Around generalized external zonotopal algebras of graphs

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Zonotopal algebras of undirected graphs have been introduced in [1] and [3] following the earlier studies [9, 6, 7]. According to the existing definitions they are subdivided into three classes: external, central and internal, see loc. cit. In what follows we concentrate only on external zonotopal algebras and their generalizations although the definitions and problems under consideration immediately extend to the other two classes.

Let G be a undirected graph without loops on the labelled vertex set $\{0, \ldots, n\}$. (Below we always assume that all graphs might have multiple edges, but no loops.) Throughout the whole paper, we fix a field $\mathbb{K}$ of zero characteristic and assume that all algebras contain 1.

We define the *external zonotopal algebra* $\mathcal{C}_G^{\mathcal{E}x}$ introduced in [6] and [9], see also [1] and [3]. Let Φ_G be the graded commutative algebra over $\mathbb{K}$ generated by the variables $\phi_e, e \in G$, with the defining relations:

$$(\phi_e)^2 = 0, \text{ for any edge } e \in G.$$

The *external zonotopal algebra* $\mathcal{C}_G^{\mathcal{E}x}$ *of* G is the subalgebra of Φ_G generated by the elements

$$X_i = \sum_{e \in G} c_{i,e} \phi_e, \quad i = 1, \ldots, n,$$

where

$$c_{i,e} = \begin{cases} 1 & \text{if } e = (i,j), i < j; \\ -1 & \text{if } e = (i,j), i > j; \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Remark

Observe that since $\sum_{i=0}^{n} X_i = 0$, we can define $\mathcal{C}_G^{\mathcal{E}x}$ as the subalgebra of Φ_G generated by all $X_0, X_1, \ldots, X_n$. Notice that any change of vertex labelling in a given undirected vertex labelled (multi)graph G will produce an external zonotopal algebra which is graded isomorphic to the above $\mathcal{C}_G^{\mathcal{E}x}$. Thus, up to a graded isomorphism, one can define and speak about an external zonotopal algebra of any undirected (multi)graph without loops.

Its Hilbert series and the set of defining relations have been calculated in [9, 7]. Namely, let $\mathcal{J}_G^{\mathcal{E}x}$ be the ideal in $\mathbb{K}[x_1, \dots, x_n]$ generated by the polynomials

$$p_I = \left(\sum_{i \in I} x_i\right)^{D_I+1}, \tag{1.2}$$

where I ranges over all nonempty subsets in $\{1, \dots, n\}$ and $D_I = \sum_{i \in I} d_I(i)$, where $d_I(i)$ is the total number of edges connecting a given vertex $i \in I$ with all vertices outside I. Thus, D_I is the total number of edges between I and the complementary set of vertices $\bar{I}$. Set $B_G^{\mathcal{E}x} := \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}_G^{\mathcal{E}x}$.

We can also define $B_G^{\mathcal{E}x}$ as the quotient algebra of $\mathbb{K}[x_0, \ldots, x_n]$ by the ideal generated by p_I, where I runs over all subsets of $\{x_0, x_1, \cdots, x_n\}$. This follows from the relation

$$p_I = \left(\sum_{i \in I} x_i\right)^{D_I+1} = \left(p_{\{0,1,\ldots,n\}} - \sum_{i \in \bar{I}} x_i\right)^{D_{\bar{I}}+1}.$$

To describe the Hilbert polynomial of $\mathcal{C}_G^{\mathcal{E}x}$, we need the classical notion of the Tutte polynomial of a graph G. Given a simple graph G, fix an arbitrary linear order of its edges. Now, given a spanning forest F in G (i.e., a subgraph without cycles which includes all vertices of G) and an edge $e \in G \setminus F$ in its complement, we say that e is *externally active* for F, if there exists a cycle C in G such that all edges in $C \setminus \{e\}$ belong to F and e is minimal in C with respect to the chosen linear order. The total number of external edges is called the *external activity* of F. The total number of forests with a given external activity is independent of this ordering. The Tutte polynomial $T_G(x, y)$ of a connected graph G is defined as

$$T_G(x, y) = \sum_{i,j} t_{i,j} x^i y^j, \qquad (1.3)$$

where t_{ij} denotes the number of spanning trees of internal activity i and external activity j.

Theorem (Theorems 3 and 4 of [7])

For any (multi)graph G without loops,
(i) the algebras $\mathcal{C}_G^{\mathcal{E}x}$ and $B_G^{\mathcal{E}x}$ are isomorphic;
(ii) the total dimension of these algebras (as vector spaces over $\mathbb{K}$) is equal to the number of all spanning subforests in G. The dimension of the k-th graded component of these algebras equals the number of subforests F in G with external activity $|G| - |F| - k$, where $|G|$ (resp. $|F|$) stands for the number of edges in G (resp. F). The latter statement is equivalent to the formula

$$HS_{\mathcal{C}_G^{\mathcal{E}x}}(t) = t^{n+1-m} T_G \left(1 + t, \frac{1}{t}\right), \tag{1.4}$$

where $n + 1$ is the number of vertices and m is the number of connected components of G.

Furthermore, in [5] G. Nenashev has shown that $\mathcal{C}_G^{\mathcal{E}x}$ contains all information about the graphical matroid of G and only it.

Proposition (Theorem 5 of [5])

Given two undirected (multi)graphs G_1 and G_2, algebras $\mathcal{C}_{G_1}^{\mathcal{E}x}$ and $\mathcal{C}_{G_2}^{\mathcal{E}x}$ are isomorphic if and only if the graphical matroids of G_1 and G_2 coincide. (The latter isomorphism can be thought of either as graded or as non-graded, the statement holds in both cases.)

The main object of our study is a certain family of filtered algebras which we call *generalized external zonotopal algebras* introduced in [NSh]. Let $f(u)$ be a univariate polynomial or a formal power series over $\mathbb{K}$.

Definition

A univariate polynomial/power series $f(u)$ is called **unipotent** if it has a non-vanishing linear term and **nillpotent**, otherwise.

Definition

Given f and G as above, we define the *generalized external zonotope algebra* $\mathcal{C}_G^{[f]} \subset \Phi_G$ *associated to* $f(u)$ as the subalgebra generated by

$$f(X_i) = f\left(\sum c_{i,e}\phi_e\right),\ i = 0, \ldots, n.$$

The algebra $\mathcal{C}_G^{[f]}$ is endowed with an increasing filtration where the j-th term of this filtration is generated by all monomials of degree at most j in the chosen generators $f(X_i)$. Notice that this filtration is independent of the constant term of $f(u)$ since for any g such that $f-g$ is constant, the filtered algebras $\mathcal{C}_G^{[f]}$ and $\mathcal{C}_G^{[g]}$ are the same. Because of that, from now on, we assume that $f(u)$ has no constant term. Let us present some basic properties of $\mathcal{C}_G^{[f]}$ proven in [NSh].

Proposition (Proposition 2 of [NSh])

Let $f(u)$ be any polynomial with a non-vanishing linear term. Then the graded algebra $\mathcal{C}_G^{\mathcal{E}x}$ and the filtered algebra $\mathcal{C}_G^{[f]}$ coincide as subalgebras of Φ_G. In other words, they coincide as linear subspaces of Φ_G closed under multiplication in Φ_G, but have different filtration/grading.

Theorem (Theorem 6 of [NSh])

Let $f(u)$ be any polynomial with non-vanishing linear and quadratic terms. Then given two simple graphs G_1 and G_2 without isolated vertices, $\mathcal{C}_{G_1}^{[f]}$ and $\mathcal{C}_{G_2}^{[f]}$ are isomorphic as filtered algebras if and only if the graphs G_1 and G_2 are isomorphic.

Given a graph G, consider the space $\mathcal{A}_G$ of univariate complex-valued polynomials without constant term, with coefficient 1 of the linear term, and of degree less than or equal to mv_G, where $mv_G := \max_{v \in V(G)} \mathrm{val}(v)$ is the maximal valency of vertices in G. (Here $\mathrm{val}(v)$ denotes the valency of the vertex v.) For a polynomial $f \in \mathcal{A}_G$, we will be interested in the Hilbert–Samuel sequence $HS_G^{[f]} = \{\dim_{\mathbb{K}} F_j\}_{j \geq 0}$ of the filtered algebra $\mathcal{C}_G^{[f]}$. This coincides with the Hilbert–Samuel function $k \mapsto \sum_{j=0}^{k} \mathrm{gr}_j(\mathcal{C}_G^{[f]})$ of the associated graded ring of the filtration. One can easily show that for any polynomial f of degree exceeding mv_G or for any power series f, the algebra $\mathcal{C}_G^{[f]}$ coincides with the algebra $\mathcal{C}_G^{[\tilde{f}]}$ where $\tilde{f}$ is the truncation of f at degree mv_G, see Lemma 3 below.

Proposition (Proposition 3 of [NSh] or Theorem 9)

There is a Hilbert–Samuel sequence $HS_G^{[f]}$ which holds for a generic polynomial f of degree at most mv_G. This generic Hilbert–Samuel sequence (denoted by HS_G below) is maximal in the lexicographic order among all $HS_G^{[f]}$, where f runs over the set of all formal power series with non-vanishing linear term, see below.

Here by generic polynomials of degree at most mv_G we mean polynomials belonging to some Zariski open subset in the linear space of all polynomials of degree at most mv_G.

Lemma

For any power series or polynomial f of degree exceeding mv_G, the algebra $\mathcal{C}_G^{[f]}$ coincides with the algebra $\mathcal{C}_G^{[\tilde{f}]}$ where $\tilde{f}$ is the truncation of f at degree mv_G.

Proof.

Let us show that in the above notation and for any vertex v_i of G, one has that $f(X_i) = \tilde{f}(X_i)$ as elements of $\Phi(G)$. Recall that $X_i = \sum_{e \in E_i} \pm\phi_e$, where E_i is the set of all edges adjacent to the vertex v_i. Then since $\phi_e^2 = 0$ for any edge e of G and all edges in the presentation of X_i are adjacent to v_i, one has that $X_i^j = 0$ for any $j > val(v_i)$. Thus for $f(u) = \sum_j a_j u^j$, in the expansion $f(X_i) = \sum_j a_j X^j$ all terms of degree j exceeding $val(v_i)$ vanish which implies the claim. □

Proposition

Given an artinian finite-dimensional algebra A over a field $\mathbb{K}$ with a generating set $x_1, \ldots, x_N$ and let $f \in \mathbb{K}[[u]]$ such that $f \in (u) \setminus (u^2)$. Then the following facts hold.

(1) Define the map $f : A \to A$ given by $a \mapsto f(a)$; this map is invertible.

(2) Define $y_1 = f(x_1), \ldots, y_N = f(x_N)$; then $y_1, \ldots, y_N$ is a generating set of A.

(3) $L(x_1, \ldots, x_N) = 0$ is a relation in A if and only if $L(f^{-1}(y_1), \ldots, f^{-1}(y_N)) = 0$ is a relation in A.

Corollary

Let G be a graph on the vertex set $\{0, 1, \ldots, n\}$ and let $f \in \mathbb{K}[[u]]$ such that $f \in (u) \setminus (u^2)$. Then the algebra $\mathcal{C}_G^{[f]}$ is given by the quotient

$$\mathcal{C}_G^{[f]} = \mathbb{K}[Y_0, \ldots, Y_n]/I_G^{[f]}, \tag{1.5}$$

where $Y_0 = f(X_0), \ldots, Y_n = f(X_n)$ and $I_G^{[f]}$ is the ideal generated by

$$Y_0^{d(0)+1}, \; Y_1^{d(1)+1}, \; \ldots, \; Y_n^{d(n)+1} \tag{1.6}$$

and

$$\left(\sum_{i \in I} f^{-1}(Y_i) \right)^{D_I + 1} \tag{1.7}$$

for all subsets $I \subseteq \{0, 1, \ldots, n\}$ of cardinality at least 2. Here $d(0) = val(0), \ldots, d(n) = val(n)$ are the valencies of the vertices.

Remark

Observe that the relation corresponding to $I = \{0, 1, \dots, n\}$ reads as

$$f^{-1}(Y_0) + f^{-1}(Y_1) + \cdots + f^{-1}(Y_n) = 0. \qquad (1.8)$$

Using (1.8) we can remove half of the generators of the ideal I_f of the form (1.7). Namely, for any subset $I \subset \{0, 1, \dots, n\}$ with $2 \leq |I| \leq n$, consider its complement $I^c = \{0, 1, \dots, n\} \setminus I$. If $|I| < |I^c|$, then keep the generator (1.7) corresponding to I and remove the one corresponding to I^c. If $|I^c| > |I|$, then keep the generator (1.7) corresponding to I^c and remove the one corresponding to I. Finally, if $|I| = |I^c|$ (which can happen only if $n + 1$ is even), then keep whichever of the two you want and remove the other. The generator (1.8) should be always kept in the ideal $I_G^{[f]}$.

Given a non-empty (multi-)graph G without loops, we earlier introduced the space

$$\mathcal{A}_G = \{f(u)\,|\,f(u) = u + \dots,\ \deg f \leq m v_G\}.$$

We will refer to $\mathcal{A}_G$ as *the space of parameters of generalized PS-algebras of a given graph* G. Our ambition is to study the stratification of $\mathcal{A}_G$ according to the Hilbert–Samuel sequence of $\mathcal{C}_G^{[f]}$.

Notice that for every $f(u)$ as above, $\mathcal{C}_G^{[f]}$ is a finite-dimensional vector space over $\mathbb{K}$, i.e. the ideal $I_G^{[f]}$ is zero-dimensional. Thus for every graph G and every finite sequence $M = \{1, m_1, m_2, \dots, m_n\}$ of positive non-decreasing integers, we can consider the *Hilbert–Samuel stratum* $S_G^M \subseteq \mathcal{A}_G$ (HS-stratum, for short) consisting of all f such that $HS^{[f]} = M$. (Observe that S_G^M might be empty.) We will show that each S_G^M is a constructible algebraic variety and we will be interested in the decomposition of its closure into irreducible closed algebraic components. We will call their intersection with S_G^M as *irreducible HS-strata corresponding to* M. Let us describe the adjacency of these strata.

Definition

Let X be a topological space and $(\Lambda, \prec)$ be a partially ordered set. We say that a function $g\colon X \to \Lambda$ is upper semicontinuous if for every $\lambda \in \Lambda$ the set

$$g^{-1}(\prec \lambda) := \{x \in X \mid g(x) \prec \lambda\}$$

is open.

Lemma (Nagata's criterion)

Let R be a commutative ring and $U \subseteq \operatorname{Spec} R$. Then U is open if and only if the following two conditions hold:

1. *if $\mathfrak{q} \in U$ and $\mathfrak{p} \subset \mathfrak{q}$ then $\mathfrak{p} \in U$,*
2. *if $\mathfrak{q} \in U$ then there exists an element $s \notin \mathfrak{q}$ such that $V(\mathfrak{p}) \cap D_s \subseteq U$.*

Lemma

Let A be a commutative ring and M be a matrix with entries in A. For a prime $\mathfrak{p} \in A$, let $M(\mathfrak{p})$ be the matrix obtained from M by replacing the entries by their images in $k(\mathfrak{p})$. Then the real-valued function $\mathfrak{p} \mapsto \operatorname{rk} M(\mathfrak{p})$ (respectively, $\mathfrak{p} \mapsto \dim\ker M(\mathfrak{p})$) is lower (resp., upper) semicontinuous.

Proof.

We use Nagata's criterion and the fact that non-vanishing of a minor is an open condition. Namely, if B is a square matrix then $\det B(\mathfrak{p}) \neq 0$ if and only if $\det B \notin \mathfrak{p}$. □

Theorem

Let A be a commutative ring and $R = A[X_1, \dots, X_N]/I$ be a finite A-module. Then

1. $\mathfrak{p} \mapsto \dim_{k(\mathfrak{p})} R \otimes_A k(\mathfrak{p})$ *is upper semicontinuous;*
2. *for any positive integer k,* $H_k : \mathfrak{p} \mapsto \dim_{k(\mathfrak{p})} \langle X_1^{\alpha_1} \cdots X_N^{\alpha_N} \mid \alpha_1 + \cdots \alpha_n \leq k \rangle$ *is a lower semicontinuous function.*

Corollary

Let A be a commutative ring and $R = A[X_1, \ldots, X_N]/I$ be a finite A-module. Suppose that R is generated (as a module) by monomials of degree at most D and set $\Lambda = \mathbb{Z}^{\oplus D+1}$ with the lexicographic order. Then the function $H\colon \operatorname{Spec} R \to \Lambda, \mathfrak{p} \mapsto (H_0, H_1, \ldots, H_D)$ is a lower semicontinuous. (H_k were defined in Theorem 9). Moreover, for any vector $\lambda = (\lambda_0, \lambda_1, \ldots)$, the set

$$H^{-1}(\lambda) = \{\mathfrak{p} \in \operatorname{Spec} R \mid H(\mathfrak{p}) = \lambda\}$$

is the intersection of an open set $H^{-1}(\succeq_{lex} \lambda)$ and a closed set $H^{-1}(\preceq_{lex} \lambda)$.
In particular, the closure of $H^{-1}(\lambda)$ coincides with $H^{-1}(\preceq_{lex} \lambda)$.

Corollary

The stratum S_G^M is constructive and, in the lexicographic partial order, we have $\overline{S_G^M} = S_G^{\leq M}$.

Next we introduce a natural $\mathbb{K}^\star$-action on $\mathcal{A}_G$.

Lemma

For any (multi)graph G, the natural $\mathbb{K}^\star$-action on $\mathcal{A}_G$ given by $f(u) \mapsto \frac{1}{\epsilon} f(\epsilon u)$ with $0 \neq \epsilon \in \mathbb{K}$ preserves the HS-stratification. This action is free on $\mathcal{A}_G \setminus \{u\}$ and $\{u\}$ lies in the closure of every HS-stratum.

Proof.

We substitute $Y_i = X_i/\epsilon$ to get an isomorphism of algebras

$$\mathbb{K}[X_1, \dots, X_n]/I_G^{[f(u)]} \cong \mathbb{K}[Y_1, \dots, Y_n]/(I_G^{[f(\epsilon u)]}).$$

The second claim follows the fact that if $f(u) = u + \dots$ contains at least one more monomial except u, then $f(u) \neq \frac{1}{\epsilon} f(\epsilon u)$ for $\epsilon \neq 1$. □

Denote by $P\mathcal{A}_G = (\mathcal{A}_G \setminus \{u\}) / \mathbb{K}^\star$ the weighted projective space obtained as the latter quotient. We get the HS-stratification of $P\mathcal{A}_G$ which we will be interested in. Consider the finite poset of HS-strata. This poset has the minimal element corresponding to $\{u\}$ and the maximal element corresponding to the stratum with a generic Hilbert–Samuel sequence.

Corollary

The Hilbert–Samuel sequence of the original algebra $\mathcal{C}_G^{\mathcal{E}x}$ is minimal. The maximal length of the Hilbert–Samuel sequence among the generalized zonotopal algebras from $\mathcal{A}_G$ is attained for $f(u) = u$ and equals $|G|$ which is the total number of edges in G.

Example. For $G = K_4$, $P\mathcal{A}_{K_4}$ is a weighted projective line (topologically S^2). Its HS-stratification consists of two points corresponding to $f(u) = u + u^2$ and $f(u) = u + u^3$ and a complex 1-dimensional stratum (coinciding with S^2 minus two points) which is the factor of $u + au^2 + cu^3$ with $b \neq 0$ and $c \neq 0$ mod the above $\mathbb{K}^\star$-action.

Observe that if $H \subset G$ is a subgraph of a (multi)graph G on the same set of labelled vertices, then, for any function f as above, we have a surjective homomorphism $\mathcal{C}_G^{[f]} \to \mathcal{C}_H^{[f]}$ induced by the obvious map $Tr\colon X_i^G \to X_i^H$ sending the generators corresponding to vertices on G to that of H (vertex by vertex).

Proving the next claim would be very crucial for our consideration.

Conjecture

The map Tr induces the surjective map $\mathcal{A}_G \to \mathcal{A}_H$ preserving the HS-stratifications.

Corollary

(Conjectural) For the usual graphs it is enough to study HS-stratification of $\mathcal{A}_{K_n}$.

Generalized external zonotopal algebras

Example

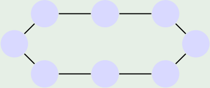
G_n = the n-chain (i.e. the Dynkin diagram of A_n),

, one only needs to consider $f(u) = u + u^2$.

1	3						
1	4	3					
1	5	10					
1	6	16	9				
1	7	23	33				
1	8	31	61	27			
1	9	40	98	108			
1	10	50	145	225	81		
1	11	61	203	397	351		
1	12	73	273	636	810	243	
1	13	86	356	955	1551	1134	
1	14	100	453	1368	2665	2862	729

Example

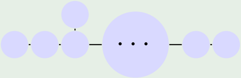
G_n = the n-cycle (i.e. the Dynkin diagram of $\widehat{A}_n$, the affine A_n),



, one only needs to consider $f(u) = u + u^2$.

1	3	2	1					
1	4	7	3					
1	5	14	10	1				
1	6	20	31	5				
1	7	27	63	28	1			
1	8	35	96	106	9			
1	9	44	138	243	75	1		
1	10	54	190	405	346	17		
1	11	65	253	627	891	198	1	
1	12	77	328	921	1620	1103	33	
1	13	90	416	1300	2691	3159	520	1

Example

$G_n =$  (A_n+leg on 3rd place) and $f(u) = u + u^2$. (In case $f(u) = u + u^3$ the Hilbert series is as for D_n, see example 6).

1	5	10			
1	6	15	10		
1	7	22	34		
1	8	30	59	30	
1	9	39	95	112	
1	10	49	141	221	90
1	11	60	198	388	366

Example

$G_n = A_n$+leg 4th place and $f(u) = u + u^2$. (For $f(u) = u + u^3$ and $f(u) = u + u^2 + u^3$, the Hilbert series is the same as for D_n, see Example 6 and A_n, see Example 1, resp.)

1	7	22	34		
1	8	30	62	27	
1	9	39	96	111	
1	10	49	142	229	81
1	11	60	199	393	360

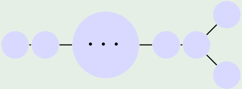
Example

$G_n = A_n$+leg on 5th place and $f(u) = u + u^2$. (In case $f(u) = u + u^3$ one gets the same Hilbert series as for D_n and $f(u) = u + u^2 + u^3$ the same as A_n).

1	6	16	9		
1	7	23	33		
1	8	30	59	30	
1	9	39	96	111	
1	10	49	140	222	90
1	11	60	197	392	363

Generalized external zonotopal algebras

Example

$G_n = D_n$, , and $f(u) = u + u^3$. (In case $f(u) = u + u^2$ and $f(u) = u + u^2 + u^3$ the Hilbert series is the same as for A_n).

1	4	3							
1	5	7	3						
1	6	12	10	3					
1	7	18	22	13	3				
1	8	25	40	35	16	3			
1	9	33	65	75	51	19	3		
1	10	42	98	140	126	70	22	3	
1	11	52	140	238	266	196	92	25	3

Guess: Pascal-type triangle.

Example

$G_n = \widehat{D_n}$ (affine D_n) and $f(u) = u + u^3$. (In case $f(u) = u + u^2$ and $f(u) = u + u^2 + u^3$ the Hilbert series is the same as for A_n).

1	5	7	3						
1	6	14	10	1					
1	7	22	25	9					
1	8	30	47	33	9				
1	9	39	77	79	42	9			
1	10	49	116	155	121	51	9		
1	11	60	165	270	276	172	60	9	
1	12	72	225	434	546	448	232	69	9

Guess: Pascal-type except for the 3rd and the 5th rows and element 25.

Generalized external zonotopal algebras

Example

$G_n = D_n$+leg at $n-2$nd place and $f(u) = u + u^3$. (In case $f(u) = u + u^2$ and $f(u) = u + u^2 + u^3$ the Hilbert series is the same as A_n+leg in 3rd place and A_n resp.)

1	7	20	24	11	1	
1	8	29	47	34	9	
1	9	38	77	80	42	9

Guess: Looks like Pascal-type plus/minus **1** in the available places.

Example

$G_n = D_n$+leg at 3rd place and $f(u) = u + u^3$. (In case $f(u) = u + u^2$ and $f(u) = u + u^2 + u^3$ the Hilbert series is the same as for A_n+leg in the 4th place and A_n resp)

1	5	7	3					
1	6	14	10	1				
1	7	20	24	11	1			
1	8	27	44	35	12	1		
1	9	35	71	79	47	13	1	
1	10	44	106	150	126	60	14	1

Guess: Pascal-type triangle except for the number 14.

Generalized external zonotopal algebras

Example

$G_n = D_n$+leg at 4th place and $f(u) = u + u^3$.

1	7	22	25	9			
1	8	29	47	34	9		
1	9	37	76	81	43	9	
1	10	46	113	157	124	52	9

Guess: Pascal-type triangle.

Generalized external zonotopal algebras

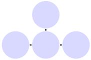
Example

$G_n = D_n$+leg at 5th place and $f(u) = u + u^3$.

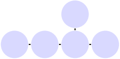
1	8	30	47	33	9		
1	9	38	77	80	42	9,	
1	10	47	115	157	122	51	9

Guess: Pascal-type triangle.

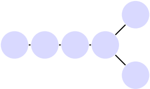
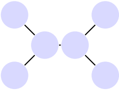
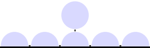
Notice that by Theorem A, the original Hilbert series and Hilbert-Samuel function of any tree (which correspond to $f(u) = u$) depend only on its number of vertices.
The homogeneous Hilbert-Samuel function is $(1, 4, 7, 8)$.

- has the Hilbert-Samuel function $(1, 5, 8)$ for all $f(u) = u + au^2$ with $a \neq 0$.
-  has the Hilbert-Samuel function $(1, 5, 8)$ for all $f(u) = u + au^2 + bu^3$ unless $a = b = 0$.

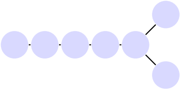
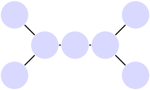
The homogeneous Hilbert-Samuel function is $(1, 5, 11, 15, 16)$.

- has the Hilbert-Samuel function $(1, 6, 16)$ for all $f(u) = u + au^2$ with $a \neq 0$.
-  has the Hilbert-Samuel function $(1, 6, 16)$ for all $f(u) = u + au^2 + bu^3$ unless $a = 0$. For $f(u) = u + bu^3$ with $b \neq 0$, the Hilbert–Samuel function is $(1, 6, 13, 16)$.

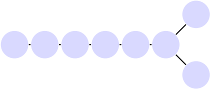
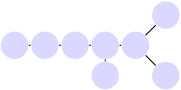
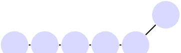
The homogeneous Hilbert-Samuel function is $(1, 6, 16, 26, 31, 32)$. Surprisingly all 4 non-isomorphic trees on 6 vertices have the same general Hilbert-Samuel function $(1, 7, 23, 32)$ for $f(u) = u + au^2 + bu^3$ with $a, b \neq 0$.

- has the Hilbert-Samuel function $(1, 7, 23, 32)$ for all $f(u) = u + au^2$ with $a \neq 0$.
-  has special value at $a = 0$; the family $f(u) = u + bu^3$, $b \neq 0$ gives the Hilbert-Samuel function $(1, 7, 19, 29, 32)$.
-  also has special value at $a = 0$: the family $f(u) = u + bu^3$, $b \neq 0$ gives the Hilbert-Samuel function $(1, 7, 21, 31, 32)$.
- 

The homogeneous Hilbert–Samuel function is $(1, 7, 22, 42, 57, 63, 64)$. A general Hilbert–Samuel function is no longer shared by all trees with 7 vertices.

- has the Hilbert–Samuel function $(1, 8, 31, 64)$ for all $f(u) = u + au^2$ with $a \neq 0$.
-  has the (general) Hilbert–Samuel function $(1, 8, 31, 64)$ and one special value $a = 0$ which for $f(u) = u + bu^3,\ b \neq 0$ gives the Hilbert-Samuel function $(1, 8, 26, 48, 61, 64)$.
-  has the (general) Hilbert–Samuel function $(1, 8, 31, 64)$ and one special value $a = 0$ which for $f(u) = u + bu^3, b \neq 0$ gives the Hilbert–Samuel function $(1, 8, 30, 55, 64)$.

The homogeneous Hilbert–Samuel function is $(1, 8, 29, 64, 99, 120, 127, 128)$.

- For $f(u) = u + au^2$, $a \neq 0$ the Hilbert-Samuel function equals $(1, 9, 40, 101, 128)$.
-  For $f(u) = u + u^2$ and $f(u) = u + u^2 + u^3$, the Hilbert-Samuel function equals $(1, 9, 40, 101, 128)$. For $f(u) = u + u^3$, it is $(1, 9, 34, 74, 109, 125, 128)$.
-  For $f(u) = u + u^2$, the Hilbert-Samuel function equals $(1, 9, 39, 101, 128)$; for $f(u) = u + u^2 + u^3$, it equals $(1, 9, 40, 101, 128)$, and for $f(u) = u + u^3$, it equals $(1, 9, 36, 80, 115, 127, 128)$.
- 

f(u)	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8
u	1	5	15	35	70	121	185	245	280
$u+u^2$	1	6	20	50	105	185	262	277	286
$u+u^3$	1	6	21	54	114	190	250	277	286
$u+u^4$	1	6	20	50	103	176	236	277	286
$u+u^2+u^3$	1	6	21	55	119	209	262	277	286
$u+u^2+u^4$	1	6	21	56	123	214	262	277	286
$u+u^3+u^4$	1	6	21	55	118	200	256	277	286
$u+u^2+u^3+u^4$	1	6	21	56	123	214	262	277	286
$u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{t^4}{4}$	1	6	21	56	119	203	262	277	286

(A) The above table shows the Hilbert-Samuel functions for several generalized external zonotope algebras of K_5.

(B) The table below shows the Hilbert-Samuel functions for several generalized external zonotope algebras of $K_5 \setminus e$ where e is any edge.

f(u)	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8
u	1	5	15	35	68	113	159	188	197
$u+u^2$	1	6	20	50	103	165	198		
$u+u^3$	1	6	21	54	109	168	196	198	
$u+u^4$	1	6	20	50	98	148	185	198	
$u+u^2+u^3$	1	6	21	55	117	185	198		
$u+u^2+u^4$	1	6	21	56	121	185	198		
$u+u^3+u^4$	1	6	21	56	121	185	198		
$u+u^2+u^3+u^4$	1	6	21	56	121	185	198		
$u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}$	1	6	21	56	116	175	198		

(C) The table below shows the Hilbert-Samuel functions for several generalized external zonotope algebras of $K_5 \setminus (e_1 \cup e_2)$ where e_1 and e_2 are any two disjoint edges of K_5.

f(u)	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8
u	1	5	15	35	66	101	125	133	134
$u+u^2$	1	6	20	50	101	127	134		
$u+u^3$	1	6	21	54	99	128	134		
$u+u^4$	1	6	19	45	79	112	130	134	
$u+u^2+u^3$	1	6	21	55	111	130	134		
$u+u^2+u^4$	1	6	21	55	111	130	134		
$u+u^3+u^4$	1	6	21	55	102	130	134		
$u+u^2+u^3+u^4$	1	6	21	56	112	130	134		
$u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}$	1	6	21	56	107	127	134		

(D) The table below shows the Hilbert-Samuel functions for several generalized external zonotope algebras of the square with parallel double edges, see left part of Fig. 1 in [NSh].

f(u)	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6
u	1	4	10	19	27	31	32
$u+u^2$	1	5	14	29	32		
$u+u^3$	1	5	15	26	31	32	
$u+u^2+u^3$	1	5	15	30	32		
$u-\frac{u^2}{2}+\frac{u^3}{3}$	1	5	15	29	32		

f(u)	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6
u	1	4	10	19	27	31	32
$u+u^2$	1	5	14	29	32		
$u+u^3$	1	5	15	27	32		
$u+u^4$	1	5	13	23	30	32	
$u+u^2+u^3$	1	5	15	31	32		
$u+u^2+u^4$	1	5	15	31	32		
$u+u^3+u^4$	1	5	15	28	32		
$u+u^2+u^3+u^4$	1	5	15	31	32		
$u-\frac{u^2}{2}+\frac{u^3}{3}$	1	5	15	31	32		
$u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}$	1	5	15	30	32		

(E) The above table shows the Hilbert-Samuel functions for several generalized external zonotope algebras of the square with adjacent double edges, see right part of Fig. 1 in [NSh].

Below we present two small example with complete analysis of their spaces $P\mathcal{A}_G$.

Proposition

For $G = K_3 + e$, the space $P\mathcal{A}_G$ is stratified by the following functions over a field $\mathbb{K}$ of characteristic 0.

$f(u)$	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4
u	1	3	6	9	10
$u + bu^2 + cu^3$	1	4	9	10	10
$*$	1	4	8	10	10

In the last row either $b = 0$, or $c = 0$, or $3c = 4b^2$.

We have a linear order $u < u + u^3 < u + u^2 < u + u^2 + u^3$.

Proposition

For $G = K_4$, the space $\mathcal{PA}_G$ is stratified by the following functions with $b, c \neq 0$ over a field $\mathbb{K}$ of characteristic 0.

$f(u)$	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6
u	1	4	10	20	31	37	38
$u + bu^2$	1	5	14	29	34	37	38
$u + cu^3$	1	5	14	26	34	37	38
$u + bu^2 + cu^3$	1	5	15	29	34	37	38

Here we formulate some concrete questions/conjectures which we hope can be solved.

Conjecture

For any graph G and any unipotent function f, the Hilbert series $HS^{\mathcal{E}x}_{G,f}(t)$ is log-concave.

Problem

In the case of trees, can we show that the stratification is given by the coordinate planes? In particular, e^u is general. In particular, the value in a natural 1-parameter family does not change unless the parameter turns in 0.

Problem

Find a graph-theoretical interpretation of HS_G? If possible, find a graph-theoretical interpretation of $HS_G^{[f]}$ at least for some special cases. Say, for $f = e^u$.

Problem

Is it true that if $HS_{G_1}^{[f]} = HS_{G_2}^{[f]}$ for every function f, then G_1 is isomorphic to G_2?

Problem

For a given graph $G = (V, E)$, $k \geq 1$, and $S \subseteq V$ describe for which a_v, we have $\prod_{v \in S} X_v^{a_v} = 0$.

Problem

Understand when $\mathcal{C}_G^{[f]}$ is Gorenstein. Can we say when is it Koszul?

Problem

Understand how does $HS_G^{[f]}$ change when graph operations are performed.

Problem

Give formulas for $HS_G^{[f]}$ of nice graphs.

- F. Ardila, A. Postnikov, Combinatorics and geometry of power ideals, Trans. Amer. Math. Soc. 362 (8) (2010) 4357–4384.
- G.H. Hardy, J. E. Littlewood,and G. Pólya, Inequalitites, 2nd edition, Cambridge Univ. Press, Cambridge, England, 1952.
- O. Holtz, A. Ron, Zonotopal algebra, Adv. Math. 227 (2011) 847–894.
- A. W. Marshall, I. Olkin, B. Arnold, Inequalities: Theory of Majorization and Its Applications, Second edition. Springer Series in Statistics. Springer, New York, 2011.
- G. Nenashev, Classification of external zonotopal algebras. Electron. J. Combin. 26 (2019), no. 1, Paper No. 1.32, 10 pp.
- G. Nenashev, B. Shapiro, "K-theoretic" analogs of Postnikov-Shapiro algebra distinguishes graphs, Journal of Combinatorial Theory, Series A 148 (2017) 316–332.

A. Postnikov, B. Shapiro, Trees, parking functions, syzygies, and deformations of monomial ideals, Trans. Amer. Math. Soc. 356 (8) (2004) 3109–3142.

A. Postnikov, B. Shapiro, M. Shapiro, Algebras of curvature forms on homogeneous manifolds. Differential topology, infinite-dimensional Lie algebras, and applications, 227–235, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., Providence, RI, 1999.

B. Shapiro, M. Shapiro, On ring generated by Chern 2-forms on SL_n/B. C. R. Acad. Sci. Paris Sér. I Math. **326**(1) (1998), 75–80.

D. Wagner, Algebras related to matroids represented in characteristic zero. European J. Combin. 20 (1999), no. 7, 701–711.