Lecture

μ is compactly supported measure in $\mathbb{R}^2 = \mathbb{C}$

$$m_j(\mu) := \int_{\mathbb{C}} z^j \, d\mu(z)$$

$$\overline{m}_j(\mu) := \int_{\mathbb{C}} \bar{z}^j \, d\mu(z)$$

Cauchy transform

$$C_\mu(z) := \int_{\mathbb{C}} \frac{d\mu(\zeta)}{z-\zeta} = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \dots$$

Inverse problem in logarithmic potential theory:
Given $m_0, m_1, \dots m_n, \dots$, find μ
(Usually some special type of μ)
~~Funda~~ Typical: μ is the restriction of Lebesgue measure to a domain, over Lebesgue measure $\times$ polynomial density etc.

P. S. Novikov (1938). Two different star-shaped domains can not have the same Cauchy transform ~~near~~ ∞. $\Leftrightarrow$
Sequences of harmonic moments of Lebesgue measures of 2 star-shaped domain can not coincide.

V. N. Strakhov & M. A. Brodsky (1986) Considered a similar problem for polygons (not necessarily simply connected) and domains bounded by lemniscates.
We are interested in polygons

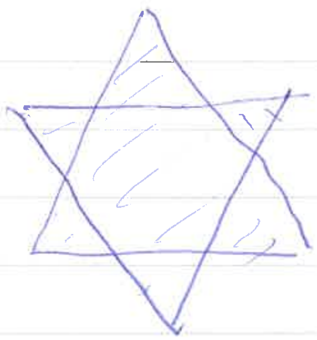
Th4 If for polygons P_1 and P_2 (with constant but might be different densities) $P_1 \cap P_2$ and $\mathbb{C} \setminus (P_1 \cup P_2)$ are connected and their Cauchy transforms near ∞ coincide, then $P_1 = P_2$ (and densities are equal)

They also shown that for polygons with coinciding Cauchy transform near ∞

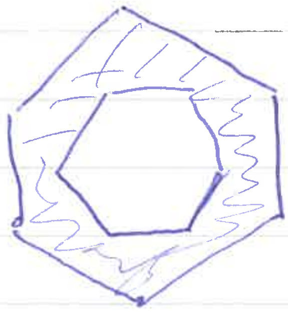
a) vertices must be equal

b) bisectors of the internal angles at each vertex must coincide

Examples of nonuniqueness



$\simeq$



$\leftarrow$ non simply connected

Our project

Describe relations between harmonic and anti-harmonic moments of simply connected polygons in $\mathbb{C}$.
Why anti-harmonic? Because the real part and imaginary part of harmonic moments are essential! But to describe them one can equivalently describe harmonic & anti-harmonic moments.

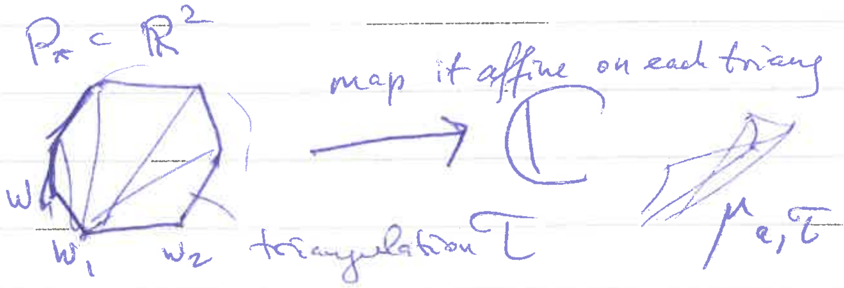
A simply-connected plane polygon is a sequence cyclicly ordered point in $\mathbb{C} \simeq \mathbb{R}^2 \subset \mathbb{C}^2$. Since we will have to complexify the real and imaginary part of coordinates we consider sequences of points in $\mathbb{C}^2$.

$$\mathbf{a} = (a_1 \dots a_n), \quad a_j \in \mathbb{C}^2 \Rightarrow a_j = (x_j, y_j)$$

We will use $z_j = x_j + i y_j$, $\bar{z}_j = x_j - i y_j$

$$\mathbb{Z} = (z_1 \dots z_n), \quad \bar{\mathbb{Z}} = (\bar{z}_1, \dots, \bar{z}_n)$$

A polygonal curve $\Gamma_{\mathbf{a}} := [\overrightarrow{a_1, a_2}] \cup [\overrightarrow{a_2, a_3}] \dots \cup [a_n, a_1]$



th (i) If $h : \mathbb{C}^2 \to \mathbb{C}$ is a holomorphic function of 2 variables then $\int_{\Delta_{a,\tau}} h \, d\mu_{a,\tau}$ is independent of τ and equals $\int_{\Gamma_a} \omega$

ω is any 1-form with $d\omega = -\frac{i}{2} h \, dz \wedge d\bar{z}$

(ii) If the vertices are real, $\mu_{a\tau}$ is independent of τ. Its value at $q \in \mathbb{R}^2 \setminus \Gamma_a$ is the linking number of Γ_a with q.

$$V_k(z,\bar z) = \binom{k}{2} m_{k-2}(\mu_{a,\tau}) \text{ and } \bar V_k(z,\bar z) = \binom{k}{2} \overline{m_{k-2}(\mu_{a,\tau})}$$

RHS is independent of τ.

Important theorem

$$V_k(\mathbb{Z},\bar{\mathbb{Z}}) = \frac{i}{4}\sum_{j=1}^{n}(\bar z_j - \bar z_{j+1})\left(z_j^{k-1} + z_j^{k-2} z_{j+1} + \dots + z_{j+1}^{k-1}\right) \quad (*)$$

$k=2,3,\dots$

and for $\bar V_k(\mathbb{Z},\bar{\mathbb{Z}})$ interchange $\mathbb{Z}$ and $\bar{\mathbb{Z}}$

Main problem Describe the algebraic relations between sequences of polynomials of $2n$ complex variables $\mathbb{Z}$ and $\bar{\mathbb{Z}}$ given by $(*)$.

Algebraic approach: Only $V_k(\mathbb{Z},\bar{\mathbb{Z}})$, $k=2,\dots$

Geometric approach: Both $V_k(\mathbb{Z},\bar{\mathbb{Z}})$ and $\bar V_k(\mathbb{Z},\bar{\mathbb{Z}})$

Theorem A (i) The field $\mathcal{F}_n$ generated by $V_k(\mathbb{Z},\bar{\mathbb{Z}})_{k=2}^{\infty}$ is ~~isomorphic~~ generated by $V_2,\dots,V_{2n-1}$ and is isomorphic to the field of rational functions in $2n-2$ independent variables.

(ii) $\mathcal{F}_n \supset \mathbb{C}(\mathbb{Z})^{S_n}$

Theorem A' The ring $R_n = \mathbb{C}[V_2, V_3, \dots]$ is infinitely generated but its localization $R_n|_{D_n} \simeq \mathbb{C}[V_2,\dots,V_{2n-1}]\left[\frac{1}{D_n}\right]$

Here D_n is the determinant of the ~~Hankel~~ Toeplitz matrix

$$U = \begin{pmatrix} V_{n-1} & V_{n-2} & \cdots & V_1 & V_0 \\ V_{n} & V_{n-1} & V_{n-2} & \cdots & V_1 \\ & & \ddots & \ddots & \\ V_{2n-2} & V_{2n-3} & \cdots & & V_{n-1} \end{pmatrix} \Big\} n$$

$v_0 = 0$, $v_1 = 0$

Theorem B (i) the field $\widetilde{\mathcal{F}}_n$ is generated by the first $2n-2$ harmonic and $2n-2$ antiharmonic (totally $4n-5$ since $V_2 = -\overline{V}_2$) — generated by all harmonic & antiharmonic

(ii) $\widetilde{\mathcal{F}}_n$ contains the subfield $H = \mathbb{C}(Z, \overline{Z})^{S_n \times S_n}$ of rational functions symmetric with respect to Z and $\overline{Z}$ separately.

(iii) $\widetilde{\mathcal{F}}_n$ is the algebraic extension of H generated by V_2

The degree of this extension equals $n!(n-1)!$ for n odd ≥ 3 and $2((n-1)!)^2$ for n even ≥ 4.

We have a somewhat explicit description of the Galois group of this extension.

Idea of the proofs

Important theorem (Enough for a triangle)

For V_k we need to integrate $h(z, \bar{z}) = \frac{k(k-1)}{2} z^{k-2}$ over a triangle

take 1 form $\frac{ik}{4} z^{k-1} d\bar{z}$ and integrate over the boundary (but inner sides will later cancel out)

If we parameterize a segment $[p, q]$ as $z = p(1-t) + tq$, $t \in [0, 1]$

then

$$\frac{ik}{4}\int_{[p,q]} z^{k-1}\,d\bar z = \frac{ik}{4}\int_0^1 \left(p(1-t)+tq\right)^{k-1}(\bar q-\bar p)\,dt =$$

$$= \frac{i}{4}\int_0^1 \frac{\bar q-\bar p}{q-p}\, d\left(\left(p(1-t)+tq\right)^k\right) = \frac{i}{4}(\bar q-\bar p)\,\frac{q^k-p^k}{q-p}$$ ⟸ what we need

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Generating function and proof of Th A.

$$\Psi_\mu(w) := \sum_{j=2}^{\infty} V_j(\mu)\, w^{j-2} = \sum_{j=0}^{\infty}\binom{j+2}{2} m_j(\mu)\, w^j$$

Lemma For a triangle $\Delta \subset \mathbb{C}^2$ with the vertices $(z_1, z_2, z_3, \bar z_1, \bar z_2, \bar z_3)$

set $D_{123} := V_2(z_1, z_2, z_3, \bar z_1, \bar z_2, \bar z_3)$

Then $$\Psi_\Delta(w) = \frac{D_{123}}{(1-z_1w)(1-z_2w)(1-z_3w)}$$

Proof $$\Psi_\Delta(w) = \frac{i}{4}\,\frac{\bar z_1-\bar z_2}{z_1-z_2}\sum_{k=2}^{\infty}(z_1^k-z_2^k)\,w^{k-2} + \text{cyclic} =$$

$$= \frac{i}{4}\left(\frac{\bar z_1-\bar z_2}{z_1-z_2}\left(\frac{z_1^2}{1-z_1w}-\frac{z_2^2}{1-z_2w}\right)+\text{cyclic}\right)$$

$$= \frac{i}{2}\,\frac{z_1\bar z_2 - z_2\bar z_1 + z_3\bar z_2 - z_2\bar z_3 + z_3\bar z_1 - z_1\bar z_3}{(1-z_1w)(1-z_2w)(1-z_3w)} = \frac{D_{123}}{(1-z_1w)(1-z_2w)(1-z_3w)}$$

Corollary For any $Z = (z_1 \ldots z_n)$

$$\Psi_{\mu_Z}(w) = \sum_{j=2}^{n-1}\frac{D_{1,j,j+1}}{(1-z_1w)(1-z_jw)(1-z_{j+1}w)} =$$

$$= \frac{AD_Z(w)}{\prod_{j=1}^{n}(1-z_jw)}$$ ⟸ polynomial of degree at most $n-3$

Part of Proof of Th A | How to express $V_{2n}, V_{2n+1}\dots$ threw $V_2 \dots V_{2n-1}$.

We have that the generating function of V_j is rational!

General result: $\sum_{j=0}^{\infty} f(j)t^j = \frac{P(t)}{Q(t)}$ with $\deg P < \deg Q$

and $Q(t) = 1 + d_1 t + d_2 t^2 + \dots + d_d t^d$, $a_d \neq 0 \Rightarrow k \geq 0$

one has a recurrence relation

$$f(k+d) + d_1 f(k+d-1) + d_2 f(k+d-2) + \dots + d_d f(k) = 0$$

We have $AD_z(w) = \prod_{j=1}^{n}(1-z_j w) \sum_{j=0}^{\infty} V_{j+2}(z) w^j =$

$$= (1 - e_1(z)w + \dots + (-1)^n e_n(z) w^n) \cdot \sum_{j=0}^{\infty} V_{j+2}(z) w^j$$

Since $\deg AD_z \leq n-3$ we get

$$(**) \quad V_{k+2+n}(z) - e_1(z) V_{k+1+n}(z) + e_2(z) V_{k+n}(z) + (-1)^n e_n(z) V_{k+2}(z) = 0$$

We need to express $e_1(z) \dots e_n(z)$ threw initial moments v_j. If we look at n first equations of $(**)$ we get

$$U \cdot E = V \Rightarrow E = U^{-1} V$$

U = Toeplitz matrix, $E = \begin{pmatrix} e_1 \\ -e_2 \\ \vdots \\ (-1)^{n} e_n \end{pmatrix}$, $V = \begin{pmatrix} V_n \\ \vdots \\ V_{2n-1} \end{pmatrix}$

Substituting back we can express all V_j, $j > 2n-1$ threw $V_2 \dots V_{2n-1}$