On the number of intersection points of the contour of an amoeba with a line

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Summary. We investigate the maximal number of intersection points of a line with the contour of a hypersurface amoeba in $\mathbb{R}^n$. We define the latter number to be the $\mathbb{R}$-degree of the contour. We also investigate the $\mathbb{R}$-degree of related sets such as the boundary of an amoeba and the amoeba of the real part of a hypersurface defined over $\mathbb{R}$. For all these objects, we provide bounds for the respective $\mathbb{R}$-degrees.

Topics to discuss

Amoebas of algebraic hypersurfaces in $(\mathbb{C}^\star)^n$ were introduced in 1994 by Gelfand-Kapranov-Zelevinskii and since then have been one of the central objects of study in tropical geometry.

Amoebas enjoy a number of beautiful and important properties such as special asymptotic at infinity and convexity of all connected components of the complement.

Formal definition

For a finite set $\mathcal{M} \subset \mathbb{Z}^n$ of Laurent monomials, denote by $|\mathcal{L}_{\mathcal{M}}|$ the space of all Laurent polynomials supported on $\mathcal{M}$, up to multiplication by a scalar.

For a hypersurface $\mathcal{H} \subset (\mathbb{C}^\star)^n$ given by $\{P = 0\}$ where $P \in |\mathcal{L}_{\mathcal{M}}|$, denote by $\mathcal{A}_{\mathcal{H}} \subset \mathbb{R}^n$ its *amoeba*, i.e. the image of $\mathcal{H}$ under the logarithmic map

$$\begin{array}{rcccl} \textbf{Log} & : & (\mathbb{C}^\star)^n & \to & \mathbb{R}^n \\ & & (z_1, \ldots, z_n) & \mapsto & (\log|z_1|, \ldots, \log|z_n|) \end{array} .$$

Example

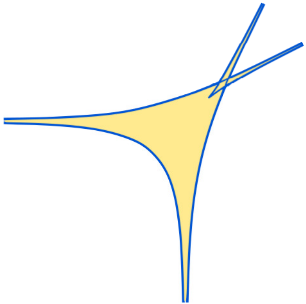

Figure: Amoeba of the discriminant $27 + 4a^3 - 18ab - a^2b^2 + 4b^3$ of the family $1 + ax + bx^2 + x^3$ and its contour (in blue).

Main definition of today

Definition (R. Thom)

Given a closed semi-analytic hypersurface $H \subset \mathbb{R}^n$ without boundary, we define the $\mathbb{R}$-*degree* $\mathbb{R}\deg(H)$ as the supremum of the cardinality of $H \cap L$ taken over all lines $L \subset \mathbb{R}^n$ such that L intersects H transversally. (Observe that we count points in $H \cap L$ without multiplicity).

Goal

Our aim is to provide estimates for the $\mathbb{R}$-degree of four closely related types of sets H, namely when H is

– a tropical hypersurface,

– the boundary of the amoeba of an algebraic hypersurface $\mathcal{H} \subset (\mathbb{C}^\star)^n$,

– the amoeba of the real locus of a real algebraic hypersurface $\mathcal{H} \subset (\mathbb{C}^\star)^n$,

– the contour of the amoeba of an algebraic hypersurface $\mathcal{H} \subset (\mathbb{C}^\star)^n$.

In particular, we will show that $\mathbb{R}\mathrm{deg}(H)$ is finite for all the above surfaces.

Remark. For a subset $H \subset \mathbb{R}^n$ that is real-algebraic (respectively piecewise real-algebraic), the $\mathbb{R}$-degree satisfies

$$\mathbb{R}\text{deg}(H) \leqslant \deg(H),$$

where $\deg(H)$ is the usual degree of H (respectively the degree of the Zariski closure of H). In particular, the $\mathbb{R}$-degree of a real-algebraic hypersurface is always finite.

More generally, if H is piecewise real-analytic, then it can happen that either $\mathbb{R}\text{deg}(H) = \infty$ or $\mathbb{R}\text{deg}(H) < \infty$ although the degree of the analytic continuation of H is always infinite.

We begin our investigation of the $\mathbb{R}$-degree with the case of tropical hypersurfaces.

Recall that for a finite set $\mathcal{M} \subset \mathbb{Z}^n$, a *tropical polynomial* supported on $\mathcal{M}$ is a convex piecewise linear function $p : \mathbb{R}^n \to \mathbb{R}$ of the form

$$p(x) = \max_{\alpha \in \mathcal{M}} \langle\, x \,|\, \alpha \,\rangle + c_\alpha$$

where $\langle\, x \,|\, \alpha \,\rangle = \alpha_1 x_1 + \cdots + \alpha_n x_n$ and $c_\alpha \in \mathbb{R}$. The *tropical hypersurface* associated to p is the set of points $x \in \mathbb{R}^n$ for which $p(x)$ is not smooth, i.e. it is equal to at least two of its *tropical monomials* $\langle\, x \,|\, \alpha \,\rangle + c_\alpha$.

First result

Proposition

Let $\mathcal{M} \subset \mathbb{Z}^n$ be any finite set. For any tropical hypersurface $H \subset \mathbb{R}^n$ defined by a tropical polynomial supported on $\mathcal{M}$, one has

$$\mathbb{R}deg(H) \leqslant \#\mathcal{M} - 1.$$

Moreover, there always exists a tropical hypersurface H supported on $\mathcal{M}$ such that $\mathbb{R}deg(H) = \#\mathcal{M} - 1$.

Introducing the spine and contour

Let $\mathcal{H}$ be the zero locus of a Laurent polynomial

$$P(z) = \sum_{\alpha \in \mathcal{M}} a_\alpha z^\alpha, \qquad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n},$$

where each a_α is a non-zero complex number and $\mathcal{M}$ is a finite subset of $\mathbb{Z}^n$. Since $\mathcal{H}$ is a subset of the complex torus, it is in fact defined by any Laurent polynomial obtained by multiplying $P(z)$ with an arbitrary monomial z^β. In particular, we may assume that $\mathcal{M} \subset N^n$, which means that $P(z)$ is an ordinary polynomial with no negative exponents. Whenever we talk about the amoeba of a Laurent polynomial $P(z)$ we have in mind the amoeba of the complex hypersurface

$$\mathcal{H} := \{P(z) = 0\}.$$

Some facts

(i) The complement $\mathbb{R}^n \setminus \mathcal{A}_{\mathcal{H}}$ of the amoeba consists of finitely many connected components E_ν, which are all open and convex.

(ii) There is a certain injective mapping $\{E_\nu\} \to \mathbb{Z}^n \cap \Delta_P$, where Δ_P is the Newton polytope of the polynomial $P(z)$. (We are not describing it.)

(iii) The integer vector $\nu \in \Delta_P$ corresponding to a component E_ν is called *the order* of that component. Multiplying $P(z)$ by a monomial z^β amounts to adding the constant vector β to each order vector. For any *vertex* ν of the Newton polytope Δ_P there is always an (unbounded) component E_ν in $\mathbb{R}^n \setminus \mathcal{A}$ having the order ν.

Amoebas with the property that the number of connected complement components is minimal, that is, equal to the number of vertices of Δ_P for any defining polynomial P, will be called *solid amoebas*. It has only unbounded components in its complement.

When $n = 1$ a solid amoeba consists of a single point, and it corresponds to a variety X with all its points having equal absolute value. From now on we will focus our attention on the case $n > 1$.

With each amoeba $\mathcal{A}_{\mathcal{H}}$ we are going to associate two real hypersurfaces $\mathcal{S}\mathcal{A}_{\mathcal{H}}$ and $\mathcal{C}\mathcal{A}_{\mathcal{H}}$ contained in $\mathcal{A}$, called respectively the *spine* and the *contour* of the amoeba.

Also denote by $\partial\mathcal{A}_{\mathcal{H}} \subset \mathcal{A}_{\mathcal{H}}$ the *boundary* of $\mathcal{A}_{\mathcal{H}}$

The spine $\mathcal{S}\mathcal{A}_{\mathcal{H}}$ is a tropical hypersurface, whereas the contour $\mathcal{C}\mathcal{A}_{\mathcal{H}}$ is real semi-analytic hypersurface containing the boundary (often with singularities).

The spine.

Let $\mathcal{M}'$ be the subset of $\mathbb{Z}^n \cap \Delta_P$ consisting of those vectors α for which the amoeba defined by the polynomial $P(z)$ has a complement component E_α of order α. The set $\mathcal{M}'$ necessarily contains all the vertices of the Newton polytope Δ_P.

For each $\alpha \in \mathcal{M}'$ we define the corresponding real number c_α to be the mean value of the function $\log|P(z)/z^\alpha|$ over any torus $\mathbf{Log}^{-1}(x)$, with $x \in E_\alpha$.

When α is a vertex of the Newton polytope Δ_P, one has $c_\alpha = \log|a_\alpha|$, but in general c_α depends in a complicated way of the coefficients of P.

Definition. The spine $\mathcal{SA}_{\mathcal{H}}$ of the amoeba $\mathcal{A}_{\mathcal{H}}$ is given by the tropical variety of the tropical polynomial

$$\max_{\alpha \in \mathcal{M}'} \{\langle x \mid \alpha \rangle + c_\alpha.\}.$$

Facts. (a) The spine $\mathcal{SA}_{\mathcal{H}}$ is contained in the amoeba $\mathcal{A}_{\mathcal{H}}$ and is in fact its strong deformation retract.

(b) The complement $\mathbb{R}^n \setminus \mathcal{SA}_{\mathcal{H}}$ consists of a finite number of convex polyhedra, each such polyhedron containing precisely one connected component of the amoeba complement $\mathbb{R}^n \setminus \mathcal{A}_{\mathcal{H}}$.

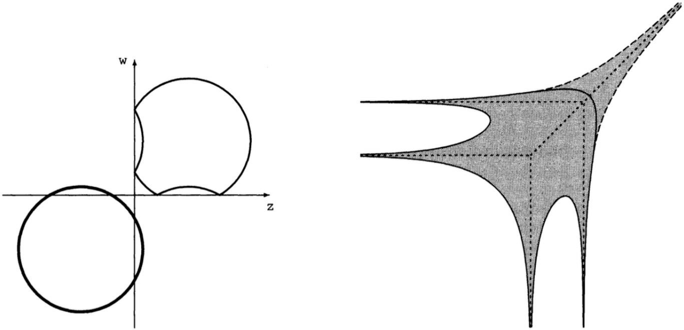

FIGURE 2. *The real circle* $1 + z + w + z^2/6 + w^2/6 = 0$ *and its reflection in the positive quadrant; the corresponding amoeba with its spine.*

Further definitions

The contour. Given an algebraic hypersurface $\mathcal{H} \subset (\mathbb{C} \setminus \{0\}^n$ we let **Log** denote also the restricted mapping **Log** : $\mathcal{H} \to \mathbb{R}^n$, whose image is precisely the amoeba $\mathcal{A}_{\mathcal{H}}$.

A closely related mapping is the *logarithmic Gauss mapping* $\gamma : \mathcal{H} \to \mathbb{C}P^{n-1}$ which can be analytically described on the regular part of $\mathcal{H}$ as

$$(z_1, \ldots, z_n) \mapsto (z_1\partial_1 P(z); \ldots; z_n\partial_n P(z)), \quad \partial_j = \partial/\partial z_j,$$

where P is a defining polynomial for $\mathcal{H}$.

Geometrically, if p is a regular point of $\mathcal{H}$ and if one chooses a local branch near p of the logarithmic transformation $(z_1, \dots, z_n) \mapsto (\log z_1, \dots, \log z_n)$, then the projective point $\gamma(p)$ equals the complex normal direction of the transformed surface $\widetilde{\mathcal{H}}$ at the corresponding point $\tilde{p}$.

It is not hard to show that a point $p \in \mathcal{H}$ is a critical point of **Log** if and only if $\gamma(p)$ belongs to the real subspace $\mathbb{R}P^{n-1} \subset \mathbb{C}P^{n-1}$.

Definition. The contour $\mathcal{C}\mathcal{A}_{\mathcal{H}}$ of the amoeba $\mathcal{A}_{\mathcal{H}}$ is defined to be the set of critical values of the mapping **Log** when restricted to $\mathcal{H}$, that is, the set **Log**$(\gamma^{-1}(\mathbb{R}P^{n-1}))$.

Illustration

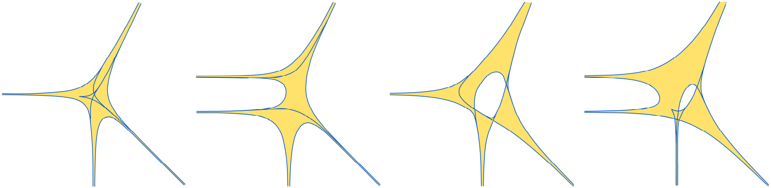

Figure: Amoebas of curves with Newton polygon $\text{conv}\{(0,0),(1,0),(2,1),(0,2)\}$ with their respective contours (in blue).

Lemma

For any algebraic hypersurface $\mathcal{H} \subset (\mathbb{C}^\star)^n$, the contour $\mathcal{C}\mathcal{A}_\mathcal{H}$ and the boundary $\partial\mathcal{A}_\mathcal{H}$ are closed semi-analytic hypersurfaces without boundary in $\mathbb{R}^n$.

Note that the contour $\mathcal{C}\mathcal{A}_\mathcal{H}$ may possibly have components of various dimensions. Although such phenomenon has not been observed by the authors, this occurs for the critical locus of the coordinatewise argument map Arg. For instance, the latter critical locus for a real algebraic curve may consist of arcs and isolated points. Since **Log** and Arg are the projections onto the real and imaginary axes in logarithmic coordinates, there is a priori no reason why the latter phenomenon should occur only for the projection Arg.

Let us now discuss the $\mathbb{R}$-degree of the boundary of the amoeba of a hypersurface. Recall that the spine $\mathcal{SA}_{\mathcal{H}} \subset \mathbb{R}^n$ is a tropical hypersurface lying inside $\mathcal{A}_{\mathcal{H}}$.

Proposition

Let $\mathcal{M} \subset \mathbb{Z}^n$ be a finite set of Laurent monomials. For any hypersurface $\mathcal{H} \subset (\mathbb{C}^\star)^n$ given by $\{P = 0\}$ where $P \in |\mathcal{L}_{\mathcal{M}}|$, one has

$$\mathbb{R}deg(\partial \mathcal{A}_{\mathcal{H}}) \leqslant 2 \cdot \mathbb{R}deg(\mathcal{SA}_{\mathcal{H}}).$$

Moreover, there always exists $P \in |\mathcal{L}_{\mathcal{M}}|$ such that $\mathbb{R}deg(\partial \mathcal{A}_{\mathcal{H}}) = 2 \cdot \mathbb{R}deg(\mathcal{SA}_{\mathcal{H}})$.

Observe that for a particular $P \in |\mathcal{L}_{\mathcal{M}}|$, the inequality in the above Proposition can be strict. Since the support of a tropical polynomial defining the spine $\mathcal{S}\mathcal{A}_{\mathcal{H}}$ can be always taken as a subset of $\Delta_P \cap \mathbb{Z}^n$, the following statement is a consequence of the previous Propositions.

Corollary

One has

$$\mathbb{R}deg(\partial \mathcal{A}_{\mathcal{H}}) \leqslant 2\big(\#(\Delta \cap \mathbb{Z}^n) - 1\big)$$

where Δ *is the convex hull of* $\mathcal{M}$.

Moreover the following special case holds.

Proposition

Let $\mathcal{M} \subset \mathbb{Z}^n$ *be a finite set of Laurent monomials and denote* $\Delta := \mathrm{conv}(\mathcal{M})$. *For a hypersurface* $\mathcal{H} \in |\mathcal{L}_{\mathcal{M}}|$ *with contractible amoeba, one has*

$$\mathbb{R}deg(\partial \mathcal{A}_{\mathcal{H}}) \leqslant 2\big(\#(\partial\Delta \cap \mathbb{Z}^n) - 1\big). \tag{1.1}$$

Since *A*-discriminants and classical discriminants have contractible amoebas we obtain

Corollary

If $\mathcal{H}$ *is is an A-determinantal or discriminantal hypersurface, then the above inequality holds, where* Δ_P *is the Newton polytope of* $\mathcal{H}$.

More results in special cases

In the case of curves, we can prove a stronger statement than the latter Corollary. Recall that for a non-degenerate lattice polygon $\Delta \subset \mathbb{R}^2$, i.e. $\mathrm{int}(\Delta) \neq \emptyset$, we can construct a toric surface X_Δ together with the tautological linear system $|\mathcal{L}_\Delta|$.

Proposition

Let $\Delta \subset \mathbb{R}^2$ be a non-degenerate lattice polygon. Then, for any $0 \leqslant g \leqslant \#(int(\Delta) \cap \mathbb{Z}^2)$ and any irreducible curve $\mathcal{H} \in |\mathcal{L}_\Delta|$ of geometric genus g, one has

$$\mathbb{R}deg(\partial \mathcal{A}_{\mathcal{H}}) \leqslant 2(\#(\partial\Delta \cap \mathbb{Z}^2) - 1 + g).$$

Furthermore, this upper bound is sharp.

Illustration

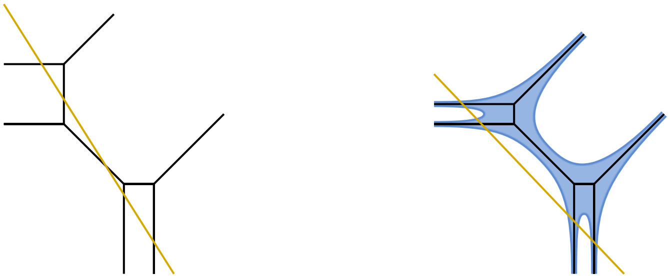

Figure: A tropical conic (left) and the amoeba of a conic (right) together with lines realizing their $\mathbb{R}$-degrees. On the right, the bounds given by the latter two Propositions coincide and neither of them is sharp in this example.

Real hypersurface

Let us now consider the situation when $\mathcal{H} \subset (\mathbb{C}^\star)^n$ is a real hypersurface, i.e. its defining polynomial P can be chosen to have real coefficients. Denote by $\mathcal{H}_\mathbb{R} := \mathcal{H} \cap (\mathbb{R}^*)^n$ the set of real point of $\mathcal{H}$ and define the *real stratum* of the amoeba $\mathcal{A}_\mathcal{H}$ to be the set $\mathcal{A}_\mathcal{H}^\mathbb{R} := \mathbf{Log}(\mathcal{H}_\mathbb{R})$.

As a consequence of the above results one has the inclusion

$$\mathcal{A}_\mathcal{H}^\mathbb{R} \subset \mathcal{C}\mathcal{A}_\mathcal{H}.$$

Our next goal is to estimate the $\mathbb{R}$-degree of $\mathcal{A}_{\mathcal{H}}^{\mathbb{R}}$ in terms of the support $\mathcal{M}$ of $\mathcal{H} \subset (\mathbb{C}^{\star})^{n}$.

To this end, we consider the action of the group $\{\pm 1\}^{n}$ of the sign changes of coordinates $(z_1, z_2, \ldots, z_n)$ on the space $\{\pm 1\}^{\mathcal{M}}$ of all possible sign distributions of the monomials in $\mathcal{M}$.

Through the identification of $\{\pm 1\}^{\mathcal{M}}$ with the space of polynomials supported on $\mathcal{M}$ with coefficients in $\{\pm 1\}$, the latter action is given by

$$(\epsilon_1, \cdots, \epsilon_n) \cdot P(z_1, \cdots, z_n) := P(\epsilon_1 z_1, \cdots, \epsilon_n z_n).$$

Since all sign distributions in $\{\pm 1\}^{\mathcal{M}}$ have the same stabilizer, each orbit has the same cardinality which we denote by $2^{\kappa_{\mathcal{M}}}$.

Proposition

Let $\mathcal{M} \subset \mathbb{Z}^n$ be a finite set of Laurent monomials. For any hypersurface $\mathcal{H} \subset (\mathbb{C}^\star)^n$ given by $\{P = 0\}$ where $P \in |\mathcal{L}_\mathcal{M}|$ is a real polynomial, one has

- *if* $\{-1\}^\mathcal{M} \notin \{\pm 1\}^n \cdot \{+1\}^\mathcal{M}$, *then*

$$\mathbb{R}deg(\mathcal{A}^\mathbb{R}_\mathcal{H}) \leqslant \begin{cases} \#\mathcal{M} - 1, & \text{for } \kappa_\mathcal{M} = 0, \\ 2^{\kappa_\mathcal{M}-1}(2\#\mathcal{M} - 3), & \text{for } \kappa_\mathcal{M} \geqslant 1, \end{cases} \tag{1.2}$$

- *if* $\{-1\}^\mathcal{M} \in \{\pm 1\}^n \cdot \{+1\}^\mathcal{M}$, *then*

$$\mathbb{R}deg(\mathcal{A}^\mathbb{R}_\mathcal{H}) \leqslant \begin{cases} \#\mathcal{M} - 1, & \text{for } \kappa_\mathcal{M} = 1, \\ 2^{\kappa_\mathcal{M}-2}(2\#\mathcal{M} - 3), & \text{for } \kappa_\mathcal{M} \geqslant 2. \end{cases} \tag{1.3}$$

The case of the contour

Finally, we consider the contour $\mathcal{CA}_{\mathcal{H}}$ of a hypersurface $\mathcal{H}$ in $(\mathbb{C}^\star)^n$.

Using Khovanskii's fewnomial theory, we obtain the following upper bound for the $\mathbb{R}$-degree of the contour.

Proposition

For any hypersurface $\mathcal{H} \subset (\mathbb{C}^\star)^n$ defined by a polynomial P of degree d, one has

$$\mathbb{R}deg(\mathcal{CA}_{\mathcal{H}}) \leqslant 2^{2n+(n-1)(n-2)/2} d^{n+1} \left(4dn + 2(n-1)^2 - 1\right)^{n-1}.$$

The upper bound of the above proposition is probably not sharp, as illustrated by the following improvement in dimension 2 in which case we take into account the combinatorics of the Newton polygon of the curve.

Proposition

For any curve $\mathcal{H} \subset (\mathbb{C}^\star)^2$ defined by a bivariate polynomial P of degree d and with Newton polygon Δ_P, one has

$$\mathbb{R}deg(\mathcal{C}\mathcal{A}_{\mathcal{H}}) \leqslant 4d^3(4d-2) + \#(\partial\Delta_P \cap \mathbb{Z}^2) - Area(\Delta_P)$$

where $Area(\Delta_P)$ is twice the Euclidean area of Δ_P.

Some proofs

Remark

For any continuous family of lines L_t intersecting H transversally, the number of points $\#(L_t \cap H)$ is a lower semi-continuous function in t. In particular, whenever $\mathbb{R}\mathrm{deg}(H)$ is finite, one can always find a line L with rational slope which is transversal to H and such that $\#(L \cap H) = \mathbb{R}\mathrm{deg}(H)$. Similarly, for any continuous family of hypersurface H_t intersecting L transversally, the number $\#(L \cap H_t)$ is a lower semi-continuous function in t.

Lemma

The number of tropical roots of any tropical univariate polynomial supported in a d points is at most $d - 1$.

Proof.

Obvious. □

Lemma

Given an arbitrary univariate d-nomial, the total number of distinct absolute values of its non-vanishing real zeros is at most $d - 1$ if all the exponents have the same parity and is at most $2d - 3$ otherwise. Both bounds are sharp.

Proof of Proposition on tropical hypersurfaces.

For any tropical hypersurface $H \subset \mathbb{R}^n$ supported on $\mathcal{M}$, the $\mathbb{R}$-degree $\mathbb{R}\text{deg}(H)$ is finite since H is contained in the union of finitely many hyperplanes. In particular, the integer $\mathbb{R}\text{deg}(H)$ is given as the number of the intersection points of H with some line $L \subset \mathbb{R}^n$ with rational slope, see above Remark.

In other words, $\mathbb{R}\text{deg}(H)$ is the number of tropical roots of the univariate tropical polynomial p_L obtained by restricting the tropical polynomial defining H to L. Obviously, the tropical polynomial p_L is the sum of at most $\#\mathcal{M}$ tropical monomials. Therefore, p_L has at most $\#\mathcal{M} - 1$ tropical roots.

To prove that $\#\mathcal{M} - 1$ is a sharp upper bound for a given support set $\mathcal{M} \subset \mathbb{Z}^n$, notice that the direction of L can be chosen so that p_L has exactly $\#\mathcal{M}$ monomials and that the coefficients of the tropical polynomial defining H can be chosen so that p_L has the maximal number of tropical roots, that is $\#\mathcal{M} - 1$.

Proof of Proposition of the boundary of amoeba.

Recall that all connected components of the complement to the amoeba $\mathcal{A}_{\mathcal{H}} \subset \mathbb{R}^n$ of the hypersurface $\mathcal{H} \subset (\mathbb{C}^*)^n$ are always convex. Moreover, the spine $\mathcal{S}\mathcal{A}_{\mathcal{H}}$ is a deformation retract of the amoeba $\mathcal{A}_{\mathcal{H}}$. Therefore, the inclusion of the connected components of $\mathbb{R}^n \setminus \mathcal{A}_{\mathcal{H}}$ in the connected components of $\mathbb{R}^n \setminus \mathcal{S}\mathcal{A}_{\mathcal{H}}$ is a 1-to-1 correspondence.

Now, the intersection of any line $L \subset \mathbb{R}^n$ with $\mathcal{A}_\mathcal{H}$ is a union of intervals and we claim that each such interval I intersects $\mathcal{S}\mathcal{A}_\mathcal{H}$ at least once. Indeed, by convexity of the connected components of $\mathbb{R}^n \setminus \mathcal{A}_\mathcal{H}$, the endpoints of I lie on the boundary of two different connected components of $\mathbb{R}^n \setminus \mathcal{A}_\mathcal{H}$. According to the above correspondence, the endpoints of I necessarily belong to different connected components of $\mathbb{R}^n \setminus \mathcal{S}\mathcal{A}_\mathcal{H}$. It implies that I meets $\mathcal{S}\mathcal{A}_\mathcal{H}$ as least once and the claim follows. Therefore, one has that $\mathbb{R}\mathrm{deg}(\partial\mathcal{A}_\mathcal{H}) \leqslant 2 \cdot \mathbb{R}\mathrm{deg}(\mathcal{S}\mathcal{A}_\mathcal{H})$.

For the second part of the statement, for any given finite set $\mathcal{M} \subset \mathbb{Z}^n$, one can find an amoeba which is arbitrarily close to its spine using the so-called Viro polynomials. In particular, any line L realizing $\mathbb{R}\deg(\mathcal{S}\mathcal{A}_\mathcal{H})$, that is such that

$$\#(L \cap \mathcal{A}_\mathcal{H}) = \mathbb{R}\deg(\mathcal{S}\mathcal{A}_\mathcal{H}),$$

has the property that

$$\#(L \cap \partial\mathcal{A}_\mathcal{H}) = 2 \cdot \mathbb{R}\deg(\mathcal{S}\mathcal{A}_\mathcal{H}).$$

Thank you for your attention!

Details can be found in

L. Lang, B. Shapiro, E. Shustin, On the number of intersection points of the contour of an amoeba with a line,

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Happy reading!

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