Dedicated to Heinrich Eduard Heine and his 140–years old riddle

Algebro-geometric aspects of Heine-Stieltjes theory

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Introduction and main results

The algebraic form of the classical **Lamé equation** is:

$$Q(z)\frac{d^2S}{dz^2}+\frac{1}{2}Q'(z)\frac{dS}{dz}+V(z)S=0, \qquad (1)$$

where $Q_l(z)$ is a real polynomial of degree l with all real and distinct roots, and $V(z)$ is a polynomial of degree at most $l-2$ whose choice depends on what we are looking for. It was introduced by Gabriel Lamé in 1830's in connection with the separation of variables in the Laplace equation in $\mathbb{R}^l$. It was studied in the second half of the 19-th century several celebrated mathematicians including M. Bôcher, E. Heine, F. Klein, T. Stieltjes.

In what follows we will concentrate on the usual polynomial solutions of (1) and its various modifications.

A **generalized Lamé equation** is the second order differential equation given by

$$Q_2(z)\frac{d^2S}{dz^2}+Q_1(z)\frac{dS}{dz}+V(z)S=0, \qquad (2)$$

where $\deg Q_2(z) = l$ and $Q_1(z) \leq l-1$. The special case $l = 3$ is widely known as the **Heun equation**.

The next fundamental proposition announced by Heine with not a quite satisfactory proof is our starting point.

Theorem 1 (Heine)**.** *If the coefficients of $Q_2(z)$ and $Q_1(z)$ are algebraically independent then for any integer $n > 0$ there exists exactly $\binom{n+l-2}{n}$ polynomials $V(z)$ of degree exactly* $(l-2)$ *such that the equation (2) has and unique (up to a constant factor) polynomial solution S of degree exactly n.*

Later a special case of (2)

$$\prod_{i=1}^{l}(z-\alpha_i)\frac{d^2S}{dz^2}+\sum_{j=1}^{l}\beta_j\prod_{j\neq i}(z-\alpha_i)\frac{dS}{dz}+V(z)S=0, \quad (3)$$

with $\alpha_1 < \alpha_2 < \ldots < \alpha_l$ real and $\beta_1, \ldots, \beta_l$ positive was considered separately by T. Stieltjes.

Theorem 2 (Stieltjes-Van Vleck-Bôcher)**.** *Under the assumptions of (3) and for any integer $n > 0$*

1. *there exist exactly $\binom{n+l-2}{n}$ distinct polynomials V of degree $(l-2)$ such that the equation (3) has a polynomial solution S of degree exactly n.*

2. *each root of each V and S is real and simple, and belongs to the interval (α_1, α_l).*

3. *none of the roots of S can coincide with some of α_i's. Moreover, $\binom{n+l-2}{n}$ polynomials S are in 1-1-correspondence with $\binom{n+l-2}{n}$ possible ways to distribute n points into the $(l-1)$ open intervals (α_1, α_2), (α_2, α_3),. . . , (α_{l-1}, α_l).*

The polynomials V and the corresponding polynomial solutions S of the equation (2) are called *Van Vleck* and *Stieltjes* (or *Heine-Stieltjes*) polynomials resp.

The case when α_i's and/or β_j's are complex is substantially less studied. One nice result in this set-up is due to G. Pólya.

Theorem 3 (Pólya)**.** *If in the notation of (3) all α_i's are complex and all β_j's are positive that all the roots of each V and S belong to the convex hull $Conv_{Q_2}$ of the set of roots $(\alpha_1, \ldots, \alpha_l)$ of $Q_2(z)$.*

Our set-up

Consider an arbitrary linear ordinary differential operator

$$\mathfrak{d}(z) = \sum_{i=1}^{k} Q_i(z) \frac{d^i}{dz^i}, \qquad (4)$$

with polynomial coefficients. The number $r = max_{i=1,\dots,k}(\deg Q_i(z) - i)$ will be called the **Fuchs index** of $\mathfrak{d}(z)$. The operator $\mathfrak{d}(z)$ is called a **higher Lamé operator** if $r \geq 0$. The operator $\mathfrak{d}(z)$ is called **non-degenerate** if $\deg Q_k(z) = k + r$.

Problem. For each positive integer n find all polynomials $V(z)$ of degree at most r such that the equation

$$\mathfrak{d}(z)S(z) + V(z)S(z) = 0 \qquad (5)$$

has a polynomial solution $S(z)$ of degree n.

$V(z)$ is called a *higher Van Vleck polynomial*, and the corresponding polynomial $S(z)$ is called a *higher Stieltjes polynomial*.

Generalizations of Heine's theorem, degeneracies and nonresonance condition

Theorem 4. *For any non-degenerate higher Lamé operator $\mathfrak{d}(z)$ with algebraically independent coefficients of its polynomial coefficients $Q_i(z)$, $i = 1, \dots, k$ and for any $n \geq 0$ there exist exactly $\binom{n+r}{r}$ distinct Van Vleck polynomials $V(z)$'s whose corresponding Stieltjes polynomials $S(z)$'s are unique (up to a constant factor) and of degree n.*

Theorem 5. *For any non-degenerate operator $\mathfrak{d}(z)$ with a Fuchs index $r \geq 0$ and any positive integer n the total number of Van Vleck polynomials $V(z)$ (counted with natural multiplicities) having a Stieltjes polynomial $S(z)$ of degree less than or equal to n equals $\binom{n+r+1}{r+1}$.*

Remark 1. Note that in Theorem 5 we do not require that there is a unique (up to constant factor) Stieltjes polynomial corresponding to a given Van Vleck polynomial.

Below we formulate a simple sufficient condition which allows us to avoid many of the above degeneracies and guarantees the existence of Stieltjes polynomials of a given degree. Namely, consider an arbitrary non-degenerate operator $\mathfrak{d}(z)$ of the form (4) with the Fuchs index r. Denote by $A_k, A_{k-1}, ..., A_1$ the coefficients at the highest possible degrees $k+r, k+r-1, ..., r+1$ in the polynomials $Q_k(z), Q_{k-1}(z), ..., Q_1(z)$ resp. (Notice that any subset of A_j's can vanish but $A_k \neq 0$ due to the non-degeneracy of $\mathfrak{d}(z)$.) In what follows we will often use the notation

$$(j)_i = j(j-1)(j-2)...(j-i+1),$$

where j is a non-negative and i is a positive integer. In case $j = i$ one has $(j)_i = j!$ and in case $j < i$ one gets $(j)_i = 0$. For any non-negative n we call by *the n-th diagonal coefficient* $\mathbb{L}_n$ the expression:

$$\mathbb{L}_n = (n)_k A_k + (n)_{k-1} A_{k-1} + ... + (n)_1 A_1. \quad (6)$$

Proposition 1. *If in the above notation and for a given positive integer n the n-***th non-resonance condition**

$$\mathbb{L}_n \neq \mathbb{L}_j,\ j = 0, 1, ..., n-1 \qquad (7)$$

holds then there exist Van Vleck polynomials which possess Stieltjes polynomials of degree exactly n and no other Stieltjes polynomials of degree smaller than n. In this case the total number of such Van Vleck polynomials (counted with natural multiplicities) equals $\binom{n+r}{r}$.

Explicit formula (6) for $\mathbb{L}_n$ immediately shows that Theorem 5 and Proposition 1 are valid for any non-degenerate $\mathfrak{d}(z)$ and all sufficiently large n.

Corollary 1. *For any non-degenerate higher Lamé operator $\mathfrak{d}(z)$ and all sufficiently large n the n-th nonresonance condition holds. In particular, for any problem (5) there exist and finitely many (up to a scalar multiple) Stieltjes polynomials of any sufficiently large degree.*

Generalizations of Stieltjes's theorem

We continue with a conceptually new generalization of Theorem 2. It was proved by P. Bränden.

Definition 1. A differential operator $\mathfrak{d}(z) = \sum_{i=m}^{k} Q_i(z)\frac{d^i}{dz^i}$, $1 \leq m \leq k$ where all $Q_i(z)$'s are polynomials with real coefficients is called a **strict hyperbolicity preserver** if for any real polynomial $P(z)$ with all real and simple roots the image $\mathfrak{d}(P(z))$ either vanishes identically or is a polynomial with only real and simple roots.

Theorem 6. *For any strict hyperbolicity preserving non-degenerate Lamé operator $\mathfrak{d}(z)$ with the Fuchs index r as above and any integer $n \geq m$*

1. *there exist exactly $\binom{n+r}{n}$ distinct polynomials $V(z)$ of degree exactly r such that*

the equation (5) has a polynomial solution $S(z)$ of degree exactly n.

2. *all roots of each such $V(z)$ and $S(z)$ are real, simple, coprime.*

3. *$\binom{n+r}{n}$ polynomials $S(z)$ correspond exactly to $\binom{n+r}{n}$ possible arrangements of r real roots of polynomials V(z) and n real roots of the corresponding polynomials $S(z)$.*

Using Theorem 5 one immediately sees that the latter result describes the set of all possible pairs (V, S) with $m \leq n = \deg S$ for any hyperbolicity preserver $\mathfrak{d}(z)$.

Generalizations of Polya's theorem

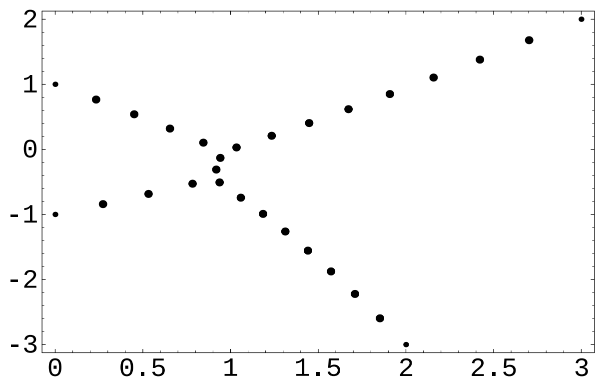
Theorem 7. *For any non-degenerate higher Lamé operator* $\mathfrak{d}(z)$ *and any* $\epsilon > 0$ *there exists a positive integer* N_ϵ *such that the zeros of all Van Vleck polynomials* $V(z)$ *possessing a Heine-Stieltjes polynomial* $S(z)$ *of degree* $n \geq N_\epsilon$ *and well as all zeros of these Stieltjes polynomials belong to* $Conv^\epsilon_{Q_k}$*. Here* $Conv_{Q_k}$ *is the convex hull of all zeros of the leading coefficient* Q_k *and* $Conv^\epsilon_{Q_k}$ *is its* ϵ*-neighborhood in the usual Euclidean distance on* $\mathbb{C}$*.*

The latter theorem is closely related to the next somewhat simpler localization result having independent interest.

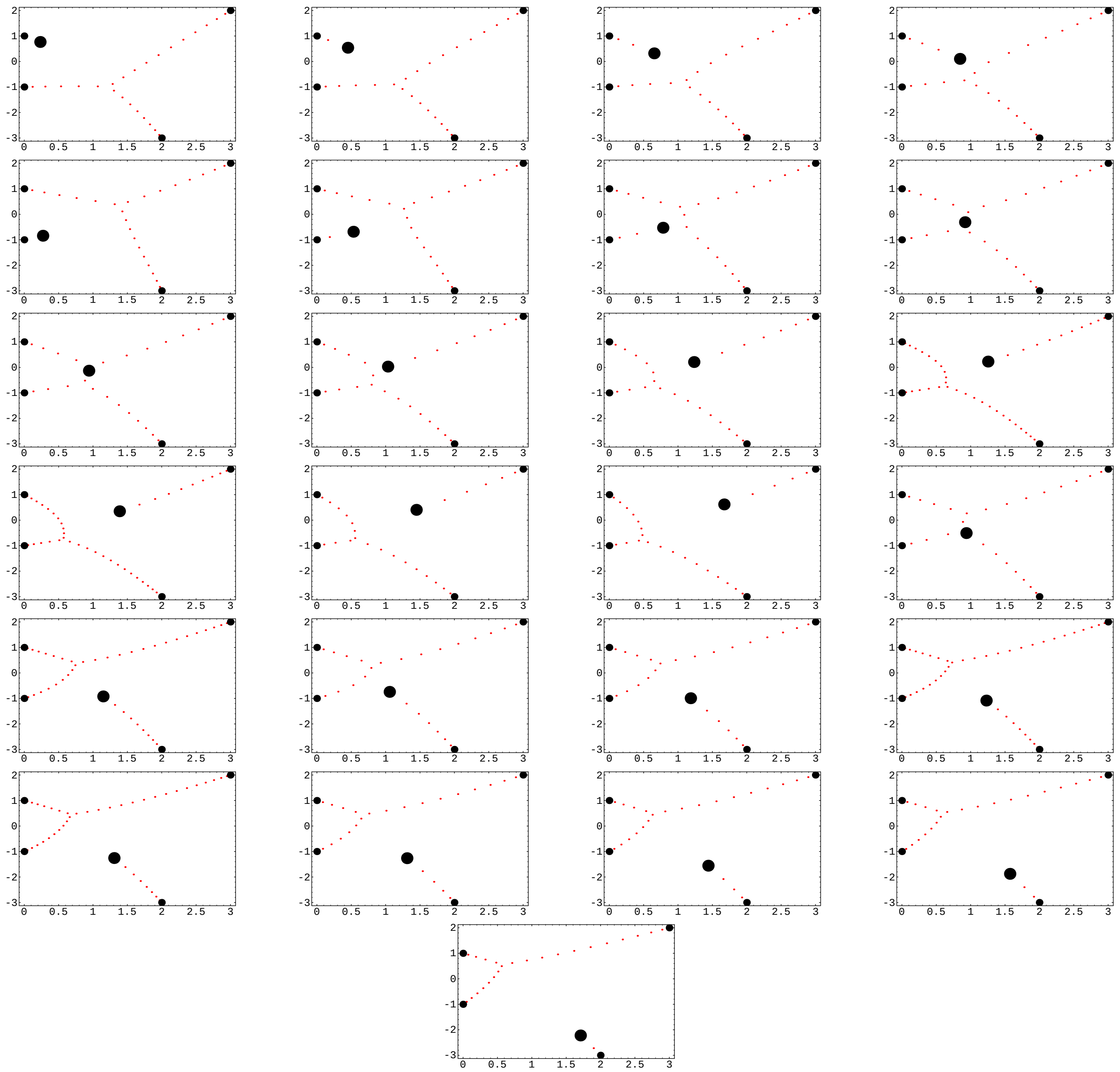
Proposition 2. *For any non-degenerate higher Lamé operator* $\mathfrak{d}(z)$ *there exist a positive integer* N_0 *and a positive number* R_0 *such that all zeros of all Van Vleck polynomials* $V(z)$

possessing a Stieltjes polynomial $S(z)$ *of degree* $n \geq N_0$ *as well as all zeros of these Stieltjes polynomials lie in the disk* $|z| \leq R_0$.

Consider as an example the operator $\mathfrak{d}(z) = Q(z)\frac{d^3}{dz^3}$ with $Q(z) = (z^2+1)(z-3I-2)(z+2I-3)$. For $n = 24$ we calculate all 25 pairs (V, S) with $\deg S = 24$. (Notice that V in this case is linear.)



Zeros of 25 different linear Van Vleck polynomials whose Stieltjes polynomials are of degree 24.



Zeros of 25 different Stieltjes polynomials of degree 24 for the above $\mathfrak{d}(z)$.

Proof of generalized Heine's theorems

We start with Theorem 4.

Proof. Substituting $V(z) = v_r z^r + v_{r-1} z^{r-1} + \ldots + v_0$ and $S(z) = s_n z^n + s_{n-1} z^{n-1} + \ldots + s_0$ in (5) we get the following system of $(n+r+1)$ equations of a band shape (i.e. only a fixed and independent of n number of diagonals is non-vanishing in this system):

$$
\begin{cases}
0 = s_n(v_r + L_{n,n+r}); \\
0 = s_n(v_{r-1} + L_{n,n+r-1}) + s_{n-1}(v_r + L_{n-1,n+r-1}); \\
0 = s_n(v_{r-2} + L_{n,n+r-2}) + s_{n-1}(v_{r-1} + L_{n-1,n+r-2} \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
0 = s_n(v_0 + L_{n,n}) + s_{n-1}(v_1 + L_{n-1,n}) + \ldots + s_{n-r}(\\
0 = s_nL_{n,n-1} + s_{n-1}(v_0 + L_{n-1,n-1}) + \ldots + s_{n-r-1}(\\
0 = s_nL_{n,n-2} + s_1L_{1,r+1} + s_2(v_r + L_{2,r}) + \ldots + s_{n-} \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
0 = s_nL_{n,r} + s_{n-1}L_{n-1,r} + s_{n-2}L_{n-2,r} + \ldots + s_0(v_r \\
0 = s_nL_{n,r-1} + s_{n-1}L_{n-1,r-1} + s_{n-2}L_{n-2,r-1} + \ldots \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
0 = s_nL_{n,1} + s_{n-1}L_{n-1,1} + \ldots + s_1(v_0 + L_{1,1}) + s_0(\\
0 = s_nL_{n,0} + s_{n-1}L_{n-1,0} + \ldots + s_1L_{0,1} + s_0(v_0 + L_0
\end{cases}
\tag{8}
$$

Here $L_{p,q}$ is a polynomial which expresses the coefficient containing s_p at the power z^q in $\sum_{i=1}^{k} Q_i(z)S^{(i)}$. Obviously, it is linear in the coefficients of $Q_k(z), \ldots, Q_1(z)$ and is explicitly given by the relation

$$L_{p,q} = \sum_{r=1}^{k} (p)_r A_{r,q-p+r},$$

where $A_{r,q-p+r}$ is the coefficient at z^{q-p+r} in $Q_r(z)$. In the notation used in the definition (6) we have $L_{m,m+r} = \mathbb{L}_m, m = 0, ..., n$. We use the convention that $L_{p,q}$ vanishes outside the admissible range of indices and, therefore many of the above coefficients $L_{p,q}$ are in fact equal to 0. (In the system (8) we assumed that $n \geq r$ for simplicity.) Notice that all equations in (8) depend linearly on the variables $v_r, ..., v_0$ and $s_n, ..., s_0$ as well as on the coefficients of polynomials $Q_i(z)$, $i = 1, \ldots, k$. Note additionally, that (8) is lower-triangular w.r.t the coefficients $s_n, ..., s_0$ which allows us to perform the following important elimination. Let us enumerate the equations of (8) from 0 to $n + r$ assigning the number j to the equation describing the vanishing of the coefficient at the power z^{n+r-j}. Then if $\mathbb{L}_n = L_{n,n+r} \neq 0$ one has that the 0-th equation has a solution $s_n = 1$ and $v_r = -L_{n,n+r} \neq 0$. The next n equations are triangular w.r.t the coefficients $s_n, ..., s_0$, i.e. j-th equation in this group contains only the variables $s_n, s_{n-1}, \ldots, s_{n-j}$ (among all s_j's) along with other types of variables. Thus under the assumption that all the diagonal terms $v_r + L_{n-i,n+r-i} = \mathbb{L}_{n-i} - \mathbb{L}_n$, $i = 0, 1, ..., n$ are nonvanishing we can express all s_{n-i}, $i = 0, 1, ..., n$ consecutively as rational functions of the remaining variables and get the reduced system of r rational equations containing only $(v_{r-1}, \ldots, v_0)$ as unknowns. Notice that in view of $v_r = -L_{n,n+r} \neq 0$ the non-vanishing of the diagonal entries $v_r + L_{n-i,n+r-i}$, $i = 0, 1, ..., n$ coincides exactly with the nonresonance condition (7).

Cleaning the common denominators we get a reduced system of polynomial equations. We show now that this polynomial system is quasi-homogeneous in the variables v_j with the quasi-homogeneous weights $w(v_j)$ given by $w(v_j) = r - j$. Thus using the weighted-homogeneous version of the Bezout theorem we get that if the system under consideration defines a complete intersection, i.e. has only isolated solutions then their number (counted with multiplicities) equals $\binom{n+r}{r}$ in the corresponding weighted projective space. To check the quasi-homogenuity note that the standard action of $\mathbb{C}^*$ on the set of roots of the polynomial $V(z)$ by simultaneous multiplication assigns the weight $r-j$ to its coefficient v_j. These weights are still valid in the reduced system with the variables $s_n, ..., s_0$ eliminated. Finally, we have to show that if the coefficients of $Q_k(z), ..., Q_1(z)$ are algebraically independent then the eliminated system has exactly $\binom{n+r}{r}$ simple solutions. Indeed, consider the linear space EQ of all systems of r quasi-homogeneous equations in the variables $(v_r, ..., v_0)$ with the weights $w(v_j) = r - j$ and where the i-th equation is weighted-homogeneous of degree $n + i$. We equip this space with the standard monomial basis.

$\square$

On eigenvalues for rectangular matrices

We start with the following natural question.

Problem. Given a $(l+1)$-tuple of $(m_1\times m_2)$-matrices $A, B_1, ..., B_l$ where $m_1 \leq m_2$ describe the set of all values of parameters $\lambda_1, ...\lambda_l$ for which the rank of the linear combination $A+\lambda_1 B_1+...+\lambda_l B_l$ is less than m_1 i.e. when the linear system $v*(A+\lambda_1 B_1+...+\lambda_l B_l) = 0$ has a nontrivial (left) solution $v \neq 0$ which we call an *eigenvector of A wrt the linear span of $B_1, ..., B_l$*.

Let $\mathcal{M}_{m_1,m_2}$ denote the linear space of all $(m_1\times m_2)$-matrices with complex entries. Below we will consider l-tuples of $(m_1 \times m_2)$-matrices $B_1, ..., B_l$ which are linearly independent in $\mathcal{M}_{m_1,m_2}$ and denote their linear span by $\mathcal{L} = \mathcal{L}(B_1, ..., B_l)$. Given a matrix pencil $\mathcal{P} = A + \mathcal{L}$ where $A \in \mathcal{M}_{m_1,m_2}$ denote by $\mathcal{E}_{\mathcal{P}} \subset \mathcal{P}$ its *eigenvalue locus*, i.e. the set of matrices in $\mathcal{P}$ whose rank is less than the maximal one. Denote by $\mathcal{M}^1 \subset \mathcal{M}_{m_1,m_2}$ the set of all $(m_1 \times m_2)$ matrices with positive corank, i.e whose rank is less than m_1. Its co-dimension equals $m_2 - m_1 + 1$ and its degree as an algebraic variety equals $\binom{m_2}{m_1-1}$. Consider the natural left-right action of the group $GL_{m_1} \times GL_{m_2}$ on $\mathcal{M}_{m_1,m_2}$, where GL_{m_1} (resp. GL_{m_2}) acts on $(m_1 \times m_2)$-matrices by the left (resp. right) multiplication. This action on $\mathcal{M}_{m_1,m_2}$ has finitely many orbits, each orbit being the set of all matrices of a given (co)rank. Notice that due to the well-known formula of the product of coranks the codimension of the set of matrices of rank $\leq r$ equals $(m_1 - r)(m_2 - r)$. Obviously, for any pencil $\mathcal{P}$ one has that the eigenvalue locus coincides with $\mathcal{E}_{\mathcal{P}} = \mathcal{M}^1 \cap \mathcal{P}$. Thus for a generic pencil $\mathcal{P}$ of dimension l the eigenvalue locus $\mathcal{E}_{\mathcal{P}}$ is a subvariety of $\mathcal{P}$ of codimension $m_2 - m_1 + 1$ if $l \geq m_2 - m_1 + 1$ and it is empty otherwise. The most interesting situation for applications occurs when $l = m_2 - m_1 + 1$ in which case $\mathcal{E}_{\mathcal{P}}$ is generically a finite set. From now on let us assume that $l = m_2 - m_1 + 1$. Denoting as above by $\mathcal{L}$ the linear span of $B_1, ..., B_l$ we say that $\mathcal{L}$ is *transversal to* $\mathcal{M}^1$ if the intersection $\mathcal{L} \cap \mathcal{M}^1$ is finite and *non-transversal to* $\mathcal{M}^1$ otherwise. Notice that due to homogeneity of $\mathcal{M}^1$ any $(m_2 - m_1 + 1)$-dimensional linear subspace $\mathcal{L}$ transversal to it intersects $\mathcal{M}^1$ only at 0 and that the multiplicity of this intersection at 0 equals $\binom{m_2}{m_1-1}$.

Proof of generalized Pólya's theorems

To settle Theorem 7 we will prove a number of localization results having an independent interest.

Definition 2. Given a finite (complex-valued) measure μ supported on $\mathbb{C}$ we call by its *total mass* the integral $\int_{\mathbb{C}} d\mu(\zeta)$. The Cauchy transform $\mathcal{C}_\mu(z)$ of μ is standardly defined as

$$\mathcal{C}_\mu(z) = \int_{\mathbb{C}} \frac{d\mu(\zeta)}{z-\zeta}. \tag{9}$$

Obviously, $\mathcal{C}_\mu(z)$ is analytic outside the support of μ and has a number of important properties, e.g. that $\mu = \frac{1}{\pi}\frac{\mathcal{C}_\mu(z)}{\partial \bar{z}}$ understood in the distributional sense.

Definition 3. Given a (monic) polynomial $P(z)$ of some degree m we associate with $P(z)$ its *root-counting measure* $\mu_P(z) = \frac{1}{m}\sum_j \delta(z-z_j)$ where $\{z_1, ..., z_m\}$ stands for the set of all roots of $P(z)$ with repetitions and $\delta(z-z_j)$ is the usual Dirac delta-function supported at z_j.

Directly from the definition of $\mu_P(z)$ one has that for any given polynomial $P(z)$ of degree m its Cauchy transform is given by $\mathcal{C}_{\mu_P}(z) = \frac{P'(z)}{mP(z)}$.

Proof. Take a pair $(V(z), S(z))$ where $V(z)$ is some Van Vleck polynomial and $S(z)$ is its corresponding Stieltjes polynomial of degree n. Let ξ be the root of either $V(z)$ or $S(z)$ which has the maximal modulus among all roots of the chosen $V(z)$ and $S(z)$. We want to show that there exists a radius $R > 0$ such that $|\xi| \leq R$ for any ξ as above and as soon as n is large enough. Substituting $V(z), S(z), \xi$ in (5) and using (4) we get the relation:

$$Q_k(\xi)S^{(k)}(\xi)+Q_{k-1}(\xi)S^{(k-1)}(\xi)+...+Q_1(\xi)S'(\xi) = 0$$

dividing which by its first term we obtain:

$$1 + \sum_{j=1}^{k-1} \frac{Q_j(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)} = 0. \qquad (10)$$

Notice that the rational function $b_i(z) := \frac{S^{(i+1)}(z)}{(n-i)S^{(i)}(z)}$ is the Cauchy transform of the polynomial $S^{(i)}(z)$. Easy arithmetic shows that

$$S^{(i)}(z) = \frac{S^{(k)}(z)}{(n-k+1)...(n-i)\prod_{j=i}^{k-1} b_j(z)}.$$

Notice additionally, that by the usual Gauss-Lucas theorem all roots of any $S^{(i)}(z)$ lie within the convex hull of the set of roots of $S(z)$. In particular, all these roots lie within the disk of radius $|\xi|$. Therefore, we get

$$\left|\frac{Q_i(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| \leq \frac{|Q_i(\xi)|}{|Q_k(\xi)|}\frac{(n-k)!}{(n-i)!}2^{k-i}|\xi|^{k-i}. \tag{11}$$

Notice that since $Q_k(z)$ is a monic polynomial of degree $k+r$ (recall that r is the Fuchs index of the operator $\mathfrak{d}(z)$) then one can choose a radius R such that for any z with $|z| > R$ one has $|Q_k(z)| \geq \frac{|z|^{k+r}}{2}$. Now since for any $i = 1, ..., k-1$ one has $\deg Q_i(z) \leq i+r$ we can choose a positive constant K such that $|Q_i(z)| \leq K|z|^{i+r}$ for all $i = 1, ..., k-1$ and $|z| > R$. We want to show that ξ can not be too large for a sufficiently large n.

Using our previous assumptions and assuming additionally that $|\xi| > R$ we get

$$\left|\frac{Q_i(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| \leq \frac{|Q_i(\xi)|}{|Q_k(\xi)|}\frac{(n-k)!}{(n-i)!}2^{k-i}|\xi|^{k-i}.$$

Now we can finally choose N_0 large enough such that for all $n \geq N_0$, all $i = 1, ..., k-1$ and any $|\xi| > R$ one has that

$$\left|\frac{Q_i(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| \leq \frac{K \cdot 2^{k-i+1}}{(n-i)...(n-k+1)} < \frac{1}{k-1}.$$

But then obviously the relation (10) can not hold for all $n \geq N_0$ and any $|\xi| > R$ since

$$\left|\sum_{j=1}^{k-1}\frac{Q_j(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| \leq \sum_{j=1}^{k-1}\left|\frac{Q_j(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| < \sum_{i=1}^{k-1}\frac{1}{k-1}.$$

$\square$

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