New results in Heine-Stieltjes theory

Boris Shapiro and Milos Tater

Dedicated to Heinrich Eduard Heine and his 140 years old riddle

Prague, March 17, 2011

Introduction

The algebraic form of the classical **Lamé equation** is:

$$Q(z)\frac{d^2S}{dz^2} + \frac{1}{2}Q'(z)\frac{dS}{dz} + V(z)S = 0,$$

where $Q_l(z)$ is a real polynomial of degree l with all real and distinct roots, and $V(z)$ is a polynomial of degree at most $l-2$ whose choice depends on what we are looking for. It was introduced by Gabriel Lamé in 1830's in connection with the separation of variables in the Laplace equation in $\mathbb{R}^l$. It was studied in the second half of the 19-th century by several celebrated mathematicians including M. Bôcher, E. Heine, F. Klein, T. Stieltjes.

Generalized Lame

A **generalized Lamé equation** is the second order differential equation given by

$$Q(z)\frac{d^2S}{dz^2} + P(z)\frac{dS}{dz} + V(z)S = 0, \qquad (1)$$

where $\deg Q(z) = l$ and $P(z) \leq l - 1$. The special case $l = 3$ is widely known as the **Heun equation**.

In what follows we will concentrate on the usual polynomial solutions of (1) and its special cases.

Heine's result

The next fundamental proposition announced by Heine with not a quite satisfactory proof is our starting point.

Theorem (Heine)

If the coefficients of $Q(z)$ and $P(z)$ are algebraically independent then for any integer $n > 0$ there exists exactly $\binom{n+l-2}{n}$ polynomials $V(z)$ of degree exactly $(l-2)$ such that the equation (1) has and unique (up to a constant factor) polynomial solution S of degree exactly n.

Stieltjes-Van Vleck

A special case of (1) with $\alpha_1 < \alpha_2 < \ldots < \alpha_l$ real and $\beta_1, \ldots, \beta_l$ positive was considered separately by T. Stieltjes.

$$\prod_{i=1}^{l}(z-\alpha_i)\frac{d^2S}{dz^2} + \sum_{j=1}^{l}\beta_j \prod_{j\neq i}(z-\alpha_i)\frac{dS}{dz} + V(z)S = 0, \qquad (2)$$

Theorem (Stieltjes-Van Vleck-Bôcher)

Under the assumptions of (2) and for any integer $n > 0$

1. *there exist exactly $\binom{n+l-2}{n}$ distinct polynomials V of degree $(l-2)$ such that the equation (2) has a polynomial solution S of degree exactly n.*
2. *each root of each V and S is real and simple, and belongs to the interval (α_1, α_l).*
3. *The $\binom{n+l-2}{n}$ polynomials S are in 1-1-correspondence with $\binom{n+l-2}{n}$ possible ways to distribute n points into the $(l-1)$ open intervals (α_1, α_2), (α_2, α_3), $\ldots$, (α_{l-1}, α_l).*

Pólya

The polynomials V and the corresponding polynomial solutions S of the equation (1) are called *Van Vleck* and *Stieltjes* polynomials resp.

The case when α_i's and/or β_j's are complex is substantially less studied. One nice result in this set-up is due to G. Pólya.

Theorem (Pólya)

If in the notation of (2) all α_i's are complex and all β_j's are positive that all the roots of each V and S belong to the convex hull $Conv_Q$ of the set of roots $(\alpha_1, \dots, \alpha_l)$ of $Q(z)$.

Illustration

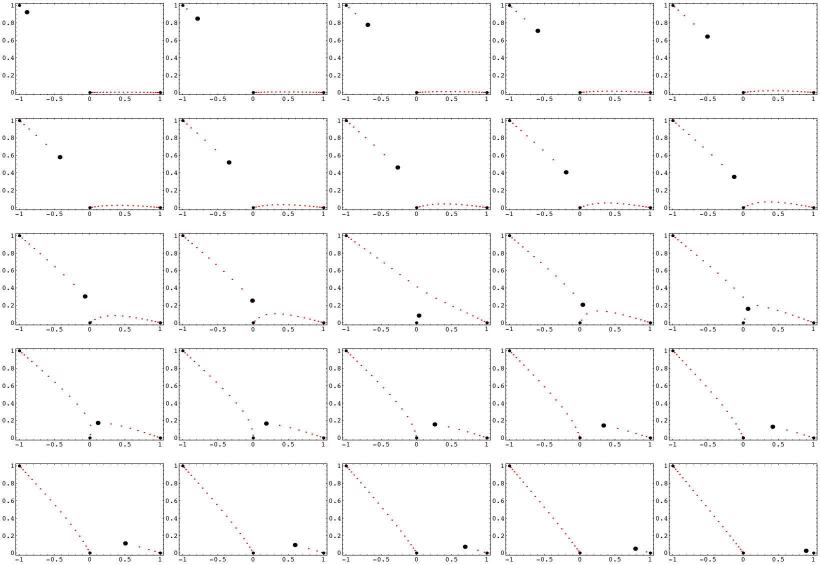


Figure: 25 Stieltjes polynomials of degree 24 for $Q(z)S''(z) + V(z)S(z) = 0$ with $Q(z) = z(z-1)(z+1-I)$.

root localization

Theorem

For any generalized Lamé operator (1) and any $\epsilon > 0$ there exists a positive integer N_ϵ such that the zeros of all its Van Vleck polynomials $V(z)$ possessing a Heine-Stieltjes polynomial $S(z)$ of degree $n \geq N_\epsilon$ as well as all zeros of these Heine-Stieltjes polynomials belong to $Conv_Q^\epsilon$. Here $Conv_Q$ is the convex hull of all zeros of the leading coefficient $Q(z)$ and $Conv_Q^\epsilon$ is its ϵ-neighborhood in the usual Euclidean distance on $\mathbb{C}$.

Main Problems

The above Theorem shows that the zeros of all Van Vleck polynomials having Stieltjes polynomials of degree greater or equal to a sufficiently large positive integer n are confined in some fixed ϵ-neighborhood $Conv_Q^\epsilon$ of $Conv_Q$. Therefore, there exist plenty of converging subsequences $\{\widetilde{V}_{n,i_n}\}$ of normalized Van Vleck polynomials. Here V_{n,i_n} is some Van Vleck polynomial having a Stieltjes polynomial of degree n and $\widetilde{V}_{n,i_n}$ is a monic polynomial proportional to V_{n,i_n}. These limits will be called *asymptotic Van Vleck polynomials*!

It seems natural to pose the following two questions.

Problem 1. What happens with the set $\{\widetilde{V}_{n,i}\}$ of normalized Van Vleck polynomials having a Stieltjes polynomial of degree exactly n when $n \to \infty$?

Problem 2. What happens with the sequence $\{S_{n,i_n}\}$ of Stieltjes polynomials whose corresponding sequence $\{\widetilde{V}_{n,i_n}\}$ of normalized Van Vleck polynomials has a limit?

Standard notions

Given a probability measure μ supported on some subset of $\mathbb{C}$ we define its Cauchy transform $\mathcal{C}_\mu$ as

$$\mathcal{C}_\mu(z) = \int_{\mathbb{C}} \frac{d\mu(\zeta)}{z - \zeta}.$$

Obviously, $\mathcal{C}_\mu$ is analytic in the complement to the support of μ.

Given a polynomial $W(z)$ of degree m we define its root-counting measure μ_W as the finite probability measure given by

$$\mu_W(z) = \sum_j \frac{k_j \delta(z - z_j)}{m}$$

where j runs over the index set of the set of all distinct zeros $\{z_j\}$ of $W(z)$, $\delta(z - z_j)$ is the usual Dirac delta function concentrated at z_j and k_j is the multiplicity of the zero z_j of $W(z)$.

Asymptotics of Stieltjes polynomials

Theorem

For any generalized Lamé operator (1) take any subsequence $\{\widetilde{V}_{n,i_n}(z)\}$ of its normalized Van Vleck polynomials converging to some monic polynomial $\widetilde{V}(z)$. Then the sequence $\{\mu_{n,i_n}\}$ of the root-counting measures of the corresponding Stieltjes polynomials $\{S_{n,j_n}(z)\}$ weakly converges to a probability measure $\mu_{\widetilde{V}}$ whose Cauchy transform $\mathcal{C}_{\widetilde{V}}(z)$ satisfies almost everywhere in $\mathbb{C}$ the equation

$$\mathcal{C}_{\widetilde{V}}^2(z) = \frac{\widetilde{V}(z)}{Q(z)}. \tag{3}$$

Remark. From a number of results in orthogonal polynomials one knows that asymptotic root distributions are often related to quadratic differentials.

quadratic differentials

A (meromorphic) quadratic differential Ψ on a compact orientable Riemann surface Y without boundary is a (meromorphic) section of the tensor square $(T^*_{\mathbb{C}}Y)^{\otimes 2}$ of the holomorphic cotangent bundle $T^*_{\mathbb{C}}Y$. The zeros and the poles of Ψ constitute the set of *singular points* of Ψ denoted by $Sing_\Psi$. (Non-singular points of Ψ are usually called *regular*.)

If Ψ is locally represented in two intersecting charts by $h(z)dz^2$ and by $\tilde{h}(\tilde{z})d\tilde{z}^2$ resp. with a transition function $\tilde{z}(z)$, then $h(z) = \tilde{h}(\tilde{z})\,(d\tilde{z}/dz)^2$. Any quadratic differential induces a canonical metric on its Riemann surface, whose length element in local coordinates is given by

$$|dw| = |h(z)|^{\frac{1}{2}}|dz|.$$

quad. diff. 2

The above canonical metric $|dw| = |h(z)|^{\frac{1}{2}}|dz|$ on Y is closely related to two distinguished line fields given by the condition that $h(z)dz^2$ is either positive or negative. The first field is given by $h(z)dz^2 > 0$ and its integral curves are called *horizontal trajectories* of Ψ, while the second field is given by $h(z)dz^2 < 0$ and its integral curves are called *vertical trajectories* of Ψ.

Since we only consider rational quadratic differentials then any such quadratic differential Ψ will be globally given in $\mathbb{C}$ by $R(z)dz^2$, where $R(z)$ is a complex-valued rational function.

quad. diff. 3

A trajectory of a meromorphic quadratic differential Ψ given on a compact Y without boundary is called *singular* if there exists a singular point of Ψ belonging to its closure. The closure of the set of singular trajectories of Ψ is denoted by K_Ψ.

A non-singular trajectory $\gamma_{z_0}(t)$ of a meromorphic Ψ is called *closed* if $\exists\ T > 0$ such that $\gamma_{z_0}(t+T) = \gamma_{z_0}(t)$ for all $t \in \mathbb{R}$. The least such T is called the *period* of γ_{z_0}.

A quadratic differential Ψ on a compact Riemann surface Y without boundary is called *Strebel* if the set of its closed trajectories covers Y up to a set of Lebesgue measure zero.

Lemma

If a meromorphic quadratic differential Ψ is Strebel, then it has no poles of order greater than 2. If it has a pole of order 2, then the coefficient at leading term of Ψ at this pole is negative.

example of the set of singular trajectories

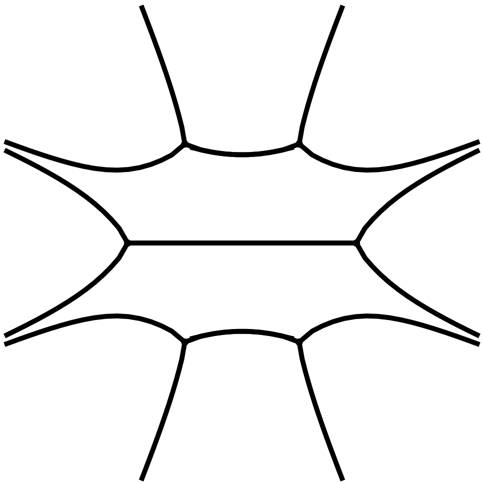


Figure: Singular trajectories for $\Psi = (1 - z^6)dz^2$.

main result

Theorem

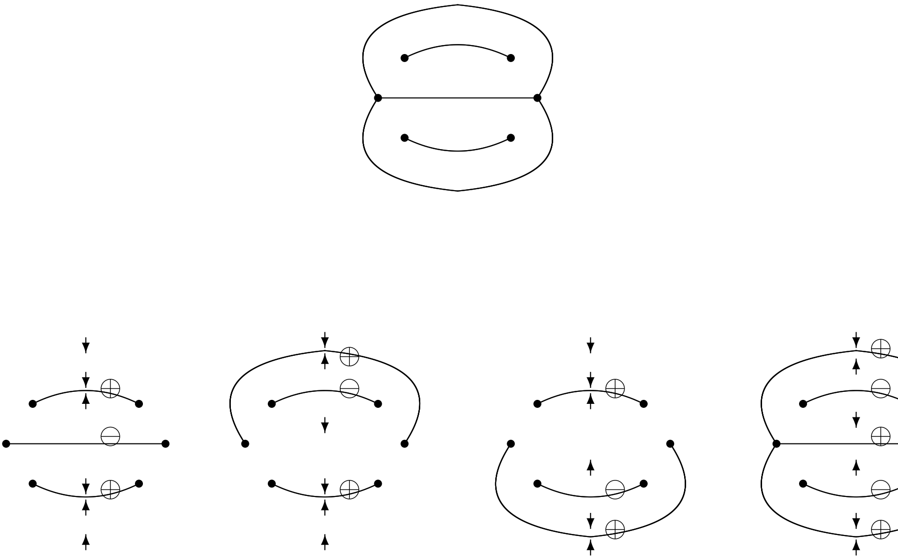
Let $U_1(z)$ and $U_2(z)$ be arbitrary monic complex polynomials with $\deg U_2 - \deg U_1 = 2$. *Then*

1. *the rational quadratic differential $\Psi = -U_1(z)dz^2/U_2(z)$ on $\mathbb{CP}^1$ is Strebel if and only if there exists a real and compactly supported in $\mathbb{C}$ measure μ of total mass 1 (i.e. $\int_{\mathbb{C}} d\mu = 1$) whose Cauchy transform C_μ satisfies a.e. in $\mathbb{C}$ the equation:*

$$C_\mu^2(z) = U_1(z)/U_2(z). \tag{4}$$

2. *for any Ψ as in (1) there exists exactly 2^{d-1} real measures whose Cauchy transforms satisfy (4) a.e. and whose support is contained in K_ψ where d is the total number of connected components in $\mathbb{CP}^1 \setminus K_\Psi$. These measures are in $1-1$-correspondence with 2^{d-1} possible choices of the branches of $\sqrt{U_1(z)/U_2(z)}$ in the union of $(d-1)$ bounded components of $\mathbb{CP}^1 \setminus K_\Psi$.*

example



positive Strebel differentials

We say that a rational Strebel differential with a double pole at ∞ is *positive* if one of its 2^{d-1} measures is positive in $\mathbb{C}$.

Theorem

For any Strebel differential $\Psi = -U_1(z)dz^2/U_2(z)$ *on* $\mathbb{CP}^1$ *there exists at most one positive measure satisfying* (4) *a.e. in* $\mathbb{C}$*. Its support necessarily belongs to* K_Ψ*, and, therefore, among* 2^{d-1} *real measures described in the previous Theorem at most one is positive.*

Moreover, we can formulate an exact criterion of the existence of a positive measure in terms of rather simple topological properties of K_Ψ. To do this we need a few definitions.

Basic properties of K_Ψ

K_Ψ is a planar multigraph with the following properties.

a) The vertices of K_Ψ are the finite singular points of Ψ (i.e. excluding ∞) and its edges are singular trajectories connecting these finite singular points.

b) Each (open) connected component of $\mathbb{C} \setminus K_\Psi$ is homeomorphic to an (open) annulus. K_Ψ might have isolated vertices which are the finite double poles of Ψ. Vertices of K_Ψ having valency 1 (i.e. hanging vertices) are exactly the simple poles of Ψ.

c) Vertices different from the isolated and hanging vertices are the zeros of Ψ. The number of edges adjacent to a given vertex minus 2 equals the order of the zero of Ψ at this point.

d) Finally, the sum of the multiplicities of all poles (including the one at ∞) minus the sum of the multiplicities of all zeros equals 4.

positivity criterion

By a *simple cycle* in a planar multigraph K_Ψ we mean any closed non-selfintersecting curve formed by the edges of K_Ψ. (Obviously, any simple cycle bounds an open domain homeomorphic to a disk which we call the *interior of the cycle*.)

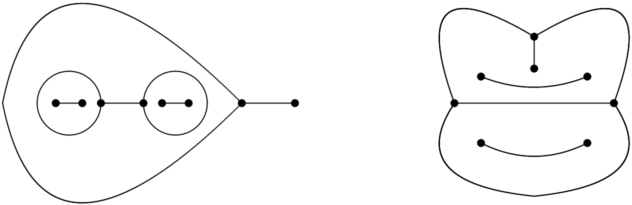


Figure 4. Examples of K_Ψ admitting/non-admitting a positive measure

crit. continued

Theorem

A Strebel differential $\Psi = -U_1(z)dz^2/U_2(z)$ *admits a positive measure satisfying* (4) *if and only if no edge of* K_Ψ *is attached to a simple cycle from inside. In other words, for any simple cycle in* K_Ψ *and any edge not in the cycle but adjacent to some vertex in the cycle this edge does not belong to its interior. The support of the positive measure coincides with the forest obtained from* K_Ψ *after the removal of all its simple cycles.*

Notice that under the above assumptions all simple cycles of K_Ψ are pairwise non-intersecting and, therefore, their removal is well-defined in an unambiguous way.

In particular, the compacts K_Ψ shown on top of Fig. 3 and on the right part of Fig. 4 admit no positive measure since both contain an edge cutting a simple cycle (the outer boundary) in two smaller cycles. The left picture on Fig. 4 has no such edges and, therefore, admits a positive measure whose support consists of the four horizontal edges of K_Ψ.

cont. 2

Now returning back to consideration of the asymptotics of Stieltjes polynomials.

Corollary

If the linear monic polynomial $\widetilde{V}(z)$ is the limit of some sequence of normalized Van Vleck polynomials for the Heun equation then the quadratic differential $\Phi = -\widetilde{V}(z)dz^2/Q(z)$ is a positive Strebel differential.

Heun's case

For $\Psi = -(z-\alpha)dz^2/(z-a_1)(z-a_2)(z-a_3)$ one generically has the following two situations.

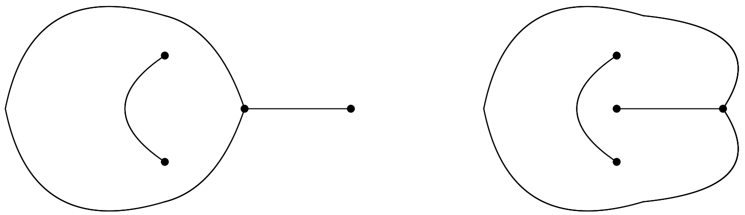


Figure 5. K_Ψ admitting/non-admitting positive measures for Ψ

What about the asymptotic

Van Vleck polynomials?

Heun case

In the case of the Heun equation we can introduce the set $\mathcal{V}_n$ consisting of polynomials $V(z)$ giving a polynomial solution $S(z)$ of the Lamé equation of degree n; each such $V(z)$ appearing the number of times equal to its multiplicity. The set $\mathcal{V}_n$ will contain exactly $n+1$ linear polynomials for all sufficiently large n.

We introduce a sequence $\{Sp_n(\lambda)\}$ of *spectral polynomials* where the n-th spectral polynomial is defined by

$$Sp_n(\lambda) = \prod_{j=1}^{n+1} (\lambda - t_{n,j}),$$

where $t_{n,j}$ is the unique root of the j-th polynomial in $\mathcal{V}_n$ in any fixed ordering. ($Sp_n(\lambda)$ will be well-defined for all sufficiently large n.)

Associate to $Sp_n(\lambda)$ the finite measure

$$\mu_n = \frac{1}{n+1} \sum_{j=1}^{n+1} \delta(z - t_{n,j}),$$

where $\delta(z - a)$ is the Dirac measure supported at a. The measure μ_n obtained in this way is clearly a real probability measure which one usually refers to as the *root-counting measure* of the polynomial $Sp_n(\lambda)$.

Numerics and conjecture

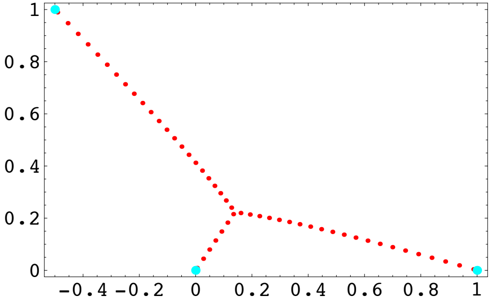


Figure: The roots of the spectral polynomial $Sp_{50}(\lambda)$ for the classical Lamé equation $\left\{Q(z)\frac{d^2}{dz^2} + \frac{1}{2}Q'(z)\frac{d}{dz} + V(z)\right\}S(z) = 0$, with $Q(z) = z(z-1)\left(z + \frac{1}{2} - i\right)$.

These numerical experiments strongly suggested that the following statement holds.

Theorem

For any Heun equation the sequence $\{\mu_n\}$ *of the root-counting measures of its spectral polynomials converges to a probability measure* μ *supported on the union of three curved segments located inside* $Conv_Q$ *and connecting the three roots of* $Q(z)$ *with a certain interior point. Moreover, the limiting measure* μ *depends only on* $Q(z)$*, i.e. is* independent *of* $P(z)$*.*

Description of support

An elegant explicit description of the support of μ was suggested by Kouchi Takemura,

Denote the three roots of $Q(z)$ by a_1, a_2, a_3. For $i \in \{1,2,3\}$ consider the curve γ_i given as the set of all b satisfying the relation:

$$\int_{a_j}^{a_k} \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}}\,dt \in \mathbf{R}, \qquad (5)$$

here j and k are the remaining two indices in $\{1,2,3\}$ in any order and the integration is taken over the straight interval connecting a_j and a_k. One can see that a_i belong to γ_i and that these three curves connect the corresponding a_i with a common point within $Conv_Q$. Take a segment of γ_i connecting a_i with the common intersection point of all γ's. Let us denote the union of these three segments by Γ_Q.

The curve Γ_Q

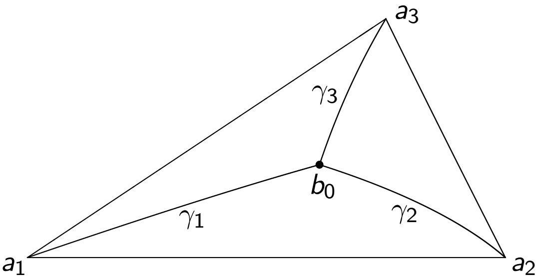


Figure. An example of Γ_Q

Final observation

Theorem

The support of the limiting root-counting measure μ coincides with the above Γ_Q.

Important observation. Detailed consideration show that the set Γ_Q coincides with the whole set of positive Strebel differentials for the above form.

General conjecture. For an arbitrary generalized Lamé equation there is a $1-1$-correspondence between asymptotic Van Vleck polynomials and positive Strebel differentials of the form $-V(z)dz^2/Q(z)$ with $\deg V(z) = l-2$.

Thanks a lot for your patience!