# Random matrix pencils and level crossings

## Albeverio Fest

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# Topics to discuss

1. Basic level crossing problem
2. Level crossing for random matrices
3. Monodromy statistics

# Main references

(i) B. Shapiro, M. Tater, On spectral asymptotics of quasi-exactly solvable sextic, Experimental Mathematics, https://doi.org/10.1080/10586458.2017.1325792.

(ii) B. Shapiro, M. Tater, On spectral asymptotics of quasi-exactly solvable quartic and Yablonskii-Vorob'ev polynomials, arXiv:1412.3026, submitted.

(iii) B. Shapiro, K. Zarembo, On level crossing in random matrix pencils. I. Random perturbation of a fixed matrix, Journal of Physics A: Mathematical and Theoretical, Volume 50(4).

(iv) T. Grøsfjeld, B. Shapiro, K. Zarembo, On level crossing in random matrix pencils. II. Random perturbation of a random matrix, arXiv: 1806.08732, submitted.

*Fundamental problem.* Given two linear operators $A$ and $B$ with discrete spectrum, consider the pencil $A + \lambda B$, where $\lambda$ is a complex parameter.

(i) Determine the level crossing set $LC \subset \mathbb{C}$ consisting of all values of $\lambda$ for which the operator $A + \lambda B$ has a multiple eigenvalue;

(ii) In case when $LC$ is discrete, determine the spectral monodromy of the latter pencil, i.e., the permutations of the eigenvalues when $\lambda$ traverses different closed loops in $\mathbb{C} \setminus LC$.

The problem seems quite difficult, in general. Especially its monodromy part!

# Complex Gaussian ensembles

Recall that the complex (non-symmetric) Gaussian ensemble $GE_n^{\mathbb{C}}$ is the distribution on the space $Mat_n^{\mathbb{C}}$ of all complex-valued $n \times n$-matrices, where each entry of a random $n \times n$-matrix is an independent complex Gaussian variable distributed as $N(0, \frac{1}{2}) + iN(0, \frac{1}{2})$. Our initial result is as follows.

## Theorem

*For any positive integer n, if the matrices A and B are independently chosen from $GE_n^{\mathbb{C}}$, then the distribution of level crossings in $A + \lambda B$ with respect to the affine coordinate $\lambda = x + iy$ of $\mathbb{C}$ is given by*

$$\mathcal{P}_{GE_n^{\mathbb{C}}}(\lambda) := \mathcal{P}_{GE_n^{\mathbb{C}}}(x, y)dxdy = \frac{dxdy}{\pi(1 + x^2 + y^2)^2} = \frac{dxdy}{\pi(1 + |\lambda|^2)^2}. \quad (1)$$

## Remark

In polar coordinates $(r, \theta)$ in the complex plane of parameter $\lambda$, the above distribution $\mathcal{P}_{GE_n^{\mathbb{C}}}(\lambda)$ has the form

$$\mathcal{P}_{GE_n^{\mathbb{C}}}(r, \theta) dr d\psi = \frac{r dr d\theta}{\pi(1 + r^2)^2},$$

giving the radial CDF of the form

$$\Psi_{GE_n^{\mathbb{C}}}(r) = \frac{r^2}{1 + r^2}.$$

Let us realize $\mathbb{CP}^1 \simeq S^2$ as the unit sphere in $\mathbb{R}^3$ with coordinates $(X, Y, Z)$ and identify the complex plane of parameter $\lambda = x + iy$ with the horizontal coordinate $(X, Y)$-plane, where $X$ corresponds to the real axis and $Y$ corresponds to the imaginary axis in $\mathbb{C}$. If we use the standard stereographic projection of the unit sphere in $\mathbb{R}^3$ from its north pole, i.e. from the point $(0, 0, 1)$ onto the $(X, Y)$-plane, then the usual area element of the sphere induced from the standard Euclidean structure in $\mathbb{R}^3$ is given by

$$dA = \frac{4dxdy}{(1 + x^2 + y^2)^2} = \frac{4dxdy}{(1 + |\lambda|^2)^2}.$$

The latter fact implies that the r.h.s. of (1) presents the constant density $\frac{1}{4\pi}$ with respect to the standard Euclidean area measure on $S^2 \simeq \mathbb{C}P^1$ compactifying the complex plane of parameter $\lambda$.

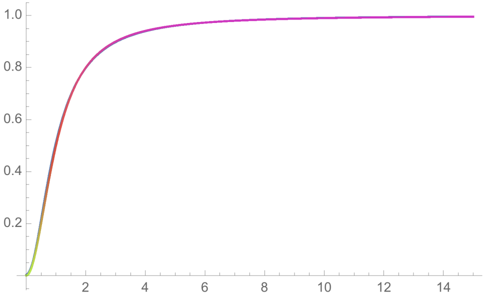



Figure: Radial density of level crossings for $A + \lambda B$, where $A$ and $B$ are independently sampled from $GE_6^{\mathbb{C}}$; (we used 100 random pairs). The above diagram shows a perfect match of the numerical distribution of the absolute values of level crossings obtained in our sampling with the theoretical radial CDF $\frac{r^2}{1+r^2}$.

In fact, we can prove the following generalisation of Theorem 1.

## Proposition

*Conclusion of Theorem 1 holds, if A and B are independently chosen from the scaled complex Gaussian ensemble* $GE^{\mathbb{C}}_{\sigma^2,n}$, *i.e., the ensemble whose off-diagonal entries are i.i.d. standard normal complex variables and whose diagonal entries are i.i.d. normal complex variables with an arbitrary fixed positive variance* $\sigma^2$.

Moreover consider the following $SU_2$-action on $Mat_n^{\mathbb{C}} \times Mat_n^{\mathbb{C}}$. A matrix $\mathfrak{U} \in SU_2$ given by $\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}, \ |u|^2 + |v|^2 = 1$ acts on the latter product space by:

$$(A, B) * \mathfrak{U} \mapsto (uA + vB, -\bar{v}A + \bar{u}B). \tag{2}$$

Take any complex linear subspace $W_n \subset Mat_n^{\mathbb{C}}$ such that the product space $W_n \times W_n \subset Mat_n^{\mathbb{C}} \times Mat_n^{\mathbb{C}}$ is preserved by the action (2). Given $\sigma > 0$, denote by $W_{\sigma^2,n}$ the space $W_n$ with the measure induced from the scaled complex Gaussian ensemble $GE^{\mathbb{C}}_{\sigma^2,n}$.

## Proposition

*In the above notation, level crossings of $A + \lambda B$ with the random matrices A and B independently chosen from $W_{\sigma^2,n}$ are uniformly distributed on $\mathbb{C}P^1$, i.e., their probability measure is given by the right-hand side of* (1).

To give an example of such $W$, recall that $GOE_n^{\mathbb{C}}$ is the distribution on the space $Sym_n^{\mathbb{C}}$ of complex-valued symmetric matrices, where each entry $e_{i,j} = e_{j,i},\ i < j$ of a $n \times n$-matrix has a normal distribution $N(0, 1/2) + iN(0, 1/2)$, and each diagonal entry $e_{i,i}$ is distributed as $\sqrt{2}(N(0, 1/2) + iN(0, 1/2))$.

Further interesting examples of linear subspaces $W$ covered by Proposition 4 include Toeplitz matrices, band matrices, band Toeplitz matrices, diagonal matrices, etc.

# Gaussian orthogonal and Gaussian unitary ensembles

A very essential feature of these cases is that their level crossings distribution is invariant under the action of the subgroup $SO_2 \subset SU_2$ given by the same formula (2), but with real $u$ and $v$ satisfying $u^2 + v^2 = 1$.

In the above realization of $\mathbb{C}P^1$ as the unit sphere $S^2 \subset \mathbb{R}^3$, $SO_2$ acts on it by rotation around the $Y$-axis. This circumstance implies that the family of orbits of the $SO_2$-action on the unit sphere $S^2 \simeq \mathbb{C}P^1$ projected to the complex plane of parameter $\lambda = x + iy$ will coincide with the family of circles given by

$$x^2 + (y - t)^2 = t^2 - 1, \quad |t| \geq 1.$$

Let us introduce the cylindrical coordinates $(\rho, \psi, Y)$ where $X = \rho\cos\psi,\ \ Y = Y,\ Z = \rho\sin\psi$. Then $(\psi, Y)$, $0 \le \psi \le 2\pi,\ -1 \le Y \le 1$ again parameterises the unit sphere $S^2 \simeq \mathbb{C}P^1$. $SO_2$-action implies that in the cylindrical coordinates $(\psi, Y)$ the distributions of level crossings of the above ensembles on $\mathbb{C}P^1$ are of the form:

$$dens(\psi, Y)d\psi dY = \rho(Y)d\psi dY,$$

for some univariate function $\rho$, i.e., its density depends only on $Y$ and is independent of the angle variable $\psi$. (In general, $\rho(Y)dY$ can be a 1-dimensional measure which not necessarily has a smooth density function.) In the affine coordinates,

$$dens(x, y)dxdy = \rho\left(\frac{2y}{x^2+y^2+1}\right)\frac{4dxdy}{(x^2+y^2+1)^2}, \qquad (3)$$

with the same $\rho$ as above.

In the case of $GOE_n^{\mathbb{R}}$ we can show the following.

## Theorem

*If the matrices A and B are independently chosen from $GOE_2^{\mathbb{R}}$, then the distribution of level crossings in $A + \lambda B$ is uniform on $\mathbb{CP}^1 \supset \mathbb{C}$, i.e., their density is given by the right-hand side of (1).*

Extensive numerical experiments strongly support the following guess.

## Conjecture

*For any size $n > 2$, if the matrices A and B are independently chosen from $GOE_n^{\mathbb{R}}$, then the distribution of level crossings in $A + \lambda B$ is uniform on $\mathbb{CP}^1 \supset \mathbb{C}$.*

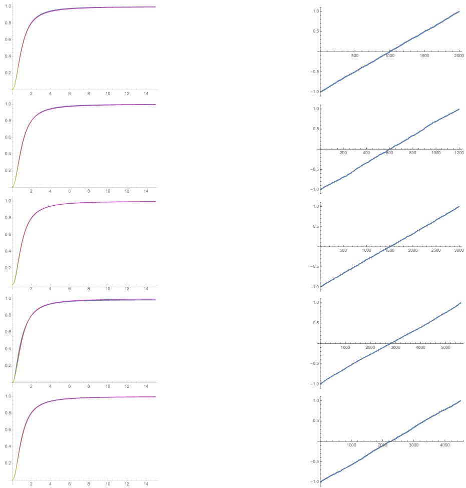


Figure: Numerical and theoretical radial and angle CDF for $A + \lambda B$ with $A$ and $B$ from $GOE_n$, for $n = 2, 4, 6, 8, 10$.

# Gaussian unitary ensemble

## Theorem

*If the matrices A and B are independently chosen from $GUE_2$, then the distribution of level crossings in $\mathbb{C}$ is given by*

$$\mathcal{P}_{GUE_2}(x,y) = \frac{4|y|dxdy}{\pi(1+x^2+y^2)^3} = \frac{1}{\pi}\left|\frac{y}{1+x^2+y^2}\right| \frac{4dxdy}{(1+x^2+y^2)^2}. \quad (4)$$

In the cylindrical coordinates $(\psi, Y)$ on $\mathbb{C}P^1$, where $0 \leq \psi \leq 2\pi$ and $-1 \leq Y \leq 1$, one has

$$\mathcal{P}_{GUE_2}(\psi, Y)d\psi dY = \frac{|Y|d\psi dY}{2\pi}. \quad (5)$$

Unfortunately, we do not have explicit (conjectural) formulas for the densities $\mathcal{P}_{GUE_n}(x, y)$, for $n \geq 3$ similar to (5). However, we carried out substantial numerical experiments for matrix sizes up to 6. They were conducted as follows. For each $n \in \{2, ..., 6\}$, sampling independently pairs of $GUE_n$-matrices, we calculated 12,000 level crossing points for every $n$ and plotted the values of $|Y|$ for obtained level crossings in increasing order, see next Figure. These numerical experiments strongly suggest the following.

**Conjecture**

*There exists a limiting distribution* $\mathcal{P}_{GUE_\infty}(Y) := \lim_{n\to\infty} \mathcal{P}_{GUE_n}(Y)$.

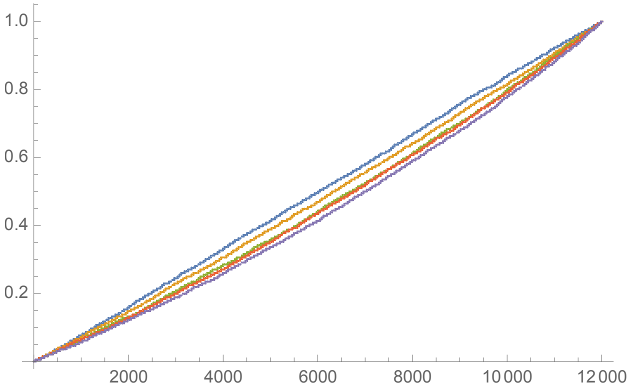



Figure: Empirical distributions of $|Y|$ for $A + \lambda B$ taken from $GUE_n$ with $n = 2, 3, 4, 5, 6$. (Curves corresponding to the increasing values of $n$ lie one below the other; the blue straight line corresponds to $n = 2$, see (5).)

# Real Gaussian ensemble

Similarly to the above, we were not able to get the explicit formulas for the distributions of level crossings of $GE_n^{\mathbb{R}}$ with $n \geq 3$, But the numerical experiments strongly suggest the validity of the following guess.

## Conjecture

*The average of the number real crossing points for A and B independently sampled from $GE_n^{\mathbb{R}}$ equals $\sqrt{n(n-1)}$.*

## Conjecture

*When $n \to \infty$, the level crossing distribution for A and B independently sampled from $GE_n^{\mathbb{R}}$ approaches the uniform distribution on $\mathbb{CP}^1$.*

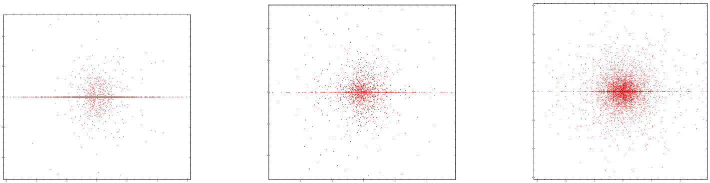


Figure: Distributions of level crossings in the $\lambda$-plane when $A$ and $B$ are sampled from $GE_n^{\mathbb{R}}$ for $n = 2, 5, 10$ apparently approaching the uniform distribution on $\mathbb{C}P^1$.

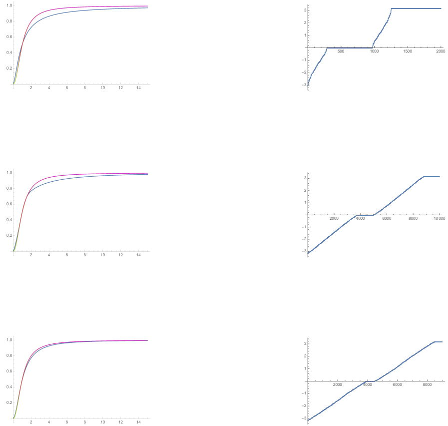


Figure: Radial and angle distributions of level crossings with $A$ and $B$ sampled from $GE_n^{\mathbb{R}}$ with $n = 2, 5, 10$ approaching that of the uniform distribution on $\mathbb{C}P^1$.

# Monodromy in the $GOE_3$- and $GUE_3$-cases

If $\lambda_i$ is the *i*-th level crossing point in the upper half-plane in the order of increasing real parts, consider the path in the $\lambda$-plane starting on the real axis at $\tau = Re(\lambda_i)$, going straight up to $\lambda_i$, making a small loop encircling $\lambda_i$ counterclockwise, and returning back to $\tau_i$.

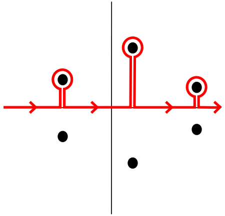
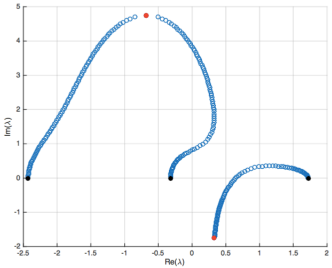


Figure: Creating the monodromy sequence

As a result, one gets a transposition $\sigma_i$ of two real eigenvalues corresponding to $\tau_i = Re(\lambda_i)$. Doing this for each $\lambda_i$, $i = 1, \dots, \binom{n}{2}$, we obtain a sequence of $\binom{n}{2}$ transpositions $\left(\sigma_1, \sigma_2, \dots, \sigma_{\binom{n}{2}}\right)$, $\sigma_i \in S_n$.
One can easily check that the obtained sequence $\left(\sigma_1, \sigma_2, \dots, \sigma_{\binom{n}{2}}\right)$ of transpositions satisfies the following two conditions:

(i) for general $A$ and $B$, they generate the symmetric group $S_n$;
(ii) the product $\sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_{\binom{n}{2}}$ coincides with the inverse permutation $(n, n-1, \dots, 1)$.

For $n = 3$, it is easy to check that there are only 8 triples of transpositions in $S_3$ satisfying conditions (i) and (ii). These triples are: (12)(12)(13); (12)(13)(23); (12)(23)(12); (13)(12)(12);
(13)(23)(23); (23)(12)(23); (23)(13)(12); (23)(23)(13). (For comparison, for $n = 4$, there are already 3840 6-tuples of transpositions in $S_4$ satisfying (i) and (ii).)



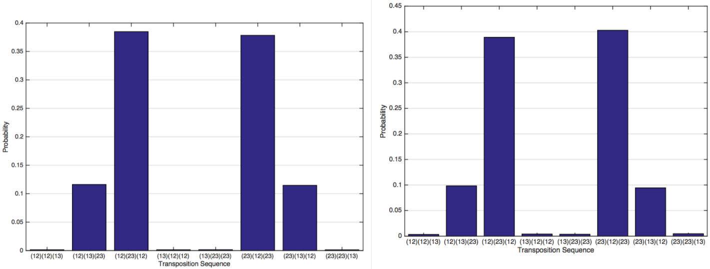



Figure: The probabilities of the monodromy triples of transpositions for $GUE_3$- and $GOE_3$-matrices.

# Monodromy in the $GE_3^{\mathbb{C}}$-case.

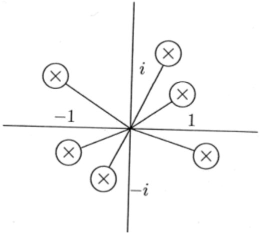



Figure: An example of paths in the $\lambda$-plane chosen to determine the monodromy for pairs $(A, B)$ from $GE_3^{\mathbb{C}}$.

For $A$ and $B$ in $GE_3^{\mathbb{C}}$, there are 240 sequences of 6-tuples of transpositions $(\sigma_1, \sigma_2, \dots, \sigma_6)$ from $S_3$ satisfying the conditions:
(i) they generate the symmetric group $S_3$;
(ii) the product $\sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_6$ coincides with the identity permutation $(1, 2, 3)$.

We generated 150000 random matrix pairs in $GE_3^{\mathbb{C}}$ and calculated their monodromy sequences. Our numerical results show the following:

(i) Of the 240 possible cases, only 209 were realized and only 204 were realized more than once.
(ii) The most common monodromy sequences were of the form (23)(12)(23)(12)(23)(12), which occurred 2.43 % of the time and of the form (12)(13)(13)(23)(23)(12) which occured 2.29 % of the time.
(iii) Monodromy sequences in which one permutation occurs four times in a row followed by two occurrences of another permutation and their cyclic permutations (for example, (12)(12)(12)(12)(13)(13) or (12)(23)(23)(23)(23)(12)) were the most rare, occurring only once or not at all.

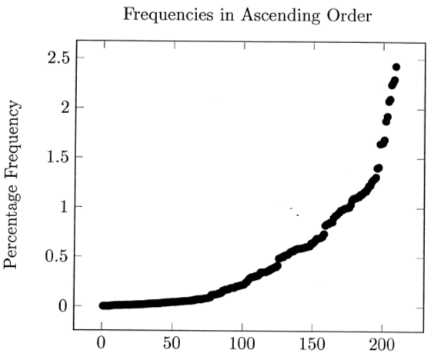



Figure: Frequencies of 240 possible 6-tuples of transpositions from $S_3$ in the ascending order.

# Open problems

There were many unsolved questions during the talk. In particular, although the simple (conjectural) answer for level crossing distribution in the GOE-case presented in Theorem 2 and Conjecture 1 indicates the possible existence of some extra symmetry complementing the above $SO_2$-action, we were not able to find such.

Tack för ert tålamod!