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Classical examples and questions

$\mathcal{P}_d^{\mathbb{C}} = \{x^d + a_1 x^{d-1} + \dots + a_d\}$ $a_i \in \mathbb{C}$.

$\mathcal{P}_d^{\mathbb{C}}$ is stratified into $\mathcal{P}_d^{\text{open},\lambda}$, $\lambda \vdash d$

consisting of all polys with the collection of root multiplicities given by λ.

Define $\mathcal{P}_d^{\lambda} = \overline{\mathcal{P}_d^{\text{open},\lambda}}$

Lemma For any $\lambda \vdash n$, $\mathcal{P}_d^{\lambda}$ is contractible (no interesting topology).

But $\mathcal{P}_d^{c\lambda} = \mathcal{P}_d \setminus \mathcal{P}_d^{\lambda}$ has a lot of interesting topology.

Basic example $\lambda = (2, 1^{d-2}) \vdash d$. Then $\mathcal{P}_d^{c\lambda}$ is the famous braid space which is $K(\pi_1)$ with π_1 being the braid group.

There is some information about $\mathcal{P}_d^{c\lambda}$ for $\lambda = (k, 1^{d-k})$ (Vassiliev)

In general, the question about topology of $\mathcal{P}_d^{c\lambda}$ is widely open and important

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We will talk about similar question over $\mathbb{R}$ where one can move much further and such spaces are useful as well.

Definition let $\mathcal{P}_d = \{x^d + a_1 x^{d-1} + \dots + a_n\}$, $a_i \in \mathbb{R}$. For $1 \le k \le d$ Denote by $\mathcal{P}_{d,k}$ – the space of $\mathcal{P}_n$ of polys with no real root of multiplicity $\geq k$.

Th A (of Arnold) If $k < n < 2k+1$, then $\mathcal{P}_{d,k}$ is diff. to $S^{k-1} \times$ Euclidean space of dim $d-k+1$

Th B. (Arnold–Vassiliev) The Homology groups with $\mathbb{Z}$-coeffs of $\mathcal{P}_{d,k}$ vanish in all dimensions which are not multiples of $k-1$

More precisely for $(k-1)r \le d$

$$H_{r(k-1)}(\mathcal{P}_{d,k}, \mathbb{Z}) \cong \mathbb{Z}.$$

(Arnold described the additive structure, Vassiliev the multiplicate. The cohomology ring is truncated ring of univ. polys.)

Our project

Study the topology of the space of real monic polys whose multiplicities of real roots avoid a given set of patterns.

Given a real monic polynomial p we associate to it its (real) multiplicity pattern $\omega = (\omega_1, \dots, \omega_\ell)$ multiplicities of real roots from left to right.

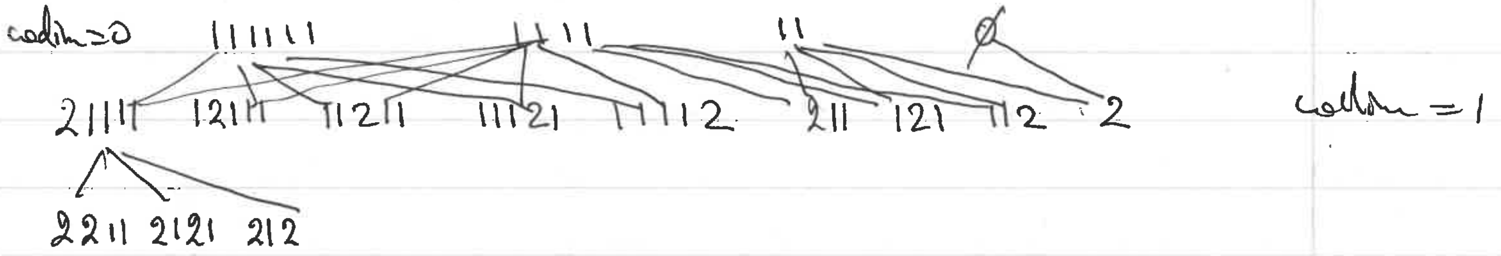
If $\deg p = d$, then $\sum_{i=1}^{\ell} \omega_i \equiv d \mod 2$ and $\sum_{i=1}^{\ell} \omega_i \leq d$.

Stratification of $\mathcal{P}_d$.

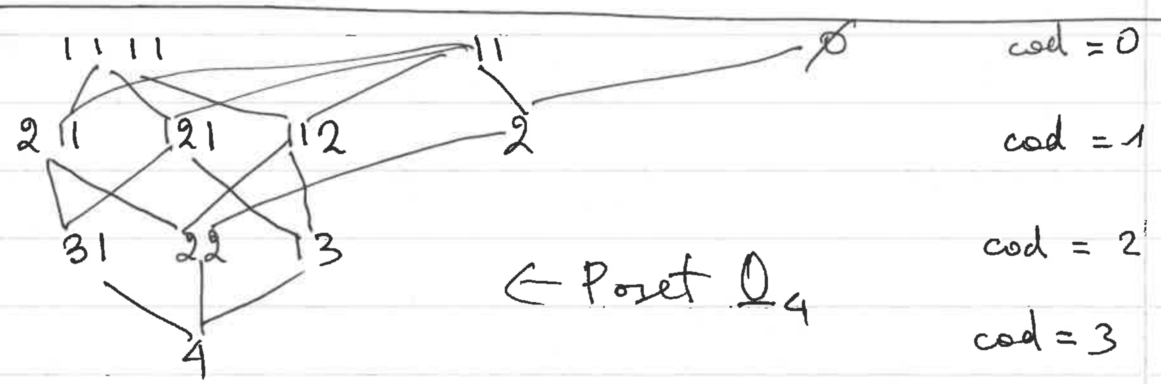
Example $d = 6$

Strata are labelled by compositions $\omega = (\omega_1, \dots \omega_\ell)$ with $\sum \omega_i \leq d$ and $\sum \omega_i \equiv d \mod 2$.

$\sum \omega_i$ is the length, $\sum \omega_i - \ell$ reduced length (codimension of the resp. stratum)



$d = 6$



cell decomposition. Adjacencies are given by 2 operations

a) merge two neighboring entries

b) insert a 2 at any position if the total length allows

~~Problem~~ We have a poset $\underline{O}$ of all compositions of all lengths. It splits into $\underline{O}_{even}$ and $\underline{O}_{odd}$.

Partial order is defined by merge & insert operations

Problem Given a subset $\Theta \subset \underline{O}$ (we assume Θ closed)

Consider the space P_d^{Θ} and study the topology of $\bar{P}_d^{\Theta}$ (or $P_d^{c\Theta} := P_d \setminus P_d^{\Theta}$)

Rmk. $\bar{P}_d^{\Theta}$ is a compact CW-complex with open cells labelled by $\omega \in \Theta$ with $|\omega| \leq d$ and $|\omega| \equiv d \mod 2$.

the unique 0-cell represents the point at infty.

Calculate $\pi_i(\bar{P}_d^{\Theta})$ & $\pi_i(P_d^{c\Theta})$ and (co)homology.

Fundamental group

$\pi_1(\bar{P}_d^{\Theta})$

Th1 For any closed subposet $\Theta \leq \underline{O}_{[d]}$, $\pi_1(\bar{P}_d^{\Theta}) = 0$ unless $\Theta = \{d\}$, in which case $\bar{P}_d^{\Theta} = S^1$

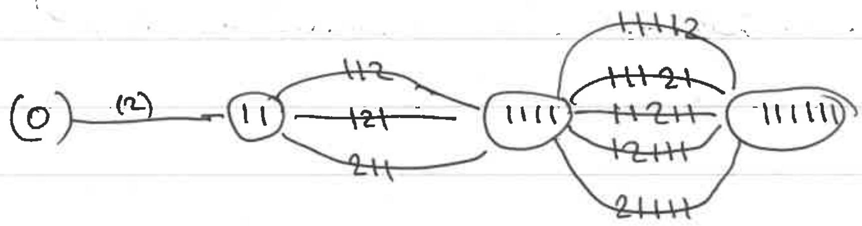
~~Th2~~. π_1 of the complement to the 2-skeleton

Th 2. For d even (resp. odd) $P_d^{c \, | \, |\geq 2}$ is homotopy equivalent to a wedge of $\frac{d(d-2)}{4}$ (resp $\frac{(d-1)^2}{4}$) circles.

In particular π_1 is a free group with that many generators,

Idea of the proof Contract $P_d^{c \, | \, |\geq 2}$ to the dual graph.

Example of the dual graph for $d=6$



6 loops

For an arbitrary closed poset $\Theta \subset \Omega_{\langle d \rangle}$

Lemma For any subposet inside the skeleton of cod k
$\pi_i(P_d^{(\Theta)})$ vanish for $i < k-1$.
$\pi_1(P_d^{(\Theta)})$ vanishes if Θ is inside the skeleton of cod 3.

Th ~~For~~ any closed subposet $\Theta \subset \Omega_{\langle d}$ $\pi_1(P_d^{(\Theta)})$
is a quotient of the free group (for the skeleton of codim 2
mod explicit relations (I can describe explicitly to
an interested listener)
(Everything is done in the alphabet of edges = codimension 2 cells ⟺ composition with two 2 ochan?
Corollary π_1 is always free, but we do not know
if $P_d^{c\Theta}$ is a $K(\pi_1)$ space in other cases.

A combinatorial differential complex computing $H_*(\overline{P}_d^\theta, \mathbb{Z})$

Decreasing filtration of P_d^θ given by

$$P_d^\theta = F_0(P_d^\theta) \supset F_1(P_d^\theta) \supset \dots \supset F_{d-1}(P_d^\theta)$$

by the closed subsets

$$F_k(P_d^\theta) := P_d^{\theta\, |w|' \geq k} = \bigcup_{\{w \in \theta : |w|' \geq k\}} P_d^w$$

Extends to $\overline{P}_d^\theta$

$$F_0^\bullet(\overline{P}_d^\theta) \supset \quad - \quad - \quad - \supset F_{d-1}^\bullet \supset F_d^\bullet := \bullet$$

$$F_k(P_d^\theta) \setminus F_{k+1}(P_d^\theta) = F_k^\bullet(\overline{P}_d^\theta) \setminus F_{k+1}^\bullet(\overline{P}_d^\theta) -$$

– disjoint union of open cells P_d^w, $w \in \theta$ over all compositions with $|w|' = k$.

Consider the spectral sequence $\{\tilde{E}^s_{p,q}\}$ calculating reduced homology of $\overline{P}_d^\theta$ associated with above decreasing filtration.

Its $E^2_{p,q}$-term is given by

$$E^2_{p,q} \cong \bigoplus_{\{w \in \theta, |w|' = p\}} \overline{H}_{p+q}(S^{d-p}, \mathbb{Z})$$

Rmk $E^2_{p,q}$ is nontrivial only when $p+q = d-p \iff 2p+q = d$

$$E^2_{p,d-2p} \cong \mathbb{Z}\left[\theta_{[w]' = p}\right]$$

Th $E^s_{p,q}$ degenerates at E^3

(Since $2p+q = d$ all differentials with $s \geq 0$ have trivial targets.)

There is an explicit differential combinatorial complex $(\mathbb{Z}[\Theta], \partial)$ coincides with the E^2-term and converging to $\bar{H}_*(\bar{P}^\Theta_d, \mathbb{Z})$

Some results & conjectures

I. $(d-k)$ skeleta of P_d.

Proposition $\bar{P}_d^{|\ |\geq k}$ is a wedge of $(d-k)$ dim spheres.
(the number is known, but clumsy)

II Θ is 1 pattern
Conjecture (i) For any single pattern ω with at least 1 entry > 2 $\bar{P}^{(\omega)}_d$ is contractible - unless $\omega = (d)$ in which case it is a circle

A weaker statement is proven.

III (i) For any ~~single~~ pattern ω with at least 2 entries $\bar{P}_d^{\langle\langle\omega\rangle\rangle}$ is contractible
(The remaining case is given by Arnold's theorem)

$M_j : \Omega \to \Omega$ define on $\omega = (\omega_1 \ldots \omega_\ell)$
For all $j \geq \ell$ $M_j(\omega) = \omega$. For $1 \leq j \leq \ell$
$M_j(\omega)_i = \omega_i, \quad i < j$
$M_j(\omega)_j = \omega_j + \omega_{j+1}$
$M_j(\omega)_i = \omega_{i+1}, \; i+1 < j \leq \ell - 1$

$I_j : \Omega \to \Omega$
$I_j(\omega)_i = \omega_i$ for $i < j$
$I_j(\omega)_j = 2$
$I_j(\omega)_i = \omega_{i-1}, \; j < i \leq \ell + 1$

$$\partial_M(\omega) := -\sum_{k=1}^{\#\omega - 1} (-1)^k M_k(\omega) \quad \text{and} \quad \partial_I(\omega) = \sum_{k=0}^{s_\omega} (-1)^k I_k(\omega)$$
$s_\omega = |\omega| - |\omega'|$

$$\partial = \partial_M + \partial_I : \mathbb{Z}[\Omega_d] \to \mathbb{Z}[\Omega_d]$$

$$\partial(\omega) = \begin{cases} \partial_M(\omega) + \partial_I(\omega) & \text{for } |\omega| < d \\ \partial_M(\omega), & \text{for } |\omega| = d \end{cases}$$

Stabilization

Th Given any finite collection $\Delta = \{w^{(1)} \ldots w^{(m)}\}$ of compositions with norms of the same parity, the following holds:

(i) For any non-negative i, $H_i(\overline{P}_d^{(\Delta)}, \mathbb{Z})$ vanishes for sufficiently large d; the limit space $\overline{P}_\infty^{(\Delta)}$ is weakly homotopically trivial

(ii) $\forall i$, $H^i(P_d^{(\Delta)}, \mathbb{Z})$ stabilizes for d large enough.

Under some condition on Θ (infinite) the limiting space $\overline{P}_\infty^{\theta}$ is an H-space In particular in Arnold-Vassiliev's case.

Still a lot to do, many explicit calculations and guesses using a program by V. Welker written in gap.