On inverse problem in Pólya–Schur theory

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Resume. Given a linear ordinary differential operator T with polynomial coefficients, we study the class of closed subsets of the complex plane such that T sends any polynomial (resp. any polynomial of degree exceeding a given positive integer) with all roots in a given subset to a polynomial with all roots in the same subset or to 0. Below we discuss some general properties of such invariant subsets as well as the problem of existence of the minimal under inclusion invariant subset.

Historical background

In 1914, generalizing some earlier results of E. Laguerre, G. Pólya and I. Schur created a new branch of mathematics now referred to as the Pólya–Schur theory. Their main result was a complete characterization of linear operators acting diagonally in the monomial basis of $\mathbb{R}[x]$ and sending any polynomial with all real roots to a polynomial with all real roots (or to 0). Without the requirement of diagonality of the action a characterization of such linear operators was obtained by P.Brändén jointly with late J. Borcea.

The main question considered in the Pólya–Schur theory can be formulated as follows.

Problem

Given a subset $S \subset \mathbb{C}$ of the complex plane, describe the semigroup of all linear operators $T : \mathbb{C}[z] \to \mathbb{C}[z]$ sending any polynomial with all roots in S to a polynomial with all roots in S (or to 0).

Definition

If an operator T has the latter property, then we say that *S is a T-invariant set*, or that *T preserves S*.

So far the latter Problem has only been solved for the circular domains (i.e., images of the unit disk under Möbius transformations), their boundaries and more recently for strips. Even a very similar case of the unit interval is still open at present.

It seems that for a somewhat general class of subsets $S \subset \mathbb{C}$, this Problem is out of reach of all currently existing methods.

In this lecture, we consider an inverse problem in the Pólya–Schur theory which seems both natural and more accessible than the direct Problem. We will restrict ourselves to consideration of *closed* T-invariant subsets.

Problem

Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, *characterize all closed* T*-invariant subsets of the complex plane. Alternatively, find a sufficiently large class of* T*-invariant sets.*

For example, if $T = \frac{d^j}{dx^j}$, $j = 1, 2, \dots$ then a closed subset $S \subseteq \mathbb{C}$ is T-invariant if and only if it is convex.

Although it seems too optimistic to hope for a complete solution of the inverse Problem for an arbitrary linear operator T, we present below a number of relevant results valid for linear ordinary differential operators of finite order.

To move further, we need to introduce some basic notions.

Definition

Given a linear ordinary differential operator

$$T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j} \tag{0.1}$$

of finite order $k \geq 1$ with polynomial coefficients, define its *Fuchs index* as

$$\rho_T = \max_{0 \leq j \leq k} (\deg(Q_j) - j).$$

An operator T is called *non-degenerate* if $\rho_T = \deg(Q_k) - k$, and *degenerate* otherwise. In other words, T is non-degenerate if ρ_T is realized by the leading coefficient of T. We say that T is *exactly solvable* if its Fuchs index is zero.

A few operators illustrating the situation are shown in Table 1, with some of their properties listed.

Operator	Fuchs index	Properties
$(x^3+2x)\frac{d^3}{dx^3}+x\frac{d^2}{dx^2}+1$	0	Exactly solvable, non-deg.
$(x+1)\frac{d^3}{dx^3}+x^4\frac{d^2}{dx^2}+2x$	2	Degenerate
$x^2\frac{d^3}{dx^3}+4\frac{d^2}{dx^2}$	-1	Non-degenerate

Table: Three examples of differential operators.

Definition

Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, we say that a *closed* set S belongs to *the class* $\mathcal{I}^T_{\geq n}$ if for every polynomial of degree *at least* n with all roots in S, its image $T(p)$ is either 0 or has all roots in S. In this case we say that S is *$T_{\geq n}$-invariant*.

By definition, the class $\mathcal{I}^T_{\geq 0}$ coincides with the class of all T-invariant sets. We say that a $S \in \mathcal{I}^T_{\geq n}$ is *minimal* if there is no closed proper nonempty subset of S belonging to $\mathcal{I}^T_{\geq n}$.

General properties of invariant sets

Definition

Given an operator $T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j}$ with $Q_k(x)$ different from a constant, denote by $\mathrm{Conv}(Q_k) \subset \mathbb{C}$ the convex hull of the zero locus of $Q_k(x)$. We refer to $\mathrm{Conv}(Q_k)$ as the *fundamental polygon* of T.

The next proposition contains basic information about invariant sets in $\mathcal{I}_{\geq n}^{T}$.

Theorem

The following facts hold:

- *for any operator* T *as above and any non-negative integer* n*, every* $S \in \mathcal{I}^T_{\geq n}$ *is convex;*
- *for any operator* T *as above and any non-negative integer* n*, if* S *is an unbounded closed set belonging to* $\mathcal{I}^T_{\geq n}$*, then* S *is* T*-invariant, i.e.,* S *belongs to* $\mathcal{I}^T_{\geq 0}$*;*
- *for any* T *as above with* $Q_k(x)$ *different from a constant and any non-negative integer* n*, every* $S \in \mathcal{I}^T_{\geq n}$ *contains the fundamental polygon* $\mathrm{Conv}(Q_k)$*;*
- *for any* T *as above with* $Q_k(x)$ *different from a constant and any non-negative integer* n*, the set* $\mathcal{I}^T_{\geq n}$ *has a unique minimal (under inclusion) element.*

Proof. Item (1). Fix $S \in \mathcal{I}^T_{\geq n}$ and choose $x_1, x_2 \in S$. Take $p(x) = (x - x_1)^m (x - x_2)^m$ for sufficiently large m, and consider $p^{(\ell)}(x)$ of some order ℓ. Then

$$p^{(\ell)}(x) = \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{m!}{(m-j)!} \frac{m!}{(m+j-n)!} (x - x_1)^{m-j} (x - x_2)^{m+j-\ell}$$

which implies that

$$q(x) := \frac{p^{(\ell)}(x)}{(x - x_1)^{m-\ell}(x - x_2)^{m-\ell}} =$$

$$\sum_{j=0}^{\ell} \binom{\ell}{j} (m)_j (m)_{\ell-j} (x - x_1)^{\ell-j} (x - x_2)^j.$$

Dividing both sides by m^ℓ and expanding the Pochhammer symbols, we see that

$$m^{-\ell} q(x) = \left(\sum_{j=0}^{\ell} \binom{\ell}{j} (x - x_1)^{\ell-j} (x - x_2)^j \right) + \frac{R_1(x)}{m} + \frac{R_2(x)}{m^2} + \cdots$$

$$= ((x - x_1) + (x - x_2))^\ell + O(m^{-1}) R(x).$$

Using the latter expression, we obtain

$$p^{(\ell)} = m^{\ell}((x-x_1)(x-x_2))^{m-\ell}\left((2x-x_1-x_2)^{\ell}+O(m^{-1})R(x)\right).$$

Therefore,

$$\frac{T(p(x))}{m^{\ell}} = Q_k(x)((x-x_1)(x-x_2))^{m-k}\Big((2x-x_1-x_2)^{\ell}+O(m^{-1})R(x$$
$$+\sum_{j=1}^{k}\frac{Q_{k-j}((x-x_1)(x-x_2))^{m-k+j}}{m^j}\Big((2x-x_1-x_2)^{\ell}+O(m^{-1}$$

All terms in the above sum approaches 0 as m gets large, implying that the roots of $T(p(x))$ are close to that of

$$Q_k(x)((x-x_1)(x-x_2))^{m-\ell}(2x-x_1-x_2))^{\ell}.$$

Since $\frac{x_1+x_2}{2}$ is a root of the latter polynomial, the original set S is convex.

Item (2). Assume that S is an unbounded set belonging to $\mathcal{I}^T_{\geq n}$ for some positive n. Take some polynomial p of degree less that n with roots in S. Consider a 1-parameter family of polynomials of degree n of the form $P_t := (x-\alpha(t))^{n-\deg p} p(x)$, $t \in [0, +\infty)$, where $\alpha(t)$ is a variable point in S which continuously depends on t and escapes to ∞ when $t \to +\infty$. (Such a family obviously exists since S is convex and unbounded.) Consider the polynomial family $T(P_t)$. Since $S \in \mathcal{I}^T_{\geq n}$, the roots of $T(P_t)$ belong to S for any finite t and continuously depend on t. Since S is closed the same holds for the limit of the roots of $T(P_t)$ which do not escape to infinity. Notice that the set of finite limiting roots exactly coincides with the set of roots of $T(p)$ which finishes the proof of item (2).

Item (3). Take an arbitrary T with $Q_k(x)$ different from a constant, any non-negative integer n, and an arbitrary set $S \in \mathcal{I}^T_{\geq n}$. Set $p(x) = (x - \alpha)^m$, where $\alpha \in S$. Then

$$\frac{T(p(x))}{(m)_k} = \sum_{j=0}^{k} Q_j(x) \frac{(m)_j}{(m)_k} (x - \alpha)^{m-j}.$$

If $m \to \infty$, then $\frac{(m)_j}{(m)_k} \to 0$ for $j < k$. Hence, the roots of $T(p(x))$ approach those of $Q_k(x)(x - \alpha)^{m-k}$ as m grows.

Item (4). Observe that for any differential operator T as above, the set $\mathcal{I}^T_{\geq n}$ is non-empty since it at least contains the whole $\mathbb{C}$. Now notice that by items (1) – (3), the intersection of all sets in $\mathcal{I}^T_{\geq n}$ is non-empty. Indeed each of them contains all roots of $Q_k(x)$. Since this intersection is convex it also contains the convex hull $\mathrm{Conv}(Q_k)$ of the roots of $Q_k(x)$. Since $\mathcal{I}^T_{\geq n}$ consists of closed convex sets with a non-empty common intersection, there is the unique minimal set in $\mathcal{I}^T_{\geq n}$.

Let us denote by $\mathrm{M}^T_{\geq n}$ the unique minimal element in $\mathcal{I}^T_{\geq n}$. The following consequence of the previous Theorem is straightforward.

Corollary

(i) *Under the assumption that $Q_k(x)$ is not constant, one has the sequence of inclusions of closed convex sets*

$$\mathrm{M}^T_{\geq 0} \supseteq \mathrm{M}^T_{\geq 1} \supseteq \cdots . \tag{0.2}$$

(ii) *Under the same assumption, if for some n, there exists a compact set $S \in \mathcal{I}^T_{\geq n}$ then $\mathrm{M}^T_{\geq m}$ is compact for all $m \geq n$ and there exists a well-defined limit*

$$\mathrm{M}^T_{\infty} := \lim_{n\to\infty} \mathrm{M}^T_{\geq n}. \tag{0.3}$$

Obviously, M^T_{∞} is a closed convex compact set.

Remark

The assumption that $Q_k(x)$ is different from a constant is important for the existence of the unique minimal under inclusion element in $\mathcal{I}^T_{\geq n}$.

Many operators with a constant leading term violate this property. For example, for $T = \frac{d}{dx}$, every convex closed subset of $\mathbb{C}$ belongs to $\mathcal{I}^T_{\geq n}$ for every non-negative integer n. In fact, every point in $\mathbb{C}$ is a minimal set for $T = \frac{d}{dx}$.

Non-degenerate operators

The main result in this part claims that for a fixed non-denegerate differential operator T, there exists a nonnegative integer n such that $\mathcal{I}^T_{\geq n}$ contains all sufficiently large disks. This implies compactness of the minimal set $\mathrm{M}^T_{\geq n}$ for large n.

Unfortunately, at present we do have an explicit description the boundary of $\mathrm{M}^T_{\geq n}$ for a given T and n. Our best result in this direction is that the limit M^T_{∞} coincides with the fundamental polygon $Conv(Q_k)$.

The next example shows that there exist non-degenerate exactly solvable operators for which $\mathrm{M}^{T}_{\geq n}$ is non-compact for small n.

Example. Consider the non-degenerate exactly solvable operator given by

$$T = \left(-\frac{x^2}{4} + \frac{x}{4}\right)\frac{d}{dx^2} + \left(\frac{x}{4} - \frac{1}{2}\right)\frac{d}{dx} + 1. \qquad (0.4)$$

For every $z \in \mathbb{C}$,

$$T\left[(x-z)^2\right] = (x-2z)\left(x - \left(\tfrac{z}{2} + \tfrac{1}{2}\right)\right). \qquad (0.5)$$

Take any closed subset $S \in \mathcal{I}^{T}_{\geq 2}$. The first factor in (0.5) ensures that if $z \in S$, then we also have $2z \in S$.

The second factor ensures that if $z \in S$, then $\frac{1}{2}(z+1) \in S$. These two facts imply that S must contain the interval $[1, \infty)$ of the real axis.
Moreover, the image of $(x-1)^4$ has -3 as root. This then implies that the entire real line lies in $\mathrm{M}^{T}_{\geq 2}$. Finally, the image of $(x+1)^2(x-1)^2$ has two complex (conjugate) roots, and this then implies that $\mathrm{M}^{T}_{\geq 2}$ is in fact the entire $\mathbb{C}$.

Existence of invariant disks based on results of Brändén-Borcea

We will show that for any non-degenerate operator T, the collection $\mathcal{I}^T_{\geq n}$ of its n-invariant sets contains large disks centered at 0 for all sufficiently large n.

Define the *n^{th} Fuchs index* of a linear operator $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$ as given by

$$\rho = \rho_n = \max_{0 \leq j \leq n} (\deg T(x^j) - j), \tag{0.6}$$

and call T *non-degenerate* if $\deg T(x^n) - n = \rho_n$.

Set $G_T(x, y) := T[(1 + xy)^n]$ and note that there exist polynomials P^n_ℓ, $\ell = -n, \dots, \rho$, of degree at most n, such that

$$G_T(x, y) = \sum_{-n \leq \ell \leq \rho} x^\ell P^n_\ell(xy). \tag{0.7}$$

In what follows, D_R denotes the open disk $\{x \in \mathbb{C} : |x| < R\}$, and $\bar{D}_R$ is the closure of D_R. We also define Ω_R as the open set $\{(x, y) \in \mathbb{C}^2 : |x| > R \text{ and } |y| > 1/R\}$.

Proposition (Thm.7, Borcea-Branden 2009)

Let $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$ be a linear operator of rank greater than one. The disk $\bar{D}_R$ is T_n-invariant if and only if $G_T(x, y) \neq 0$ for all $(x, y) \in \Omega_R$.

Remark.

In the above notation, T is non-degenerate if and only if the degree of P^n_ρ is n. If $T = \sum_{j=0}^{k} Q_j(x)\frac{d^j}{dx^j}$ is a differential operator of order k, then

$$G_T(x,y) = \sum_{j=0}^{k} j! x^{-j} Q_j(x) \binom{n}{j} (xy)^j (1+xy)^{n-j}, \qquad (0.8)$$

and it follows that

$$P^n_\ell(x) = \sum_{j=0}^{k} j! a_{\ell,j} \binom{n}{j} x^j (1+x)^{n-j}, \qquad (0.9)$$

where $a_{\ell,j}$ is the coefficient of $x^{j+\ell}$ in $Q_j(x)$. Define

$$f^n_\ell(x) = \sum_{j=0}^{k} j! a_{\ell,j} \binom{n}{j} x^j. \qquad (0.10)$$

Theorem

Suppose $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$ *is a non-degenerate linear operator with* n^{th} *Fuchs index* ρ. *Let* $g(x)$ *be the greatest common divisor of* $\{P^n_\ell(x)\}_\ell$. *Then the closed disk* $\bar{D}_R = \{x : |x| \leq R\}$ *is* T_n*-invariant for all sufficiently large* $R > 0$ *if and only if*

1. *all zeros of* $g(x)$ *lie in* $\{x : |x| \leq 1\}$;
2. *all zeros of* $P^n_\rho(x)/g(x)$ *lie in* $\{x : |x| < 1\}$.

Proposition

Let $T : \mathbb{C}_n[x] \to \mathbb{C}_n[x]$ be a diagonal operator, i.e.,

$$T(x^i) = \lambda_i x^i, \quad 0 \leq i \leq n.$$

The following conditions are equivalent:

1. *There is a compact non-empty T_n-invariant set $K \neq \{0\}$,*
2. *$\overline{D}_1$ is T_n-invariant,*
3. *$\overline{D}_R$ is T_n-invariant for all $R > 0$,*
4. *all zeros of the polynomial*

$$\sum_{i=0}^{n} \lambda_i \binom{n}{i} x^i$$

lie in $\overline{D}_1$.

Corollary

If T is a non-degenerate differential operator, then there is an integer N_0 and a positive number R_0 such that the disk $D_R := D(0, R)$ is T_n-invariant whenever $n \geq N_0$ and $R \geq R_0$.

Example

Consider the operator $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$ given by

$$T = (x^2 - x^3)\frac{d^3}{dx^3} + (x + x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx} - 6. \qquad (0.11)$$

When $n = 3$, we have that for every $z \in \mathbb{C}$,

$$T\left[(x - z)^3\right] = 12\left(x - z^2\right)(x - z/2). \qquad (0.12)$$

In particular, if z lies in a T_3-invariant set, then z^2 is also in the set. Thus, there are no large T_3-invariant disks. However, this does not violate the previous Theorem since the 3rd Fuchs index of T is 0, but $P_\rho^n(x) = -6(1 + 2x)$. Hence, the operator T is degenerate for $n = 3$ and this Theorem does not apply.

Extension to halvplanes

Definition

A polynomial $f(z_1, \dots, z_\ell) \in \mathbb{C}[z_1, \dots, z_\ell]$ is called *stable* if for all ℓ-tuples $(z_1, \dots, z_\ell) \in \mathbb{C}^\ell$ with $\mathrm{Im}(z_j) > 0$, $1 \leq j \leq \ell$, one has $f(z_1, \dots, z_\ell) \neq 0$.

Proposition

Take a closed half-plane given by $H = \{ax + b : \mathrm{Im}\, x \leq 0\}$, $(a, b) \in \mathbb{C}^2$, $a \neq 0$ *and let* $T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j}$ *be a differential operator. Then the following facts are equivalent:*

1. *The set of positive integers* n *for which* H *is* T_n*-invariant is unbounded,*
2. H *is* T_n*-invariant for all* $n \geq 0$,
3. *The polynomial* $\sum_{j=0}^{k} Q_j(ax + b)(-y/a)^j$ *considered as an element in* $\mathbb{C}[x, y]$ *is a stable polynomial in* (x, y).

Description of the limiting minimal set M_∞^T

Recall that we proved that whenever the leading coefficient $Q_k(x)$ of an operator T is has positive degree, then there is a minimal invariant set M_∞^T containing the convex hull of the roots of $Q_k(x)$. Furthermore, if T is non-degenerate, the previous Corollary implies that M_∞^T is compact. The next result was the main motivation for the above Theorem.

Theorem (See Thm. 9, Shapiro 2010)

Given a non-degenerate operator $T = \sum_{j=0}^{k} Q_j(x)\frac{d^j}{dx^j}$ and $\epsilon > 0$, there exists a positive integer n_ϵ such that for any $n > n_\epsilon$ and any polynomial p of degree n with all roots in $\mathrm{Conv}(Q_k)$, all roots of $T(p)$ lie in the ϵ-neighborhood of $\mathrm{Conv}(Q_k)$.

The main technical tool in the proof is the next result of independent interest. It extends the previous Theorem.

Proposition

Let $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$ *be a linear operator of rank greater than one, and let* D *be a closed disk in* $\mathbb{C}$*. Then* D *is* T_n*-invariant if and only if* $G_n^T(x, y) \neq 0$ *whenever* $x \in D^c$ *and* $y \in D$*, where*

$$G_n^T(x, y) = T((x-y)^n) = \sum_{i=0}^{n} \binom{n}{i} T(x^i) y^{n-i}.$$

Existence of invariant disks with arbitrary centers

Theorem

Given a non-degenerate operator

$$T = Q_k(x)\frac{d^k}{dx^k} + Q_{k-1}(x)\frac{d^{k-1}}{dx^{k-1}} + \cdots + Q_0(x),$$

let D be any closed disk that contains $\mathrm{Conv}(Q_k)$, and is such that the distance between $\mathrm{Conv}(Q_k)$ and the boundary of D is positive. Then D is T_n-invariant for all sufficiently large degrees n.

Theorem

If T is non-degenerate, then $M_\infty^T = Conv(Q_k)$.

Proof.

We assume $Conv(Q_k)$ is not a line or a point. The proofs for those cases are similar.

Let $\epsilon > 0$. For each side L of the polygon $Conv(Q_k)$, let $D_\epsilon(L)$ be a disc containing $Conv(Q_k)$ such that the distance between L and the boundary of $D_\epsilon(L)$ is at most ϵ and at least $\epsilon/2$, see next Fig.By the latter Theorem, $D_\epsilon(L)$ is T_n-invariant for all $n \geq N(L, \epsilon)$, where $N(L, \epsilon)$ is a positive integer. But then

$$K_\epsilon = \bigcap_L D_\epsilon(L)$$

is T_n-invariant for all $n \geq N(\epsilon)$, where $N(\epsilon) = \max_L N(L, \epsilon)$. Clearly $K_\epsilon \to Conv(Q_k)$. □

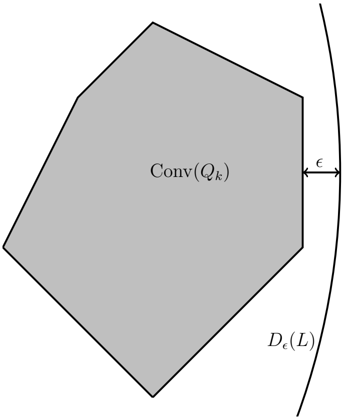

Figure: Illustration to the proof of the limit Theorem

Let us now describe a special class of non-degenerate operators for which all $\mathrm{M}^T_{\geq n}$, $n = 0, 1, \dots,$ coincide with each other and with the fundamental polygon $\mathrm{Conv}(Q_k)$.

Proposition

Take a non-degenerate operator of the form $T = Q_k(x)\frac{d^k}{dx^k} + Q_{k-1}(x)\frac{d^{k-1}}{dx^{k-1}}$ *satisfying the condition*

$$\frac{Q_{k-1}(x)}{Q_k(x)} = \sum_{i=1}^{\deg Q_k} \frac{\kappa_i}{x - x_i}, \tag{0.13}$$

where $\kappa_i \geq 0$ *and* $\{x_1, \dots, x_{\deg Q_k}\}$ *is the set of all roots of* $Q_k(x)$. *Then,*

$$\mathrm{M}^T = \mathrm{M}^T_{\geq 1} = \mathrm{M}^T_{\geq 2} = \cdots = \mathrm{M}^T_{\infty} = \mathrm{Conv}(Q_k).$$

Thank you for your attention!

Details can be found in

http://arxiv.org/abs/2404.14365

Happy reading!

We will need the following information, see e.g. Bergkvist 2007.

Given an exactly solvable operator T, observe that for each non-negative integer j,

$$T(x^j) = \lambda_j^T x^j + \text{ lower order terms.} \qquad (0.14)$$

Define the *spectrum* of an exactly solvable T as the sequence $\Lambda^T := \{\lambda_j^T\}_{j=0}^{\infty}$ of complex numbers.

Lemma

For any exactly solvable operator T and any sufficiently large positive integer n, there exists a unique (up to a constant factor) eigenpolynomial $p_n^T(x)$ of T of degree n. Additionally, the eigenvalue of p_n^T equals λ_n^T, where λ_n^T is given by (0.14).

One can easily show that for any exactly solvable operator T, the sequence $\{|\lambda_m^T|\}$ is monotone increasing to $+\infty$ which implies that for any sufficiently large positive integer m, $|\lambda_j^T| < |\lambda_m^T|$ for $0 \leq j < m$.

Remark

In addition to the latter Lemma, observe that for any exactly solvable operator T as in (0.1) and any non-negative integer n, T has a basis of eigenpolynomials in the linear space $\mathbb{C}_n[x]$ consisting of all univariate polynomials of degree at most n. This follows immediately from e.g., the fact that T is triangular in the monomial basis $\{1, x, \dots, x^n\}$. In other words, even if T has a multiple eigenvalue it has no Jordan blocks. However, the eigenpolynomial in the respective degree is no longer unique. A simple example of such situation occurs for $T = x^k \frac{d^k}{dx^k}$ in which case any polynomial of degree less than k lies in the kernel.

In what follows, we will use the following result.

Proposition

Given an exactly solvable operator T as in (0.1) and any invariant set $S \in \mathcal{I}^T_{\geq n}$, one has that S must contain the union of all roots of the eigenpolynomial p^T_m satisfying two conditions: $n \leq m$ and $|\lambda^T_j| < |\lambda^T_m|$ where $0 \leq j < m$. The latter fact implies that S contains the union of all roots of all eigenpolynomials of sufficiently large degrees.

Preliminaries on degenerate operators

An important although not very complicated result about degenerate operator which partially follows from our previous considerations is as follows.

Proposition

If T is a degenerate operator, then for any non-negative n, every set in $\mathcal{I}_{\geq n}^{T}$ is unbounded and, therefore is T-invariant.

(Tropical) algebraic preliminaries and three types of Newton polygons

In our study of invariant sets for degenerate operators we will need some classical results about root asymptotics of bivariate polynomials in the spirit of modern tropical geometry. These results will be used later.

We start by introducing the domination partial order on points in $\mathbb{R}^2$, Namely, we say that a point $p = (u, v) \in \mathbb{R}^2$ *dominates* a point $p' = (u', v')$ if $u \geq u'$ and $v \geq v'$. Given a subset $S \subseteq \mathbb{R}^2$, we call by its *northeastern border* $\mathbf{NE}_S$ the set of all points in S which are not dominated by other points in S. Observe that $\mathbf{NE}_S$ can be empty if S is non-compact, but for compact S, $\mathbf{NE}_S$ is always nonempty. Furthermore, if S is both compact and convex then $\mathbf{NE}_S$ is contractible.

Given a bivariate polynomial $R(u,v)=\sum_{(i,j)\in\Theta} a_{i,j}u^i v^j$, denote by $Conv(R)\subset\mathbb{R}^2$ its *Newton polygon*, i.e. the convex hull of the set of exponents $(i,j)\in\Theta$. The northeastern border of $Conv(R)$ will be denoted by $\mathbf{NE}_R$, see examples in Figure 2 and Figure 3. By the above, $\mathbf{NE}_R$ is connected and contractible. The point of $\mathbf{NE}_R$ with the maximal value of u will be called the *eastern vertex* and denoted by V_e and the point of $\mathbf{NE}_R$ with the maximal value of v will be called the *northern vertex* and denoted by V_n. The set $\mathbf{NE}_R$ coincides with a point if and only if $V_e=V_n$. Notice that every edge of the boundary of $Conv(R)$ included in $\mathbf{NE}_R$ has a negative slope. Finally, denote by $R^{ne}(u,v)$ the restriction of $R(u,v)$ to the subset $\Theta^{ne}\subseteq\Theta$ consisting of all monomials whose exponents are the vertices of $\mathbf{NE}_R$. We will call $R^{ne}(u,v)$ the *northeastern part* of $R(u,v)$.

Given an arbitrary bivariate polynomial

$$R(u,v)=\sum_{(i,j)\in\Theta} a_{i,j}u^i v^j=\sum_{j=0}^{m} R_j(v)u^j$$

and some number $w\in\mathbb{C}$, denote by $\mathcal{U}_R(w)$ the set of zeros of the equation $R(u,w)=0$ in the variable u considered as the divisor in $\mathbb{C}$, i.e. zeros are counted with multiplicities. Here m is the degree of R w.r.t. u. Assume that the parameter w runs over the portion of the positive half-axis $[\kappa,+\infty)$ which contains no root of $R_m(v)$; one can always choose κ sufficiently large so that the latter condition is satisfied. (Obviously, for all $w\in[\kappa,+\infty)$, the degree of the divisor $\mathcal{U}_R(w)$ equals m.) We define the subdivisor $\mathcal{U}_R^\infty(w)\subset\mathcal{U}_R(w)$ as the set of all roots $u(w)$ whose absolute values tend to ∞ when w tends to $+\infty$ along the positive half-axis.

Notice that $\mathcal{U}_R^{\infty}(w)$ is well-defined for all sufficiently large positive $\widetilde{\kappa} > \kappa$, since there exists $\widetilde{\kappa}$ such that for any $w \in [\widetilde{\kappa}, +\infty)$, the absolute value of every root in $\mathcal{U}_R^{\infty}(w)$ will be strictly larger than the absolute value of any root in the complement $\mathcal{U}_R(w) \setminus \mathcal{U}_R^{\infty}(w)$.

Our next goal is to describe $\mathcal{U}_R^\infty(w)$ in terms of $R^{ne}(u,v)$. In what follows we will frequently use the following statement.
Given an arbitrary bivariate polynomial $R(u,v)$ whose $\mathbf{NE}_R$ is not a single point, decompose $\mathbf{NE}_R$ into the (disjoint) union of consecutive edges $\mathbf{NE}_R = \cup_{s=1}^h e_s$ covering $\mathbf{NE}_R$ from north to east. That is e_1 starts at V_n, e_h ends at V_e, and each e_s is adjacent to e_{s+1}, see Figure 2. The absolute values of the slopes of $e_1, \dots, e_h$ are strictly increasing.
The following statement can be easily deduced from the known results.
(To use the latter results, one has to substitute u and v by u^{-1} and v^{-1} respectively.)

Proposition

The degree of the divisor $\mathcal{U}_R^\infty(w)$ *is equal to* $i_e - i_n$ *where* $V_e = (i_e, j_e)$ *and* $V_n = (i_n, j_n)$*. In other words,* $\deg \mathcal{U}_R^\infty(w)$ *equals the length of the projection of* $\mathbf{NE}_R$ *onto the u-axis. Additionally,* $\mathcal{U}_R^\infty(w)$ *splits into h subdivisors* $\mathcal{U}_1^\infty(w), \dots, \mathcal{U}_h^\infty(w)$ *corresponding to the edges* $e_1, \dots, e_h$ *respectively; the degree of* $\mathcal{U}_s^\infty(w),\ s = 1, \dots, h$ *equals the length of the projection of* e_s *on the u-axis. All zeros in the divisor* $\mathcal{U}_s^\infty(w)$ *have the asymptotic growth* $u \sim \epsilon w^{sl_s}$ *where* sl_s *is the absolute value of the slope of* e_s*. Possible values of* ϵ *can be found by substituting* ϵw^{sl_s} *in the restriction of* $R(u, v)$ *to the monomials contained in the edge* e_s *and finding the non-vanishing roots of this restriction.*

Definition

Given an arbitrary bivariate polynomial $R(u, v)$ whose northeastern border $\mathbf{NE}_R$ is not a single point, we will call the slopes of edges in $\mathbf{NE}_R$ the *characteristic exponents* of $R(u, v)$. For a given edge $e_s \in \mathbf{NE}_R$, all possible values of ϵ corresponding to the restriction of $R(u, v)$ to this edge will be called the *leading constants* corresponding to (the characteristic exponent of) e_s. The union of all leading constants of $R(u, v)$ will be denoted by Υ_R.

Example.

To illustrate Proposition 24 and Definition 25, take

$$R(u,v) = u^8 + u^7v^2 + u^5v^4 + (5+7\sqrt{-1})u^3v^6 - 23uv^7.$$

One can easily check that all monomials in $R(u,v)$ belong to $\mathbf{NE}_R$ which consists of three edges e_1, e_2, e_3 connecting $(1,7)$ with $(3,6)$, $(3,6)$ with $(7,2)$, and $(7,2)$ with $(8,0)$ resp. (The exponent $(5,4)$ of the second monomial belongs to e_2.) Degree of $\mathcal{U}_R^\infty(w)$ equals $8-1=7$. Restriction $R_1(u,v)$ of $R(u,v)$ to e_1 is given by $(5+7\sqrt{-1})u^3v^6 - 23uv^7$. Its nontrivial zeros with respect to the variable u are given by $(5+7\sqrt{-1})u^2 - 23w = 0$. Thus for two roots from $\mathcal{U}_1^\infty(w)$, $u \sim \epsilon w^{1/2}$ where ϵ are the two roots of the equation $(5+7\sqrt{-1})\epsilon^2 - 23 = 0$. They are approximately equal to -1.45392 ± 0.748212. (The absolute value of the slope of e_1 equals $\frac{1}{2}$.)

Restriction $R_2(u,v)$ of $R(u,v)$ to e_2 is given by $u^7v^2+u^5v^4+(5+7\sqrt{-1})u^3v^6$. Its nontrivial zeros with respect to the variable u are given by $u^4+u^2w^2+(5+7\sqrt{-1})w^4=0$; we have substituted w instead of v here to keep our notation. Thus for 4 different roots belonging to $\mathcal{U}_2^\infty(w)$, we have $u\sim\epsilon w$ where ϵ are the four roots of the equation $\epsilon^4+\epsilon^2+5+7\sqrt{-1}=0$. These are approximately equal to $-1.22651\pm0.961446\sqrt{-1}$ and $-0.809831\pm1.58673\sqrt{-1}$. (The absolute value of the slope of e_2 equals 1.) Finally, the restriction $R_3(u,v)$ of $R(u,v)$ to e_3 is given by $u^8+u^7v^2$ which gives $u=-v^2$. (The absolute value of the slope of e_3 equals 2.) Summarizing, we get that Υ_R consists of 6 complex numbers approximately given by $\{-1,-1.22651\pm0.961446\sqrt{-1},-0.809831\pm1.58673\sqrt{-1},-1.45392\pm0.748212\}$. Its convex hull contains 0 as its interior point.

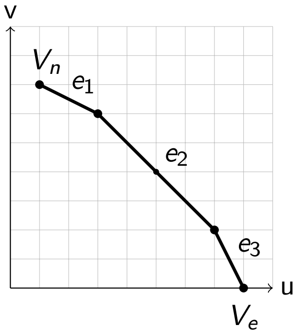

Figure: The northeastern border of the Newton polygon of $R(u, v) = u^8 + u^7v^2 + u^5v^4 + (5 + 7\sqrt{-1})u^3v^6 - 23uv^7$, see Example 49. (The Newton polygon itself is obtained by adding an edge connecting V_n with V_e.)

Corollary

In the above notation, for a given bivariate polynomial $R(u, v)$, the family of convex hulls of $\mathcal{U}_R^\infty(w)$ converges to $\mathbb{C}$ when $w \to +\infty$ if and only if the convex hull of Υ_R contains 0 as its interior point.

Let us fix a connected contractible piecewise linear curve $\mathbf{NE} \subset \mathbb{R}^2$ with integer vertices consisting of pairwise non-dominating points, see Figure 2. In other words, $\mathbf{NE}$ is a piecewise linear path with integer vertices whose edges have negative slopes whose absolute values increase when moving down along the path. Denote by $Pol(\mathbf{NE})$ the set of all bivariate polynomials whose northeastern border coincides with a given $\mathbf{NE}$. (In particular, we assume that all coefficients at the corners/endpoints of $\mathbf{NE}$ are non-vanishing. $Pol(\mathbf{NE})$ is a Zariski-open subset of a finite-dimensional linear space of bivariate polynomials.) Recall that the *integer length* of a closed straight interval $I \subset \mathbb{R}^2 \supset \mathbb{Z}^2$ is the number of points from $\mathbb{Z}^2$ contained in I, i.e. the number of integer points belonging to I.

Definition

Given **NE** $\subset \mathbb{R}^2$ as above, we call it

label=() *defining* if there exists an edge in **NE** with the slope $-\alpha/\beta$ where α and β are coprime positive integers and $\beta \geq 3$;

lbbel=() *almost defining* if there are no edges as in (i), but there either
1. exists at least one edge in **NE** with the slope $-\alpha/2$ and whose integer length is larger than 2, or
2. there exist at least two edges with the slope $-\alpha/2$ and integer length at least 2;

lcbel=() *non-defining* in the remaining case i.e., when either all edges of **NE** have negative integer slopes or all edges but one have negative integer slopes and the remaining edge has a negative half integer slope and integer length 2.

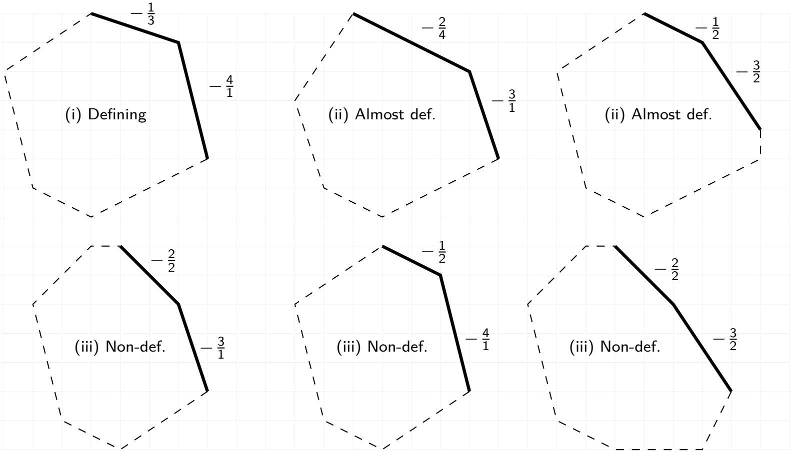

Figure: Examples of defining/almost defining/non-defining Newton polygons, see Definition 27. The slopes of the edges of the northeasten boundary are shown as fractions, such that the length of the projection is the respective denominator.

Definition

A Newton polygon $N \subset \mathbb{R}^2$ is called *defining/almost defining/non-defining* if its northeastern border contains at least one edge and is *defining/almost defining/non-defining* respectively.

In Figure 3, we show examples of Newton polytopes illustrating Definition 27 and Definition 28.

Proposition

Given $\mathbf{NE} \subset \mathbb{R}^2$ *as above, the convex hull of* $\mathcal{U}_R^\infty(w)$ *converges to* $\mathbb{C}$, *when* $w \to +\infty$

- label=() *for any* $R \in Pol(\mathbf{NE})$ *if* $\mathbf{NE}$ *is defining;*
- lbbel=() *for generic* $R \in Pol(\mathbf{NE})$ *if* $\mathbf{NE}$ *is almost defining;*
- lcbel=() *if* $\mathbf{NE}$ *is non-defining there is a full-dimensional subset of* $Pol(\mathbf{NE})$ *for which the convex hull of* $\mathcal{U}_R^\infty(w)$ *converges to* $\mathbb{C}$ *when* $w \to +\infty$ *and the complement of the latter set in* $Pol(\mathbf{NE})$ *is also full-dimensional.*

Remark

In case (ii), the condition of nongenericity is given by the fact that all $\epsilon \in \Upsilon_R$ are real proportional to each other (i.e. they lie on the same real line in $\mathbb{C}$ passing through the origin);
In case (iii) if one forces the next to the leading coefficient for some edge with integer slope and length of projection larger than 2 to vanish, i.e. one forces the sum of the respective ϵ to be equal to 0, then the conclusion of Corollary 26 will be valid for a generic choice of the remaining coefficients at the vertices belonging to this edge.
If the convex hull of $\mathcal{U}_R^\infty(w)$ does not tend to $\mathbb{C}$, but $i_n > 0$ which means that $\mathcal{U}_R(w) \setminus \mathcal{U}_R^\infty(w)$ is nonempty, then the convex hull of $\mathcal{U}_R(w)$ will tend to the convex cone with apex at 0 spanned by the elements of Υ_R.

Application of algebraic results to invariant sets of degenerate operators

We need to consider the action of $T = \sum_{j=0}^{k} Q_k(x)\frac{d^j}{dx^j}$ on polynomials of the form $(x-t)^n$ for sufficiently large n. One has

$$T(x-t)^n = (x-t)^{n-k}\sum_{j=0}^{k}(n)_j(x-t)^{k-j}Q_j(x) = (x-t)^{n-k}\psi_T(x,n,t)$$

where $\psi_T(x,n,t)$ is a trivariate polynomial. The important circumstance is that the essential part $\psi_T^{+}(x,n)$ of $\psi_T(x,n,t)$ is independent of t, see beginning of § 42. We will apply to $\psi_T^{+}(x,n)$ the results of the previous section and discuss how its zeros w.r.t x behave when $n \to +\infty$. Denote by $a_j x^{d_j}$ the leading monomial of $Q_j(x)$ and consider the polynomial

$$\widetilde{\psi}_T(x,n) = \sum_{j=0}^{k} a_j n^j x^{d_j+k-j}.$$

(It contains much fewer monomials than $\psi_T(x,n,t)$, but with exactly the same coefficients.) Notice that the essential part $\psi_T^+(x,n)$ is obtained from $\widetilde{\psi}_T(x,n)$ by removing those monomials which do not belong to $\mathbf{NE}(\psi_T)$.

Taking the symbol polynomial $G_T(x,y)=\sum_{j=0}^k Q_k(x)y^j$ of T, we introduce its truncation $\widetilde{G}_T(x,y)=\sum_{j=0}^k a_j y^j x^{d_j}$ and observe that $\widetilde{\psi}_T(x,n)$ is obtained from $\widetilde{G}_T(x,y)$ by substituting y by n and adding $k-j$ to the powers d_j of x of the respective monomial. Thus the Newton polygon of $\widetilde{\psi}_T(x,n)$ is obtained from the Newton polygon of $\widetilde{G}_T(x,y)$ by the affine transformation A sending (i,j) to $(i+k-j,j)$. Therefore $\mathbf{NE}(\psi_T)$ is obtained from the part of the boundary of the Newton polygon of $\widetilde{\psi}_T(x,y)$ under the latter affine transformation, see Fig. 4 for an example.

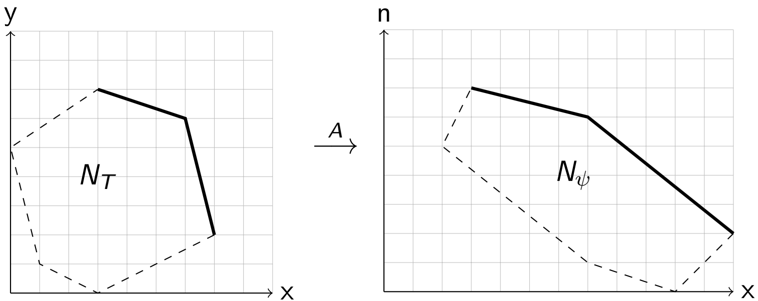

Figure: The affine transformation A sending N_T to N_ψ. Here $T = (x^3 + \dots)\frac{d^7}{dx^7} + (x^6 + \cdots)\frac{d^6}{dx^6} + \frac{d^5}{dx^5} + (x^7 + \cdots)\frac{d^2}{dx^2} + (x + \cdots)\frac{d}{dx} + (x^3 + \cdots)$,
$\widetilde{G}_T(x, y) = x^3y^7 + x^6y^6 + y^5 + x^7y^2 + xy + x^3$ and
$\widetilde{\psi}_T(x, n) = n^7x^3 + n^6x^7 + n^5x^2 + n^2x^{12} + x^{10}$.

Denote the Newton polygon of $\widetilde{G}_T(x, y)$ by N_T and the Newton polygon of $\widetilde{\psi}_T(x, y)$ by N_ψ. We have that $N_\psi = A \circ N_T$. The relation between the slopes of edges before and after the affine transformation A is as follows.

If the slope sl of an edge of N_T equals $sl = \frac{\mu}{\nu}$ where μ and ν are coprime integers and $\nu > 0$, then the slope of its image denote by asl is given by $asl = \frac{\mu}{\nu - \mu}$ which implies that $asl = \frac{sl}{1 - sl}$ or, equivalently, $sl = \frac{asl}{1 + asl}$. Therefore if asl is a negative integer then we get

$$asl = -J, J > 0 \Leftrightarrow sl = \frac{J}{J-1}.$$

Obviously any sl of the above form is positive (or $+\infty$). Analogously, if asl is a negative half-integer then we get

$$asl = -\frac{J}{2}, J > 0 \text{ and odd} \Leftrightarrow sl = \frac{J}{J-2}.$$

Again any sl of the above form is positive with the only exception $J = 1$ for which $sl = -1$.
It is easy to describe $A^{-1}(\mathbf{NE}_{\psi})$ as the part of the boundary N_T starting at V_n and going southeast till we either reach the lowest point of the polygon or till the slope of the next edge becomes smaller than or equal to 1. Denote $A^{-1}(\mathbf{NE}_{\psi})$ as $\mathfrak{B}_T$ and call it the *shifted northeastern border* of N_T.
One can easily check that for $T = \sum_{j=0}^{k} Q_k(x)\frac{d^j}{dx^j}$, the corresponding $\mathbf{NE}(\psi_T)$ is a single point if and only if T is non-degenerate. So for any degenerate T, its $\mathbf{NE}(\psi_T)$ contains at least one edge. Additionally, $asl < 0$ if and only if $\frac{1}{sl} < 1$ which means that either $sl < 0$ or $sl > 1$.
Observe that the vertex V_n of $\widetilde{\psi}$ coincides with that of $\widetilde{G}$.

Definition

A degenerate operator T is called defining/almost defining/non-defining if its Newton polygon N_ψ is defining/almost defining/non-defining resp., see Definition 27. In terms of the Newton polygon N_T this means that its shifted northeastern border $\mathfrak{B}_N$ is not a single point and in the *defining* case it contains an edge with the slope of the form $\frac{J}{J-\beta}$ with $\beta \geq 3$, in the *almost defining* case all edges of $\mathfrak{B}_N$ have slopes $\frac{J}{J-1}$ but there exists either one edge with slope $\frac{J}{J-2}$, J odd and length greater than 2 or two such edges with length 2; and in the *non-defining* case contains edges of arbitrary integer length with slopes $\frac{J}{J-1}$, J being a positive integer, except for possibly one edge of integer length 2 whose slope is $\frac{J}{J-2}$, J odd.

The following result is an easy consequence of our previous considerations.

Theorem

For any nonnegative integer n and (almost) any degenerate operator T whose N_T is (almost) defining, the only set contained in $\mathcal{I}^T_{\geq n}$ is $\mathbb{C}$.

Degenerate operators with non-defining Newton polygons

As we have seen above the convex hull of the set Υ_T of all leading constants for (almost) every degenerate T with (almost) defining N_T contains 0 as its interior point.
For degenerate T with non-defining N_T whose northeastern border we will denote by $\mathbf{NE}_T$, it might still happen that 0 is the interior point of the latter convex hull in which case the conclusion of Theorem32 holds. However for a full-dimensional subset of $Pol(\mathbf{NE})$ with a given non-defining $\mathbf{NE}$, their leading constants belong some half-plane in $\mathbb{C}$ bounded by a line passing through 0 and therefore 0 lies on the boundary of their convex hull. In this situation the conclusion of Theorem 32 fails and we will discuss this case below.

Definition

Given a finite set $\mathcal{U} = \{u_1, \ldots, u_k\}$ of (not necessarily distinct) complex numbers, we define the cone $\mathcal{C}^+\mathcal{U} \subseteq \mathbb{C}$ generated by $\mathcal{U}$ as given by

$$\mathcal{C}^+\mathcal{U} := \{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_\ell u_\ell\}, \text{where } \alpha_j \geq 0,\ j = 1, \ldots, \ell.$$

We say that a set $S \subseteq \mathbb{C}$ is *closed with respect to $\mathcal{C}^+\mathcal{U} \subseteq \mathbb{C}$* if for any complex number $z \in S$ and any $v \in \mathcal{C}^+\mathcal{U}$, $z + v$ belongs to S.

Obviously, 0 is the interior point of the convex hull of $\mathcal{U} = \{u_1, \dots, u_\ell\}$ if and only if $\mathcal{C}^+\mathcal{U} = \mathbb{C}$.
Given a degenerate operator T with non-defining polygon N_T, denote by $\Upsilon_T := \{\epsilon_1, \dots, \epsilon_m\}$ the collection of all its leading constants and set $\mathcal{C}_T^+ := \mathcal{C}^+(\Upsilon_T)$. As we mentioned above, if $\mathcal{C}_T^+ = \mathbb{C}$, then the conclusion of Theorem 32 holds. Let us assume now that $\mathcal{C}_T^+$ is a closed sector in the plane with positive angle $\leq \pi$. (We are then missing two remaining cases: $\mathcal{C}_T^+$ being a line through the origin and $\mathcal{C}_T^+$ being a half-line through the origin.)

Lemma

In the above notation, any T-invariant set S is closed with respect to $\mathcal{C}_T^+$.

Corollary

In the above notation, if the product of the leading coefficient $Q_k(x)$ and the constant term $Q_0(x)$ of the operator T is not a constant, then any T-invariant set S contains the the Minkowski sum $Conv(Q_k Q_0) \oplus \mathcal{C}_T^+ \subset \mathbb{C}$ of $\mathcal{C}_T^+$ and $Conv(Q_k Q_0)$; the latter set being the convex hull of the union of all roots of $Q_k(x)$ and $Q_0(x)$.

Degenerate operators with non-defining Newton polygon and constant leading term

The remaining case of a constant leading term is discussed below. One can easily check that the class of degenerate operators

$$T = \frac{d^k}{dx^k} + Q_{k-1}(x)\frac{d^{k-1}}{dx^{k-1}} + \cdots + Q_k(x)$$

with non-defining N_T splits into two subclasses:

- operators with constant coefficients;
- operators satisfying the following three conditions:
  - $\deg Q_{k-1} = 1$;
  - $\deg Q_j \leq 1$ for $j = 0, \ldots, k-2$;
  - if j_{min} is the smallest value of j for which $\deg Q_j = 1$, then Q_ℓ must vanish for all $\ell \leq j_{min} - 2$.

For the more interesting subclass (B) the northeastern border of such operator T can consist of 1, 2 or three edges, see Figure 5 below. If it consists of 1 edge then after an affine change of x we can reduce such an operator to

$$T = \frac{d^k}{dx^k} - x\frac{d^{k-1}}{dx^{k-1}} + \alpha\frac{d^{k-2}}{dx^{k-2}},\ \alpha \in \mathbb{C}.$$

If it consists of 2 edges then after an affine change of x we can reduce such an operator to

$$T = \frac{d^k}{dx^k} - x\left(\frac{d^{k-1}}{dx^{k-1}} + \sum_{i=1}^{\ell}\alpha_i\frac{d^{k-1-i}}{dx^{k-1-i}}\right) + \sum_{i=1}^{\ell}\beta_i\frac{d^{k-2-i}}{dx^{k-2-i}}$$

where $\ell \leq k-1$ is a positive integer and all α_i and β_i are arbitrary complex numbers with the only restriction $\alpha_\ell \neq 0$.

Finally, if it consists of 3 edges then after an affine change of x we can reduce such an operator to

$$T = \frac{d^k}{dx^k} - x\left(\frac{d^{k-1}}{dx^{k-1}} + \sum_{i=1}^{\ell} \alpha_i \frac{d^{k-1-i}}{dx^{k-1-i}}\right) + \sum_{i=1}^{\ell} \beta_i \frac{d^{k-2-i}}{dx^{k-2-i}} + \beta_{\ell+1} \frac{d^{k-3-\ell}}{dx^{k-3-\ell}}$$

where $\ell \leq k-3$ is a positive integer, all α_i and β_i are arbitrary complex numbers with the restrictions $\alpha_\ell \neq 0$ and $\beta_{\ell+1} \neq 0$. We will discuss these subcases below.

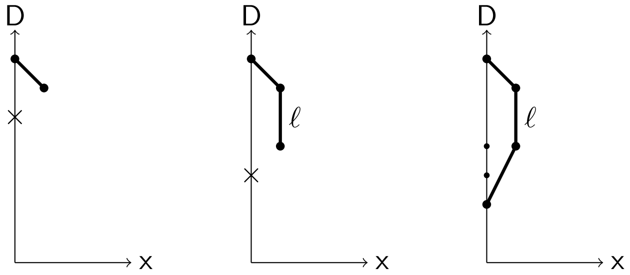

Figure: **NE** borders of the three sub-cases in subclass (B). Here, $\times$ denotes a monomial that might be present, but all monomials below $\times$ must be absent, i.e. have vanishing coefficients.

Linear differential operators with constant coefficients

Observe that in the case case of constant coefficients, if S is a T-invariant set, then for any $a \in \mathbb{C}$, $S_a := S + a$ is a T-invariant set as well. (Similarly for $T_{\geq n}$-invariant sets).

Proposition

Let

$$T = a_k \frac{d^k}{dx^k} + a_{k-1} \frac{d^{k-1}}{dx^{k-1}} + \cdots + a_0, \;\; a_k \neq 0 \qquad (0.15)$$

be a linear differential operator with constant coefficients. Let $\Lambda_T^{-1} = \{\lambda_1^{-1}, \ldots, \lambda_k^{-1}\}$ be the set of the inverses of characteristic exponents (not necessarily distinct), where

$$a_k t^k + a_{k-1} t^{k-1} + \cdots + a_0 = a_k(t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_k).$$

Then a convex set $S \subseteq \mathbb{C}$ is T-invariant if and only if S is closed with respect to $\mathcal{C}\Lambda_T^{-1}$.

Remark

We use the convention that if $\lambda_j = 0$ then its inverse disappears from the list Λ_T^{-1}. Further notice that if $\mathcal{C}\Lambda_T^{-1} = \mathbb{C}$ which happens in the open (in the usual topology) subset of linear differential operators of the form (0.15) of any given order $k \geq 3$, the only T-invariant $S \subseteq \mathbb{C}$ is the whole $\mathbb{C}$.

Lemma

For an operator $T = \frac{d}{dx} - \lambda$, a convex set $S \subseteq \mathbb{C}$ is T-invariant if and only if for any $x \in S$ and $\tau > 0$, the number $x - \tau\lambda$ belongs to S which is equivalent to S being closed with respect to $\mathcal{C}\Lambda_T^{-1}$.

Linear operators with constant leading term and $\deg Q_{k-1} = 1$

In this case we currently have only a number of sporadic results.

Let us start with operators of order 1. After an affine change of x, we only need to consider one single operator $T = \frac{d}{dx} - x$. The following statement holds.

Lemma

For $T = \frac{d}{dx} - x$, its minimal T-invariant set $\mathrm{M}^{T}_{\geq 0}$ is the real axis.

The next results describe which operators T in this class preserve a given half-plane in $\mathbb{C}$. For any such operator T, its symbol $F_T(x,y)$ is of the form $U(y) - xV(y)$ where $U(y) = y^k + \cdots$ and $V(y) = y^{k-1} + \cdots$.

Lemma

Let $H \subset \mathbb{C}$ be an open half-plane represented as

$$H = \{az + b : \mathrm{Im} z \leq 0\},$$

where $a, b \in \mathbb{C}$ and $a \neq 0$, and let

$$T = U\left(\frac{d}{dx}\right) + xV\left(\frac{d}{dx}\right),$$

where U and V are polynomials. Then the following are equivalent

1. *H is T_n-invariant for all n,*
2. *The bivariate polynomial $U(-y/a) + bV(-y/a) + aV(-y/a)x$ is stable in (x, y),*
3. *Either $V = 0$ and $U(-y/a)$ is stable, or the rational map*

Lemma

Let $f, g \in \mathbb{R}[x]$. The following are equivalent

- *the univariate polynomial $f(x) + ig(x)$ is stable,*
- *the bivariate polynomial $f(x) + yg(x)$ is stable,*
- *f and g are real-rooted, their zeros interlace, and*

$$W(f, g) = f'(x)g(x) - f(x)g'(x) \geq 0, \qquad \text{for all } x \in \mathbb{R}.$$

Also if the zeros of f and g interlace, then either $W(f, g) \geq 0$ for all x or $W(g, f) \geq 0$ for all x.

Corollary

Let

$$T = U\left(\frac{d}{dx}\right) + xV\left(\frac{d}{dx}\right),$$

where $U(y), V(y) \in \mathbb{R}[y]$*. Then* $\mathbb{R}$ *is* T_n*-invariant for all* n *if and only if*

- *there is a nonzero constant* $\xi \in \mathbb{C}$ *such that* $\xi U(y), \xi V(y) \in \mathbb{R}[y]$*, and*
- *the zeros of* $\xi U(y)$ *and* $\xi V(y)$ *are real and interlacing, and*

$$W(\xi V(y), \xi U(y)) \geq 0, \qquad \text{for all } x \in \mathbb{R}.$$

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