometric interpretation of the divisor of isodynamic points? Taking into account that the linear spaces $\mathcal{B}_n$ of degree n binary forms are the standard irreducible representation of $sl_2(\mathbb{C})$ is there a representation – theoretic meaning of our constructions.

5. Prove Conjecture 10 and its analog for rational functions.

Many thanks for your patience!