Several algebras associated to a (multi)graph

(joint with G. Nenashev, A, Postnikov, and M. Shapiro)

Boris Shapiro, Stockholm University

November 8, 2018

Topics to discuss

1. Basic algebra, SL_n/B
2. Trees counting algebras associated to directed graphs
3. Power algebras associated to undirected graphs
4. Other analogs

Main references

(i) B. Shapiro, M. Shapiro, On ring generated by Chern 2-forms on $\mathbb{SL}_n/B$, C. R. Acad. Sci. Paris Sér. I Math. vol 326, issue 1 (1998) 75–80.

(ii) A. Postnikov and B. Shapiro, Trees, parking functions, syzygies, and deformations of monomial ideals, Trans. Amer. Math. Soc. vol 356, issue 8 (2004) 3109–3142.

(iii) G. Nenashev, B. Shapiro, “K-theoretic" analogs of Postnikov-Shapiro algebra distinguishes graphs, Journal of Combinatorial Theory, Series A 148 (2017) 316–332.

(iv) A. N. Kirillov, G, Nenashev, On Q-deformations of Postnikov-Shapiro algebras Séminaire Lotharingien de Combinatoire, 78B.55, FPSAC 2017, 12 pp.

Basic algebra, SL_n/B

Interpret $SL_n/B = \mathcal{U}_n/T^n$ as the space of complete flags in $\mathbb{C}^n$ and take the standard sequence of tautological bundles

$$0 \subset E_1 \subset \ldots \subset E_n = E$$

(where E is the trivial $\mathbb{C}^n$-bundle over SL_n/B) and the corresponding n-tuple of quotient line bundles $L_i = E_i/E_{i-1}$.

Fixing some Hermitian metric on the original $\mathbb{C}^n$ one equips every bundle E_i, L_i and $E_i/E_j,\ i > j$ with the induced Hermitian metric.

Denote by w_i the curvature form of the above Hermitian metric on L_i. (Each w_i is a $\mathcal{U}_n$-invariant 2-form on SL_n/B such that $\frac{\sqrt{-1}w_i}{2\pi}$ represents the first Chern class $c_1(L_i)$ in $H^2(SL_n/B)$.)

Setting $x_i = c_1[L_i]$ one has

$$H^*(SL_n/B, \mathbb{Z}) = \frac{\mathbb{Z}[x_1, ..., x_n]}{(s_1, s_2, ..., s_n)},$$

where s_i stands for the ith elementary symmetric functions in variables $x_1, ..., x_n$.

Problem. Study the $\mathbb{Z}$-ring $\mathcal{B}_n = \mathbb{Z}(w_1, ..., w_n)$ generated by all w_is and compare it to $H^*(SL_n/B, \mathbb{Z})$.

Remark. One has the standard surjective ring homomorphism $\pi : \mathcal{B}_n \to H^*(SL_n/B, \mathbb{Z})$.

$\mathcal{B}_n$ as a subalgebra of a square-free algebra

Using results of Griffiths-Schmid (Acta Math., v.123, 1969) about the curvature forms on the homogeneous spaces, one can present w_is as follows.

Example of $\mathcal{B}_4$.

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 & +a & +b & +c \\ -a & +0 & +d & +e \\ -b & -d & +0 & +f \\ -c & -e & -f & +0 \end{pmatrix},$$

where $a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = 0$ with no other relations. Then one has $w_i^4 = 0$; $(w_i + w_j)^5 = 0$; $(w_i + w_j + w_k)^4 = 0$; $w_1 + w_2 + w_3 + w_4 = 0$.

Simpler example. The ring $\mathcal{B}_3$ is isomorphic to $\frac{\mathbb{Z}[w_1,w_2,w_3]}{I_3}$, where I_3 is generated by

$$w_1^3, w_2^3, w_3^3, (w_1+w_2)^3, (w_1+w_3)^3, (w_2+w_3)^3, w_1+w_2+w_3.$$

The Hilbert polynomial of $\mathcal{B}_3$ equals

$$H(t) = 1 + 2t + 3t^2 + t^3.$$

(For comparison, the Poincaré polynomial of SL_3/B equals $1 + 2t + 2t^2 + t^3$.)

Proposition 1. $\mathcal{B}_n$ is a graded ring isomorphic to $\frac{\mathbb{Z}[w_1,\dots,w_n]}{I_n}$, where the ideal I_n is generated by the set of $2^n - 1$ polynomials of the form

$$g^{(n)}_{i_1,\dots,i_j} = (w_{i_1} + \dots + w_{i_j})^{j(n-j)+1}, \qquad (1)$$

where $\{i_1, \dots, i_j\}$ runs over the set of all nonempty subsets in the set $\{1, \dots, n\}$.

Proposition 2. The total dimension of $\mathcal{B}_n$ equals the number of forests on n labeled vertices and there exists a natural monomial basis for $\mathcal{B}_n$ whose monomials are enumerated by the above forests.

Directed graphs

Let G be a digraph on the set of vertices $0, 1, \ldots, n$ (with possible multiple egdes, but no loops). The vertex 0 will be the root of G. The digraph G is determined by its *adjacency matrix* $A = (a_{ij})_{0 \le i,j \le n}$, where a_{ij} is the number of edges from the vertex i to the vertex j. We will regard usual graphs as a special case of digraphs with symmetric adjacency matrix A.

An *oriented spanning tree* T of the digraph G is a subgraph $T \subset G$ such that there exists a unique directed path in T from any vertex i to the root 0. The number N_G of such trees is given by the *Matrix-Tree Theorem*:

$$N_G = \det L_G, \qquad (1)$$

where $L_G = (l_{ij})_{1 \le i,j \le n}$ the *truncated Laplace matrix*,

L_G is also known as the *Kirkhoff matrix*, given by

$$l_{ij} = \begin{cases} \sum\limits_{r\in\{0,\dots,n\}\setminus\{i\}} a_{ir} & \text{for } i = j, \\ -a_{ij} & \text{for } i \neq j. \end{cases} \qquad (2)$$

If G is a graph, i.e., A is a symmetric matrix, then oriented spanning trees defined above are exactly the usual *spanning trees* of G, which are connected subgraphs of G without cycles.

For a subset I in $\{1, \dots, n\}$ and a vertex $i \in I$, let

$$d_I(i) = \sum_{j \notin I} a_{ij},$$

i.e., $d_I(i)$ is the number of edges from the vertex i to a vertex outside of the subset I.

Parking functions

A *parking function* of size n is a sequence $b = (b_1, \dots, b_n)$ of non-negative integers such that its increasing rearrangement $c_1 \le \dots \le c_n$ satisfies $c_i < i$. Equivalently, we can formulate this condition as $\#\{i \mid b_i < r\} \ge r$, for $r = 1, \dots, n$.

The parking functions of size n are known to be in bijective correspondence with trees on $n+1$ labelled vertices. Thus, according to Cayley's formula for the number of labelled trees, the total number of parking functions of size n equals $(n+1)^{n-1}$.

Let us say that a sequence $b = (b_1, \dots, b_n)$ of non-negative integers is a *G-parking function* if, for any nonempty subset $I \subseteq \{1, \dots, n\}$, there exists $i \in I$ such that $b_i < d_I(i)$.
If $G = K_{n+1}$ is the complete graph on $n+1$ vertices then K_{n+1}-parking functions are the usual parking functions of size n defined above.

Theorem

The number of G-parking functions equals the number $N_G = \det L_G$ of oriented spanning trees of the digraph G.

We can reformulate the definition of G-parking functions in algebraic terms as follows. Throughout this paper we fix a field $\mathbb{K}$. Let $\mathcal{I}_G = \langle m_I \rangle$ be the monomial ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials

$$m_I = \prod_{i \in I} x_i^{d_I(i)}, \tag{3}$$

where I ranges over all nonempty subsets $I \subseteq \{1, \ldots, n\}$.

Define the algebra $\mathcal{A}_G^T$ as the quotient

$$\mathcal{A}_G^T = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_G.$$

A integer sequence $b = (b_1, \dots, b_n)$ is a G-parking function if and only if the monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ is nonvanishing in the algebra $\mathcal{A}_G^T$.

For a monomial ideal $\mathcal{I}$, the set of all monomials that do not belong to $\mathcal{I}$ is a basis of the quotient of the polynomial ring modulo $\mathcal{I}$, called the *standard monomial basis.* Thus the monomials x^b, where b ranges over G-parking functions, form the standard monomial basis of the algebra $\mathcal{A}_G$.

Corollary

$\mathcal{A}_G^T$ is a finite-dimensional linear space over $\mathbb{K}$. Its dimension is equal to the number of oriented spanning trees of the digraph G:

$$\dim \mathcal{A}_G^T = N_G.$$

Undirected graphs

Let G be an undirected graph on the set of vertices $0, 1, \ldots, n$. In this case the dimension of the algebra $\mathcal{A}_G^T$ is equal to the number of usual spanning trees of G.

For a nonempty subset I in $\{1, \ldots, n\}$, let $D_I = \sum_{i \in I, j \notin I} a_{ij} = \sum_{i \in I} d_I(i)$ be the total number of edges that join some vertex in I with a vertex outside of I. For any nonempty subset $I \subseteq \{1, \ldots, n\}$, let

$$p_I = \left(\sum_{i \in I} x_i \right)^{D_I}. \tag{4}$$

Let $\mathcal{J}_G = \langle p_I \rangle$ be the ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials p_I for all nonempty subsets I. Define the algebra $\mathcal{B}_G^T$ as the quotient

$$\mathcal{B}_G^T = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_G.$$

The algebras $\mathcal{A}_G^T$ and $\mathcal{B}_G^T$ are graded. For a graded algebra $\mathcal{A}^T = \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \cdots$, the *Hilbert series* of $\mathcal{A}^T$ is the formal power series in q given by

$$\mathrm{Hilb}\,\mathcal{A}^T = \sum_{k \geq 0} q^k \dim \mathcal{A}^k.$$

Theorem

The monomials x^b, where b ranges over G-parking functions, form a linear basis of the algebra $\mathcal{B}_G^T$. Thus the Hilbert series of the algebras $\mathcal{A}_G^T$ and $\mathcal{B}_G^T$ coincide termwise: $\mathrm{Hilb}\,\mathcal{A}_G^T = \mathrm{Hilb}\,\mathcal{B}_G^T$. In particular, both these algebras are finite-dimensional as linear spaces over $\mathbb{K}$ and

$$\dim \mathcal{A}_G^T = \dim \mathcal{B}_G^T = N_G$$

is the number of spanning trees of the graph G.

Example

Let $n = 3$ and let G be the graph given by

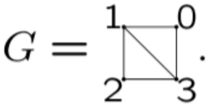

Figure: Example of a graph.

The graph G has 8 spanning trees:

The ideals $\mathcal{I}_G$ and $\mathcal{J}_G$ are given by

$$\mathcal{I}_G = \left\langle x_1^3, x_2^2, x_3^3, x_1^2 x_2, x_1^2 x_3^2, x_2 x_3^2, x_1 x_2^0 x_3 \right\rangle,$$

$$\mathcal{J}_G = \left\langle x_1^3, x_2^2, x_3^3, (x_1+x_2)^3, (x_1+x_3)^4, (x_2+x_3)^3, (x_1+x_2+x_3)^2 \right\rangle.$$

The standard monomial basis of the algebra $\mathcal{A}_G^T$ is $\{1, x_1, x_2, x_3, x_1^2, x_1 x_2, x_2 x_3, x_3^2\}$. The corresponding G-parking functions are the exponent vectors of the basis elements:

$$(0,0,0),(1,0,0),(0,1,0),(0,0,1),(2,0,0),$$

$$(1,1,0),(0,1,1),(0,0,2).$$

We have $\dim \mathcal{A}_G^T = \dim \mathcal{B}_G^T = 8$ is the number of spanning trees of G, and $\mathrm{Hilb}\, \mathcal{A}_G^T = \mathrm{Hilb}\, \mathcal{B}_G^T = 1 + 3q + 4q^2$.

We will refine Theorem 3 and interpret dimensions of graded components of the algebras $\mathcal{A}_G^T$ and $\mathcal{B}_G^T$ in terms of certain statistics on spanning trees. Let us fix a linear ordering of all edges of the graph G.

For a spanning tree T of G, an edge $e \in G \setminus T$ is called *externally active* if there exists a cycle C in the graph G such that e is the minimal edge of C and $(C \setminus \{e\}) \subset T$. The *external activity* of a spanning tree is the number of externally active edges. Let N_G^k denote the number of spanning trees $T \subset G$ of external activity k. Even though the notion of external activity depends on a particular choice of ordering of edges, the numbers N_G^k are known to be invariant on the choice of ordering.

Let $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ be the k-th graded components of the algebras $\mathcal{A}_G^T$ and $\mathcal{B}_G^T$, correspondingly.

Theorem

The dimensions of the k-th graded components $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ are equal to

$$\dim \mathcal{A}_G^k = \dim \mathcal{B}_G^k = N_G^{|G|-n-k},$$

the number of spanning trees of G of external activity $|G| - n - k$, where $|G|$ denotes the number of edges of G.

$\mathcal{C}_G^F$-algebra

We introduce the following algebra $\mathcal{C}_G^F$ associated to an arbitrary vertex-labeled undirected graph G without loops on the vertex set $[n]$. Let Φ_G be the graded commutative algebra over $\mathbb{K}$ generated by the variables ϕ_e, $e \in G$, with the defining relations: $(\phi_e)^2 = 0, \quad$ for every edge $e \in G$. Let $\mathcal{C}_G^F$ be the subalgebra of Φ_G generated by the elements

$$X_i = \sum_{e \in G} c_{i,e} \phi_e,$$

for $i \in [n]$, where

$$c_{i,e} = \begin{cases} 1 & \text{if } e = (i,j),\ i < j; \\ -1 & \text{if } e = (i,j),\ i > j; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Observe that we assume that $\mathcal{C}_G^F$ contains 1.

To describe the relations between X_i. consider the ideal J_G in the ring $\mathbb{K}[x_1, \cdots, x_n]$ generated by

$$p_I = \left(\sum_{i \in I} x_i\right)^{d_I+1},$$

where I ranges over all nonempty subsets of vertices, and d_I is the total number of edges between vertices in I and vertices outside I, i.e., belonging to $V(G) \setminus I$. Define the algebra $\mathcal{B}_G^F$ as the quotient

$$\mathbb{K}[x_1, \dots, x_n]/J_G.$$

Theorem

For any graph G, the algebras $\mathcal{B}_G^F$ and $\mathcal{C}_G^F$ are isomorphic, their total dimension over $\mathbb{K}$ is equal to the number of spanning forests in G.

Moreover, the dimension of the k-th graded component of these algebras equals the number of spanning forests F of G with external activity $e(G) - e(F) - k$.

In particular, the Hilbert polynomial of $\mathcal{C}_G^F$ is a specialization of the Tutte polynomial of G.

Corollary

Given a graph G, the Hilbert polynomial $\mathcal{H}_{\mathcal{C}_G^F}(t)$ of the algebra $\mathcal{C}_G^F$ is given by

$$\mathcal{H}_{\mathcal{C}_G^F}(t) = T_G\left(1+t, \frac{1}{t}\right) \cdot t^{e(G)-v(G)+c(G)}.$$

Theorem (G. Nenashev)

Given two graphs G_1 and G_2, the algebras $\mathcal{C}_{G_1}^F$ and $\mathcal{C}_{G_2^F}$ are isomorphic if and only if the graphical matroids of G_1 and G_2 coincide.

"K-theoretical" analog

In the above notation, our main object here will be the filtered subalgebra $\mathcal{K}_G \subset \Phi_G$ defined by the generators:

$$Y_i = \exp(X_i) = \prod_{e \in G} (1 + c_{i,e}\phi_e), \; i = 0, \dots, n.$$

Remark

Since Y_i is obtained by exponentiation of X_i, we call $\mathcal{K}_G$ the "K-theoretic" analog of $\mathcal{C}_G^F$. The original generators X_i are similar to the first Chern classes, while their exponentiations Y_i are similar to the Chern characters which are the main object of K-theory.

Relations

Define the ideal $\mathcal{I}_G$ in $\mathbb{K}[y_0, y_1, \ldots, y_n]$ as generated by the polynomials

$$q_I = \left(\prod_{i \in I} y_i - 1\right)^{D_I + 1}, \tag{6}$$

where I ranges over all nonempty subsets in $\{0, 1, \ldots, n\}$ and the number D_I is the number of edges connecting the subset I of vertices with its complement. Set

$$\mathcal{D}_G := \mathbb{K}[y_0, \ldots, y_n]/\mathcal{I}_G.$$

Theorem

For any graph G, algebras $\mathcal{B}_G^F$, $\mathcal{C}_G^F$, $\mathcal{D}_G$ and $\mathcal{K}_G$ are isomorphic as (non-filtered) algebras.

Moreover, the following stronger statement holds.

Theorem

For any graph G, algebras $\mathcal{D}_G$ and $\mathcal{K}_G$ are isomorphic as filtered algebras.

The filtered algebras $\mathcal{D}_G$ and $\mathcal{K}_G$ contain complete information about G.

Theorem

Given two graphs G_1 and G_2 without isolated vertices, $\mathcal{K}_{G_1}$ and $\mathcal{K}_{G_2}$ are isomorphic as filtered algebras if and only if G_1 and G_2 are isomorphic.

Further generalizations

Now we consider the Hilbert series of other filtered algebras similar to $\mathcal{K}_G$. (Recall that the Hilbert series of a filtered algebra is, by definition, the Hilbert series of its associated graded algebra.)

Let f be a univariate polynomial or a formal power series over $\mathbb{K}$. We define the subalgebra $\mathcal{F}[f]_G \subset \Phi_G$ as generated by 1 together with

$$f(X_i) = f\left(\sum c_{i,e}\phi_e\right), \ i = 0, \dots, n.$$

Example

For $f(x) = x$, $\mathcal{F}[f]_G$ coincides with $\mathcal{C}_G^F$. For $f(x) = \exp(x)$, $\mathcal{F}[f]_G$ coincides with $\mathcal{K}_G$.

Obviously, the filtered algebra $\mathcal{F}[f]_G$ does not depend on the constant term of f. From now on, we assume that $f(x)$ has no constant term, since for any g such that $f - g$ is constant, the filtered algebras $\mathcal{F}[f]_G$ and $\mathcal{F}[g]_G$ are the same.

Proposition

Let f be any polynomial with a non-vanishing linear term. Then the algebras $\mathcal{C}_G^F$ and $\mathcal{F}[f]_G$ coincide as subalgebras of Φ_G.

Theorem

Let f be any polynomial with non-vanishing linear and quadratic terms. Then given two simple graphs G_1 and G_2 without isolated vertices, $\mathcal{F}[f]_{G_1}$ and $\mathcal{F}[f]_{G_2}$ are isomorphic as filtered algebras if and only if G_1 and G_2 are isomorphic graphs.

Generic functions f and their Hilbert series

Since $X_i^{d_i+1} = 0$ for any i, we can always truncate any polynomial (or a formal power series) f at degree $|G| + 1$ without changing $\mathcal{F}[f]_G$. Therefore, for a given graph G, it suffices to consider f as a polynomial of degrees less than or equal to $|G|$. To simplify our notation, let us write $HS_{f,G}$ instead of $HS_{\mathcal{F}[f]_G}$.

Given a graph G, consider the space of polynomials of degree less than or equal to $|G|$ and the corresponding Hilbert series.

Proposition

In the above notation, for generic polynomials f of degree at most $|G|$, the Hilbert series $HS_{f,G}$ is the same. This generic Hilbert series (denoted by HS_G below) is maximal in the majorization partial order among all $HS_{g,G}$, where g runs over the set of all formal power series with non-vanishing linear term.

Here (as usual) by generic polynomials of degree at most $|G|$ we mean polynomials belonging to some Zariski open subset in the linear space of all polynomials of degree at most $|G|$.

Recall that, by definition, a sequence $(a_0, a_1, \dots)$ is *bigger* than $(b_0, b_1, \dots)$ in the majorization partial order if and only if, for any $k \geq 0$,

$$\sum_{i=0}^{k} a_i \geq \sum_{i=0}^{k} b_i.$$

Remark

We know that the Hilbert series of the graded algebra $\mathcal{C}_G^F$ is a specialization of the Tutte polynomial of G. However we can not calculate the Hilbert series of $\mathcal{K}_G$ from the Tutte polynomial of G, because there exists a pair of graphs (G, G') with the same Tutte polynomial and different $HS_{\mathcal{K}_G}$ and $HS_{\mathcal{K}_{G'}}$, see example on next page.
Additionally, notice that, in general, $HS_{\exp,G} := HS_{\mathcal{K}_G} \neq HS_G$.

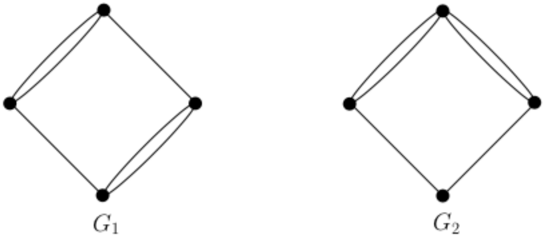

Figure: Graphs with the same matroid and different “K-theoretic" and generic Hilbert series.

G_1 and G_2 have isomorphic matroids and hence, the same Tutte polynomial. Therefore, the Hilbert series of $\mathcal{C}^F_{G_1}$ and $\mathcal{C}^F_{G_2}$ coincide. Namely,

$$HS_{\mathcal{C}^F_{G_1}}(t) = HS_{\mathcal{C}^F_{G_2}}(t) = 1 + 3t + 6t^2 + 9t^3 + 8t^4 + 4t^5 + t^6.$$

However, the Hilbert series of their “K-theoretic" algebras are distinct. Namely

$$HS_{\mathcal{K}_{G_1}}(t) = 1 + 4t + 10t^2 + 14t^3 + 3t^4,$$

$$HS_{\mathcal{K}_{G_2}}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4.$$

Moreover their generic Hilbert series are also distinct and different from their “K-theoretic" Hilbert series. Namely,

$$HS_{G_1}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4,$$

$$HS_{G_2}(t) = 1 + 4t + 10t^2 + 16t^3 + t^4.$$

Putting our information together we get,

$$HS_{\mathcal{C}^F_{G_1}} = HS_{\mathcal{C}^F_{G_2}} \prec HS_{\mathcal{K}_{G_1}} \prec HS_{\mathcal{K}_{G_2}} = HS_{G_1} \prec HS_{G_2},$$

where $\prec$ denotes the majorization partial order.

Q-deformations of Kirillov-Nenashev

Let us define a family of Q-deformations of $\mathcal{C}^F(G)$ as follows.

For a graph G and parameters $Q = \{q_e \in \mathbb{K} : \ e \in E(G)\}$, define $\Phi_{G,Q}$ as the commutative algebra generated by the variables $\{u_e : \ e \in E(G)\}$ satisfying

$$u_e^2 = q_e u_e, \text{ for every edge } \ e \in G.$$

Let $V(G) = [n]$ be the vertex set of a graph G. Define the *Q-deformation* $\Psi_{G,Q}$ of $\mathcal{C}_G^F$ as the filtered subalgebra of $\Phi_{G,Q}$ generated by the elements:

$$X_i = \sum_{e:\ i \in e} c_{i,e} u_e, \ i \in [n],$$

where $c_{i,e}$ are the same as always.

The filtered structure on $\Psi_{G,Q}$ is induced by the elements $X_i,\ i \in [n]$. More concrete, the filtered structure is an increasing sequence

$$\mathbb{K} = F_0 \subset F_1 \subset F_2 \ldots \subset F_m = \Psi_{G,Q}$$

of subspaces of $\Psi_{G,Q}$, where F_k is the linear span of all monomials $X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ such that $\alpha_1 + \ldots + \alpha_n \leq k$. Note that algebra $\Phi_{G,Q}$ has a finite dimension, then $\Psi_{G,Q}$ has a finite dimension, which gives that the increasing sequence of subspaces is finite. The Hilbert polynomial of a filtered algebra is the Hilbert polynomial of the associated graded algebra, it has the following formula

$$\mathcal{H}(t) = 1 + \sum_{i=1} (dim(F_i) - dim(F_{i-1}))t^i.$$

In case when all parameters coincide, i.e., $q_e = q$, $\forall e \in G$, we denote the corresponding algebras by $\Psi_{G,q}$ and $\Phi_{G,q}$ resp. We refer to $\Psi_{G,q}$ as the *Hecke deformation* of $\mathcal{C}_G^F$.

(i) By definition, the algebra $\Psi_{G,0}$ coincides with $\mathcal{C}_G^F$.

(ii) If we change the signs of $q_e,\ e \in E'$ for some subset $E' \subseteq E$ of edges, we obtain an isomorphic algebra.

(iii) It is possible to write relations such as $u_e^2 = \beta_e$ or $u_e^2 = q_e u_e + \beta_e$ where $\beta_e \in \mathbb{K}$.

Example 1. Let G be a graph with two vertices, a pair of (multiple) edges a, b. Consider the Hecke deformation of its $\mathcal{C}_G^F$, i.e., satisfying $q_a = q_b = q$.

The generators are $X_1 = a + b,\ X_2 = -(a + b) = -X_1$. One can easily check that the filtered structure is given by

$$F_0 = <1>;\ \ F_1 = <1,\ a + b>;\ \ F_2 = <1,\ a + b,\ ab>.$$

The Hilbert polynomial $\mathcal{H}(t)$ of $\Psi_{G,q}$ is given by

$$\mathcal{H}(t) = 1 + t + t^2.$$

The defining relation for X_1 is given by

$$X_1(X_1 - q)(X_1 - 2q) = 0.$$

Example 2. For the same graph as before, consider the case when $Q = \{q_a, q_b\},\ q_a^2 \neq q_b^2$.
The generators are the same: $X_1 = a + b$,
$X_2 = -(a + b) = -X_1$. Since

$$X_1^3 = q_a^2 a + q_b^2 b + 3(q_a + q_b)ab = \frac{3(q_a + q_b)}{2} X_1^2 - \frac{q_a^2 + 3q_b^2}{2} a - \frac{3q_a^2 + q$$

$$= \frac{3(q_a + q_b)}{2} X_1^2 - \frac{3q_a^2 + q_b^2}{2} X_1 + (q_a^2 - q_b^2)a,$$

we have

$$F_0 = <1>;\ \ F_1 = <1,\ a+b>;\ \ F_2 = <1,\ a+b,\ q_a a + q_b b + 2ab>;\ \ F_3 =$$

The Hilbert polynomial $\mathcal{H}(t)$ of $\Psi_{G,Q}$ is given by

$$\mathcal{H}(t) = 1 + t + t^2 + t^3.$$

Observe that in this case the algebra $\Psi_{G,Q}$ coincides with the whole $\Phi_{G,Q}$ as a linear space, but has a different filtration. The defining relation for X_1 is given by

$$X_1(X_1 - q_a)(X_1 - q_b)(X_1 - q_a - q_b) = 0.$$

Theorem

For any loopless graph G, filtrations of its Hecke deformation $\Psi_{G,q}$ induced by X_i and induced by the algebra $\Phi_{G,q}$ coincide. Furthermore, the Hilbert polynomial $\mathcal{H}_{\Psi_{G,q}}(t)$ of this filtration is given by

$$\mathcal{H}_{\Psi_{G,q}}(t) = T_G\left(1 + t, \frac{1}{t}\right) \cdot t^{e(G)-v(G)+c(G)},$$

i.e., it coincides with that of $\mathcal{C}_G^F$.

The latter result implies that cases when not all q_e are equal are more interesting than the case of the Hecke deformation.

Let us consider weighted graphs, i.e. when each edge e has non-zero $q_e \in \mathbb{K}$, and will simply denote the algebra for a weighted graph G by Ψ_G.

Definition

For a loopless weighted graph G on n vertices and an orientation $\vec{G}$, define the score vector $D^+_{\vec{G}} \in \mathbb{K}^n$ as follows

$$\left(\sum_{\substack{e \in E: \\ end(\vec{e})=1}} q_e, \sum_{\substack{e \in E: \\ end(\vec{e})=2}} q_e, \dots, \sum_{\substack{e \in E: \\ end(\vec{e})=n}} q_e \right),$$

where $end(\vec{e})$ is the final vertex of oriented edge $\vec{e}$.

Theorem

For any loopless weighted graph G, the dimension of the algebra Ψ_G is equal to the number of distinct score vectors, i.e.

$$dim(\Psi_G) = \#\{D \in \mathbb{K}^n : \exists \vec{G} \text{ such that } D = D^+_{\vec{G}}\}.$$

As a consequence of the above theorem, we obtain the following known property.

Corollary

For any graph G, the number of its spanning forests is equal to the number of distinct vectors of incoming degrees corresponding to its orientations.

Open problems

1. Is it true that if $HS_{f,G_1} = HS_{f,G_2}$ for any function/polynomial f, then the graphs G_1 and G_2 are isomorphic?

2. One can use the formulas for the curvature forms of all Chern classes for E_i/E_j by P. Griffiths and W. Schmid (Acta Math., v.123, 1969) and ask the following.

Problem. For a given SL_n/P study the corresponding algebra $\mathcal{B}_P$ generated by its curvature forms. In particular, what is the total dimension of $\mathcal{B}_P$ as a vector space? What about its Hilbert series?

Postnikov's conjecture

Team score sequences

The *complete multipartite graph* $K_{\bar{n}} = K_{n_1,\dots,n_k}$ is the graph on vertices $1, \dots, n$ with edges $\{i, j\}$, for any $i \in I_a$ and $j \in I_b$ with $a < b$. An *orientation* of $K_{\bar{n}}$ is a directed graph obtained by orienting each edge of $K_{\bar{n}}$. An orientation is called *acylic* if it has no directed cycles. Let us define a weaker notion of *semi-acylic orientations*.

Definition. Let us say that a directed cycle C in the multipartite graph $K_{\bar{n}}$ is *bipartite* if C contains only vertices from $I_a \cup I_b$ for some pair a, b. Let us say that an orientation of $K_{\bar{n}}$ is *semi-acyclic* if it has no bipartite directed cycles.

We can think of an orientation of $K_{\bar{n}}$ as a tournament between k teams with $n_1, ..., n_k$ players where each player of each team plays a game with each player of any other team and either wins or looses. If an edge (i, j) in a orientation of $K_{\bar{n}}$ is directed from i to j, then the player i wins and the player j looses in the corresponding tournament. The individual score of player i is the number of games the player wins, that is, the individual score of i is the outdegree of vertex i of $K_{\bar{n}}$ in the orientation.

The *team score* of team I_a is the partition

$$\lambda^{(a)} = (\lambda_1^{(a)} \geq \lambda_2^{(a)} \geq \cdots \geq \lambda_{n_a}^{(a)} \geq 0),$$

whose parts $\lambda_i^{(a)}$ are the individual scores of players from I_a arranged in the decreasing order.

Definition. The team score sequence of an orientation of $K_{\bar{n}}$ is the sequence $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ of partitions $\lambda^{(a)} = (\lambda_1^{(a)}, \ldots, \lambda_{n_a}^{(a)})$ whose parts $\lambda_i^{(a)}$ are the outdegrees of vertices $j \in I_a$ in the orientation arranged in the decreasing order.

Theorem. The number of team score sequences of acylic orientations of $K_{\bar{n}}$ equals the multinomial coefficient $\frac{n!}{n_1!\ldots n_k!}$, which is equal to the dimension of the cohomology ring $H^*(Fl(\bar{n}))$.

Conjecture. The dimension of the algebra of Chern forms $C^*(Fl(\bar{n}))$ equals the number of team score sequences of semi-acylic orientations of $K_{\bar{n}}$.

Thank you for your patience