Algebra of curvature 2-forms on G/B and its analogs

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Main references

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CONTENT

Basic example, SL_n/B.

Interprete $SL_n/B = \mathcal{U}_n/T^n$ as the space of complete flags in $\mathbb{C}^n$ and take the standard sequence of tautological bundles

$$0 \subset E_1 \subset ... \subset E_n = E$$

(where E is the trivial $\mathbb{C}^n$-bundle over SL_n/B) and the corresponding n-tuple of quotient line bundles $L_i = E_i/E_{i-1}$.

Fixing some Hermitian metric on the original $\mathbb{C}^n$ one equips every bundle E_i, L_i and E_i/E_j, $i > j$ with the induced Hermitian metric. Denote by w_i the curvature form of the above Hermitian metric on L_i. (Each w_i is a $\mathcal{U}_n$-invariant 2-form on SL_n/B such that $\frac{\sqrt{-1}w_i}{2\pi}$ represents the first Chern class $c_1(L_i)$ in $H^2(SL_n/B)$.

Setting $x_i = c_1[L_i]$ one has

$$H^*(SL_n/B, \mathbb{Z}) = \frac{\mathbb{Z}[x_1, ..., x_n]}{(s_1, s_2, ..., s_n)},$$

where s_i stands for the ith elementary symmetric functions in variables $x_1, ..., x_n$.

Problem. Study the $\mathbb{Z}$-ring $\mathcal{A}_n = \mathbb{Z}(w_1, ..., w_n)$ generated by all w_i's (which are standard) and compare it to $H^*(SL_n/B, \mathbb{Z})$.

Remark. One has the standard surjective ring homomorphism $\pi : \mathcal{A}_n \to H^*(SL_n/B, \mathbb{Z})$.

Denote by $\mathcal{A}_{k,n} = \mathbb{Z}(w_{i_1}, ..., w_{i_k})$ the subring of $\mathcal{A}_n$ generated by any k of w_i's. (One can define a natural S_n-action on $(w_1, \ldots, w_n)$ and thus show that up to an isomorphism $\mathcal{A}_{k,n}$ does not depend on a particular choice of a k-tuple.)

Remark. Experiments with Macaulay have shown that the sequence of dimensions of $\mathcal{A}_n$ (as a $\mathbb{Z}$-vector space) starts with $2, 7, 38$. Has anybody seen this before?

1.2. Example. The ring $\mathcal{A}_3 = \mathcal{A}_{3,3}$ is isomorphic to $\frac{\mathbb{Z}[w_1,w_2,w_3]}{I_{3,3}}$ where $I_{3,3}$ is generated by $w_1^3, w_2^3, w_3^3, (w_1+w_2)^3, (w_1+w_3)^3, (w_2+w_3)^3, w_1+w_2+w_3$. The Hilbert polynomial of $\mathcal{A}_3$ equals

$$H(t) = 1 + 2t + 3t^2 + t^3.$$

(For comparison, the Poincaré polynomial of SL_3/B equals $1 + 2t + 2t^2 + t^3$.)

Ring of $\mathcal{U}_n$-invariant forms on SL_n/B

Proposition. The ring of all G-invariant differential forms on a homogeneous space G/H is isomorphic to the subalgebra $\Lambda_{inv}((\mathfrak{G}/\mathfrak{H})^*)$ of the exterior algebra $\Lambda((\mathfrak{G}/\mathfrak{H})^*)$, consisting of skewsymmetric polylinear functions on $\mathfrak{G}$ which

i) vanish on $\mathfrak{H}$;

ii) are invariant under the action of internal automorphisms by the elements of H.

(Here $\mathfrak{G}$ and $\mathfrak{H}$ are the Lie algebras of G and H resp.)

In our case the linear space $(\mathfrak{U}_n/\mathfrak{T}^n)^*$ where $\mathfrak{U}_n$ (resp. $\mathfrak{T}^n$) is the Lie algebra of $\mathcal{U}_n$ (resp. of T^n) coincides with the $\binom{n}{2}$-dimensional vector space of all skew-hermitian matrices of the form

$$\begin{pmatrix} 0 & a_{1,2} & \cdots & \cdots & \cdots & a_{1,n} \\ -\bar{a}_{1,2} & 0 & a_{2,3} & \cdots & \cdots & a_{2,n} \\ -\bar{a}_{1,3} & -\bar{a}_{2,3} & 0 & a_{3,4} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\bar{a}_{1,n} & -\bar{a}_{2,n} & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Let us denote by $(e^{i\lambda_1}, e^{i\lambda_2}, ..., e^{i\lambda_n})$ the diagonal entries of elements in T^n acting by conjugation on $(\mathfrak{U}_n/\mathfrak{T}^n)^*$. Under this action each entry $a_{i,j}$ above the main diagonal is multiplied by $e^{i(\lambda_i-\lambda_j)}$ and each entry $-\bar{a}_{i,j}$ below the main diagonal is multiplied by $e^{i(\lambda_j-\lambda_i)}$. Introducing the fundamental weights $\alpha_i = \lambda_i - \lambda_{i+1}, i = 1, ..., n-1$ we get that $a_{i,j}$ is multiplied by $e^{i(\alpha_i+\alpha_{i+1}+...+\alpha_{j-1})}$. The expression $\alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$ is called *the multiweight* of the entry $a_{i,j}$. (Under this convention the entry $-\bar{a}_{i,j}$ has the opposite multiweight

$-\alpha_i-\alpha_{i+1}-...-\alpha_{j-1}$.) The multiweight of an exterior monomial $\tilde{a}_{i_1,j_1}\wedge\tilde{a}_{i_2,j_2}\cdots\wedge\tilde{a}_{i_r,j_r}$ where each $\tilde{a}_{i_l,j_l}$ is either a_{i_l,j_l} or $-\bar{a}_{i_l,j_l}$ equals to the sum of multiweights of its factors.

2.2. Corollary. The ring $\mathfrak{A}(SL_n/B)$ is the linear span of all exterior monomials $\tilde{a}_{i_1,j_1}\wedge\tilde{a}_{i_2,j_2}\cdots\wedge\tilde{a}_{i_r,j_r}$ having vanishing multiweight. (In particular, $\mathfrak{A}(SL_n/B)$ has no degree 1 elements.)

Remark. Note that such monomials with vanishing multiweight can be considered as collections of roots in $\mathfrak{U}_n$ whose sum vanishes. This notion makes sense for all root systems.

2.3. Example. The Hilbert series for $\mathfrak{A}(SL_3/B)$ is $(1,0,3,2,3,0,1)$. (We assume that $\mathfrak{A}(SL_n/B)$ contains constants.) Its degree 2 component is spanned by $a_{1,2}\wedge\bar{a}_{1,2}, a_{1,3}\wedge\bar{a}_{1,3}, a_{2,3}\wedge\bar{a}_{2,3}$; degree 3 component is spanned by $a_{1,2}\wedge\alpha_{2,3}\wedge\bar{a}_{1,3}, \bar{a}_{1,2}\wedge\bar{\alpha}_{2,3}\wedge a_{1,3}$; degree 4 component is spanned by $a_{1,2} \wedge \bar{a}_{1,2} \wedge a_{1,3} \wedge \bar{a}_{1,3}, a_{1,2} \wedge \bar{a}_{1,2} \wedge a_{2,3} \wedge \bar{a}_{2,3}, a_{1,3} \wedge \bar{a}_{1,3} \wedge a_{2,3} \wedge \bar{a}_{2,3}$ and, finally, its degree 6 component is spanned by $a_{1,2} \wedge \bar{a}_{1,2} \wedge a_{1,3} \wedge \bar{a}_{1,3} \wedge a_{2,3} \wedge \bar{a}_{2,3}$. The Hilbert series for $\mathfrak{A}(SL_4/B)$ is $(1, 0, 6, 8, 21, 24, 32, 24, 21, 8, 6, 0, 1)$.

Recall that *an Eulerian digraph* is a directed graph such that the numbers of coming and leaving edges at each vertex are equal. (We do not allow loops.)

The following proposition is a relatively simple reformulation of 2.2.

2.4. Proposition. The dimension of $\mathfrak{A}(SL_n/B)$ (as a $\mathbb{Z}$-vector space) equals the number of Eulerian digraphs on n labeled vertices, or equivalently, the number of subsets of the standard root system of A_n with vanishing sum.

Problem. For a given Weyl group count the number of subsets of its roots with vanishing sum.

Curvature 2-forms and main results

Proposition [Griffiths-Schmid]. The curvature 2-form w_i, $i = 1, ..., n$ of the tautologic line bundle $L_i = E_i/E_{i-1}$ over SL_n/B equals (up to a constant factor) to the sum of all entries in the ith row of the following matrix of 2-forms

$$\begin{pmatrix} 0 & \gamma_{1,2} & \gamma_{1,3} & \cdots & \gamma_{1,n} \\ -\gamma_{1,2} & 0 & \gamma_{2,3} & \cdots & \gamma_{2,n} \\ -\gamma_{1,3} & -\gamma_{2,3} & 0 & \cdots & \gamma_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1,n} & -\gamma_{2,n} & -\cdots & -\gamma_{n-1,n} & 0 \end{pmatrix} \qquad (3)$$

where $\gamma_{i,j} = a_{i,j} \wedge \bar{a}_{i,j}$.

Lemma. There exists a natural S_n-action on $(w_1, ..., w_n)$.

Proof. The elementary transposition τ_i, $i = 1, ..., n-1$ acts on the matrix (3) of 2-forms by the simultaneous interchanging of 1) the ith row with the $(i+1)$st row, 2) the ith column with the $(i+1)$st column and 3) changing sign of $\gamma_{i,i+1}$. One can easily check that this determines the required S_n-action. $\square$

2.7. Corollary. All rings $\mathcal{A}(w_{i_1},...,w_{i_k})$ are isomorphic to each other and, in particular, to $\mathcal{A}(w_1,...,w_k)$.

Proposition 1. $\mathcal{A}_{k,n}$ is a graded ring isomorphic to $\frac{\mathbb{Z}[w_1,...,w_k]}{I_{k,n}}$ where the ideal $I_{k,n}$ is generated by the set of 2^k-1 polynomials of the form

$$g^{(n)}_{i_1,...,i_j} = (w_{i_1}+...+w_{i_j})^{j(n-j)+1} \qquad (1)$$

where $\{i_1,...,i_j\}$ runs over the set of all nonempty subsets in the set $\{1,...,k\}$.

Proposition 2. The total dimension of $\mathcal{A}_n$ equals the number of forests on n labeled vertices and there exists a natural monomial basis for $\mathcal{A}_n$ whose monomials are enumerated by the above forests.

Remark. One can define the natural action of divided difference operators on $\mathcal{A}_n$.

Example of $\mathcal{A}_4$.

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 & a & b & b \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -b & -e & -f & 0 \end{pmatrix},$$

where $a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = 0$ with no other relations.

Then one has $w_i^4 = 0$; $(w_i + w_j)^5 = 0$; $(w_i + w_j + w_k)^4 = 0$; $w_1 + w_2 + w_3 + w_4 = 0$.

Open problems related to SL_n/P.

P. Griffiths and W. Schmid (Acta Math., v.123, 1969) have found formulas for the curvature forms of all Chern classes for E_i/E_j (and even in a more general situation).

Problem. For a given SL_n/P study the corresponding algebra $\mathcal{A}_P$ generated by its curvature forms. In particular, what is the total dimension of $\mathcal{A}_P$ as a vector space/Hilbert series? (One will certainly get some remarkable subalgebras of $\Lambda(roots)$ generated by elements of even degree.)

Grassmann case. For example, the standard $2i$-form w^{2i} corresponding to the ith Chern class c_i, $i \leq k \leq n-k$ in Grassmannian $G(k,n)$ has the following presentation.

$$w^{2i} = \sum_{g \in BEG} i_{k+1}! \dots i_n! g,$$

where g runs over the set BEG of bipartite Eulerian graphs with $2i$ edges

a) connecting the set of labelled vertices $\{1,\dots,k\}$ with the set of labelled vertices $\{k+1,\dots,n\}$;

b) the valencies=degrees/2 of all vertices among $\{1,\dots,k\}$ is at most 1. $(i_{k+1},\dots,i_n)$ stands for the set of valencies of the vertices $(k+1,\dots,n)$. (As usual, $0! = 1$.)

Notice that in the case of Grassmannian our algebra coincides with $H^*(G(k,n))$.

Problem. Derive the presentation for S_λ and Robinson-Schensted coefficients.

Example of $G_{2,4}$.

Main features of the algebra $\mathcal{A}_n$ leading to possible generalizations:

1) generated by $\mathcal{U}_n$-invariant 2-forms of linear bundles

2) generated by linear elements in the square-free commutative algebra $\frac{\mathbb{Z}[\gamma_{i,j}]}{(\gamma_{i,j}^2)}$, $1 \le i < j \le n$.

3) presented as the quotient of $\mathbb{Z}[x_1, \ldots, x_n]$ mod the ideal of powers of linear forms labelled by all nonempty subsets in $\{1, \ldots, n\}$.

Other G/B's and subalgebras generated by linear elements in $\frac{\mathbb{Z}[\gamma_1,\cdots,\gamma_m]}{(\gamma_1^2,\cdots,\gamma_m^2)}$.

Notation. G is a connected complex semisimple Lie group; B its Borel subgroup; $F = G/B$ is the flag variety; K is a maximal compact subgroup of G and $T = K \cap B$ is its maximal torus.

The group K acts transitively on F and F can be identified with the quotient space K/T.

By $\mathfrak{g}$ we denote the Lie algebra of G and by $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra. Also denote by $\mathfrak{g}_{\mathcal{R}} \subset \mathfrak{g}$ the real form of $\mathfrak{g}$ such that $i\,\mathfrak{g}_{\mathcal{R}}$ is the Lie algebra of K. Analogously, $\mathfrak{h}_{\mathcal{R}} = \mathfrak{h} \cap \mathfrak{g}_{\mathcal{R}}$ and $i\,\mathfrak{h}_{\mathcal{R}}$ is the Lie algebra of the maximal torus T. The *root system* associated with $\mathfrak{g}$ is the set Δ of nonzero vectors (roots) $\alpha \in \mathfrak{h}^*$ for which the root spaces

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

are nontrivial. Then $\mathfrak{g}$ decomposes into the direct sum of subspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

For $\alpha \in \Delta$, the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ are one-dimensional and there exists a unique element $h_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(h_\alpha) = 2$. The elements $h_\alpha \in \mathfrak{h}$ are called *coroots.* Actually, $\alpha \in \mathfrak{h}^*_{\mathcal{R}}$ and $h_\alpha \in \mathfrak{h}_{\mathcal{R}}$, for $\alpha \in \Delta$. Let us choose generators $e_\alpha \in \mathfrak{g}_{\mathcal{R}}$ of the root spaces $\mathfrak{g}_\alpha$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for any root α. Then $[h_\alpha, e_\alpha] = 2e_\alpha$ and $[h_\alpha, e_{-\alpha}] = -2e_{-\alpha}$.

The root system Δ is subdivided into a disjoint union of sets of positive roots Δ_+ and negative roots $\Delta_- = -\Delta_+$ such that the direct sum $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$ is the Lie algebra of Borel subgroup B. The Weyl group W is the group generated by the reflections $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$, $\alpha \in \Delta_+$, given by

$$s_\alpha : \lambda \mapsto \lambda - \lambda(h_\alpha)\, \alpha.$$

The lattice $\widehat{T} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta\}$ is called the *weight lattice.* Every weight $\lambda \in \hat{T}$ determines an irreducible unitary representations $\pi_\lambda : T \to \mathbb{C}^\times$ of the maximal torus T given by $\pi_\lambda(\exp(x)) = e^{\lambda(x)}$, for $x \in i\,\mathfrak{h}_{\mathcal{R}}$, and every irreducible unitary representation of T is of this form.

For a weight $\lambda \in \hat{T}$, the homomorphism $\pi_\lambda : T \to \mathbb{C}^\times$ extends uniquely to a holomorphic homomorphism $\bar{\pi}_\lambda : B \to \mathcal{C}^\times$. Thus any $\lambda \in \hat{T}$ determines a holomorphic line bundle $L_\lambda = G \times_B \mathbb{C} = K \times_T \mathbb{C}$ over $F = G/B = K/T$. The line bundle L_λ has a canonical K-invariant hermitian metric.

The classical Borel's theorem describes the cohomology ring $\mathrm{H}^*(X, \mathbb{C})$ of the homogeneous manifold F in terms of generators and relations:

$$\mathrm{H}^*(X, \mathbb{C}) \cong \mathrm{Sym}(\mathfrak{h}^*)/I_W,$$

where I_W is the ideal in the symmetric algebra Sym($\mathfrak{h}^*$) generated by the W-invariant elements without constant term. The natural projection from Sym($\mathfrak{h}^*$) to H$^*(X, \mathbb{C})$ is the homomorphism that sends a weight $\lambda \in \widehat{T}$ to the first Chern class $c_1(L_\lambda)$ of the line bundle L_λ.

It is possible to exhibit differential two-forms that represent the Chern classes $c_1(L_\lambda)$ in the de Rham cohomology of the homogeneous manifold F. Recall that for a holomorphic hermitian line bundle $L : E \to X$ there is a canonically associated connection on E. Denote by $\Theta(L)$ the *curvature form* of this connection, which is a differential two-form on F. Then the form $i\,\Theta(L)/2\pi$ represents $c_1(L)$.

In order to construct the curvature forms $\Theta(L_\lambda)$ explicitly, we define the elements $e^\alpha \in \mathfrak{g}^*_{\mathcal{R}}$, $\alpha \in \Delta$, by $e^\alpha(\mathfrak{h}) = 0$ and $e^\alpha(e_\beta) = \delta_{\alpha\beta}$, for any $\beta \in \Delta$. (Here $\delta_{\alpha\beta}$ is Kroneker's delta.)

The space of left K-invariant differential one-forms on K can be identified with the dual to its Lie algebra, t.e., with $i\,\mathfrak{g}^*_{\mathcal{R}}$. Thus the elements $i\,e^\alpha$ can be regarded as one-forms on K. The differential two-form on K given by

$$\phi_\alpha = e^\alpha \wedge e^{-\alpha}, \quad \alpha \in \Delta,$$

is invariant with respect to the right translation action of the torus T. Thus ϕ_α produces a two-form on the manifold F, for which we will use the same notation ϕ_α. It is clear from the definition that $\phi_{-\alpha} = -\phi_\alpha$.

The following statement can be derived from the results of Graffiths-Schmid.

Proposition 1. *For $\lambda \in \hat{T}$, the curvature form of the holomorphic hermitian line bundle L_λ is given by*

$$\Theta(L_\lambda) = \sum_{\alpha \in \Delta_+} \lambda(h_\alpha)\,\phi_\alpha.$$

Let Φ be the algebra generated by the two-forms ϕ_α, $\alpha \in \Delta_+$. The relations in Φ are relatively simple:

$$\phi_\alpha \phi_\beta = \phi_\beta \phi_\alpha, \qquad (\phi_\alpha)^2 = 0.$$

Thus Φ is a 2^N-dimensional algebra, where $N = |\Delta_+|$. The main object is the subalgebra of Φ generated by the curvature forms $\Theta(L_\lambda)$.

Main Results

Denote by $\mathsf{C}(X)$ the subalgebra in the algebra of differential forms on F that is generated by the curvature forms $\Theta(L_\lambda)$ of line bundles. Obviously, $\mathsf{C}(X)$ has the structure of a graded ring: $\mathsf{C}(X) = \mathsf{C}^0(X) \oplus \mathsf{C}^1(X) \oplus \mathsf{C}^2(X) \oplus \cdots$, where $\mathsf{C}^k(X)$ is the subspace of $2k$-forms in $\mathsf{C}(X)$. In order to formulate our main results about $\mathsf{C}(X)$ we need some extra notation from the matroid theory.

Let V be a collection of vectors $v_1, v_2, \ldots, v_N$ in a vector space E, say, over $\mathbb{C}$. A subset of vectors in V is called *independent* if they are linearly independent in E. By convention, the empty subset is independent. Let $\mathrm{ind}(V)$ be number of all independent subsets in V.

A *cycle* is a minimal by inclusion not independent subset. For a cycle $C = \{v_{i_1}, \ldots, v_{i_l}\}$, there is a unique, up to a factor, linear dependence $a_1 v_{i_1} + \cdots + a_l v_{i_l} = 0$ with non-zero a_i's. Let us fix a linear order $v_1 < v_2 < \cdots < v_N$ of all elements of V. For an independent subset S in V, a vector $v \in V \setminus S$, is called *externally active* if the set $S \cup \{v\}$ contains a cycle C and v is the minimal element of C. Let $\mathrm{act}(S)$ be the number of externally active vectors with respect to S.

Theorem 2. *The dimension of the algebra* $\mathrm{C}(X)$ *is equal to the number* $\mathrm{ind}(\Delta_+)$ *of independent subsets in the set of positive roots* Δ_+*. Moreover, the dimension of the* k*-th component* $\mathrm{C}^k(X)$ *is equal to the number of independent subsets* $S \subset \Delta_+$ *such that* $k = N - |S| - \mathrm{act}(S)$*, where* $N = |\Delta_+|$.

Notice that, although the number act(S) of externally active vectors depends upon a particular order of elements in V, the total number of subsets S with fixed $|S| + \text{act}(S)$ does not depend upon a choice of ordering.

We will actually prove a more general result about an arbitrary collection of vectors V. Let V and E be as above. We will assume that the elements $v_1, \ldots, v_N$ of V span the n-dimensional space E. Thus $N \geq n$. Let $F = \mathbb{C}^N$ be the linear space with a distinguished basis $\phi^1, \ldots, \phi^N$. Then V defines the projection map $p : F \to E$ that sends the ith basis element ϕ^i to v_i. The dual map $p^* : E^* \to F^*$ defines an n-dimensional plane $P = \text{Im}(p^*)$ in F^*. In other words, the collection of vectors V can be identified with an element P of the Grassmannian $G(n, N)$ of n-dimensional planes in $\mathbb{C}^N$.

Let $\phi_1, \ldots, \phi_N$ be the basis in F^* dual to the chosen basis in F. Denote by Φ_N the quotient of the symmetric algebra $\mathrm{Sym}(F^*)$ modulo the relations $(\phi_i)^2 = 0$, $i = 1, \ldots, N$. Let $\mathcal{C}_V$ be the subalgebra in Φ_N generated by the elements of the n-dimensional plane $P \subset F^*$. In other words, the algebra $\mathcal{C}_V$ is the image of the induced mapping

$$\mathrm{Sym}(E^*) \longrightarrow \Phi_N = \mathrm{Sym}(F^*)/\left\langle \phi_i^2, i = 1, \ldots, N \right\rangle.$$

The algebra $\mathcal{C}_V$ has an obvious grading $\mathcal{C}_V = \mathcal{C}_V^0 \oplus \mathcal{C}_V^1 \oplus \mathcal{C}_V^2 \oplus \cdots$ by degree of elements.

Suppose $E = \mathfrak{h}$ and V is the collections of coroots h_α, $\alpha \in \Delta_+$. For $\lambda \in \mathfrak{h}^* = E^*$, $p^*(\lambda) = \Theta(L_\lambda) \in F^*$ is the curvature form (see Proposition 1). Then $\mathcal{C}_V = \mathrm{C}(X)$ is the algebra generated by the curvature forms $\Theta(L_\lambda)$.

In general, we have the following result.

Theorem 3. *The dimension of the algebra $\mathcal{C}_V$ is equal to the number* $\mathrm{ind}(V)$ *of independent subsets in V. Moreover, the dimension of the k-th component $\mathcal{C}_V^k$ is equal to the* *number of independent subsets $S \subset V$ such that $k = N - |S| - \mathrm{act}(S)$.*

We can also describe the algebra $\mathcal{C}_V$ as a quotient of a polynomial ring. Let us say that a hyperplane H in E is a *V-essential hyperplane* if the elements of the subset $\{v_i,\, i = 1, \ldots, N \mid v_i \in H\}$ span the hyperplane H. Obviously, an essential hyperplane is uniquely determined by the subset of indices $I_H = \{i \in \{1, \ldots, N\} \mid v_i \notin H\}$. We will call such subset I_H a *V-essential index subset*. Denote by $d(H) = d_V(H) = |I_H|$ the number of its elements. A nonzero vector $\lambda \in E^*$ determines the hyperplane $H = \{x \in E \mid \lambda(x) = 0\}$ in E. Vectors $\lambda_H \in E^*$ corresponding to essential hyperplanes H will be called *V-essential vectors*. They are defined up to a nonzero factor.

Theorem 4. *The algebra $\mathcal{C}_V$ is naturally isomorphic to the quotient of the polynomial ring $\mathrm{Sym}(E^*)/\mathcal{I}_V$, where the ideal $\mathcal{I}_V$ is generated by the powers $(\lambda_H)^{d(H)+1}$ of V-essential* *vectors for all V-essential hyperplanes H in E. The isomorphism is induced by the embedding $p^* : E^* \to F^*$.*

Remark 5. There are several equivalent definitions of essential subsets, as follows:

1. An index subset $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$ is V-*essential* if and only if the following two conditions are satisfied: (i) the coordinate plane $\left\langle \phi_{i_1}, \ldots, \phi_{i_k} \right\rangle$ in F^* has one-dimensional intersection with the plane P; (ii) there is no proper subset in I that satisfies the condition (i). For an V-essential hyperplane H, the vector $p^*(\lambda_H) \in F^*$ spans the one-dimensional intersection of P and the coordinate plane associated with I_H.

2. Let $\theta_1, \ldots, \theta_n$ be any basis in P. The V-essential subsets are in one-to-one correspondence with the cycles in the vector set $\{\phi_1, \ldots, \phi_N, \theta_1, \ldots, \theta_n\}$. For a cycle $\{\phi_{i_1}, \ldots, \phi_{i_k}, \theta_{j_1}, \ldots, \theta_{j_k}\}$, the subset $\{i_1, \ldots, i_k\}$ is V-essential.

Moreover, every V-essential subset is of this form.

Note that the decomposition of the Grassmannian $G(n, N)$ of all n-dimensional planes $P \subset F^*$ into strata with the same collection of essential subsets coincides with the decomposition of $G(n, N)$ into small cells of Gelfand-Serganova because any two P_1, $P_2 \in G(n, N)$ with the same collection of essential subsets have the same dimensions of intersections with all coordinate subspaces. Equivalently, P_1 and P_2 are in the same strata if and only if the corresponding collections of vectors V_1 and V_2 define the same matroid, i.e., have the same collection of independent subsets.

Let us apply the Theorem 4 to C(X). Let $\omega_1, \omega_2, \ldots, \omega_l$ be the fundamental weights. They generate the weight lattice $\widehat{T}$. Also let d_i be the number of positive roots $\alpha \in \Delta_+$ such that $\alpha(\omega_i) \neq 0$.

Corollary 6. *The algebra* $\mathsf{C}(X)$ *is naturally isomorphic to the quotient of the polynomial ring* $\mathrm{Sym}(\mathfrak{h}^*)/\mathcal{J}$, *the ideal* $\mathcal{J} \in \mathrm{Sym}(\mathfrak{h}^*)$ *is generated by the elements* $(w \cdot \omega_i)^{d_i+1}$, *where* $i = 1, \dots, l$ *and* w *is an element of the Weyl group* W. *This isomorphism is induced by the projection* $\mathrm{Sym}(\mathfrak{h}^*) \to \mathsf{C}(X)$ *that send* $\lambda \in \hat{T}$ *to the curvature form* $\Theta(L_\lambda)$.

Two algebras associated to an undirected graph

G-parking functions and monomial ideal associated to a digraph

A *parking function* of size n is a sequence $b = (b_1, \ldots, b_n)$ of non-negative integers such that its increasing rearrangement $c_1 \leq \cdots \leq c_n$ satisfies $c_i < i$. Equivalently, we can formulate this condition as $\#\{i \mid b_i < r\} \geq r$, for $r = 1, \ldots, n$. The parking functions of size n are known to be in bijective correspondence with trees on $n+1$ labelled vertices. Thus, according to Cayley's formula for the number of labelled trees, the total number of parking functions of size n equals $(n+1)^{n-1}$. In this section we extend this statement to a more general class of functions.

Let G be a digraph on the set of vertices $0, 1, \ldots, n$ (with possible multiple egdes but no loops). The vertex 0 will be the root of

G. The digraph G is determined by its *adjacency matrix* $A = (a_{ij})_{0 \le i,j \le n}$, where a_{ij} is the number of edges from the vertex i to the vertex j. We will regard graphs as a special case of digraphs with symmetric adjacency matrix A.

An *oriented spanning tree* T of the digraph G is a subgraph $T \subset G$ such that there exists a unique directed path in T from any vertex i to the root 0. The number N_G of such trees is given by the *Matrix-Tree Theorem*:

$$N_G = \det L_G, \tag{1}$$

where $L_G = (l_{ij})_{1 \le i,j \le n}$ the *truncated Laplace matrix*, also known as the *Kirkhoff matrix*, given by

$$l_{ij} = \begin{cases} \sum\limits_{r \in \{0,\dots,n\} \setminus \{i\}} a_{ir} & \text{for } i = j, \\ -a_{ij} & \text{for } i \neq j. \end{cases} \tag{2}$$

If G is a graph, i.e., A is a symmetric matrix, then oriented spanning trees defined above are exactly the usual *spanning trees* of G, which are connected subgraphs of G without cycles.

For a subset I in $\{1, \ldots, n\}$ and a vertex $i \in I$, let

$$d_I(i) = \sum_{j \notin I} a_{ij},$$

i.e., $d_I(i)$ is the number of edges from the vertex i to a vertex outside of the subset I. Let us say that a sequence $b = (b_1, \ldots, b_n)$ of non-negative integers is a *G-parking function* if, for any nonempty subset $I \subseteq \{1, \ldots, n\}$, there exists $i \in I$ such that $b_i < d_I(i)$.

If $G = K_{n+1}$ is the complete graph on $n + 1$ vertices then K_{n+1}-parking functions are the usual parking functions of size n defined in the beginning of this section.

Theorem 7. *The number of G-parking functions equals the number $N_G = \det L_G$ of oriented spanning trees of the digraph G.*

We can reformulate the definition of G-parking functions in algebraic terms as follows. Throughout this paper we fix a field $\mathbb{K}$. Let $\mathcal{I}_G = \langle m_I \rangle$ be the monomial ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials

$$m_I = \prod_{i \in I} x_i^{d_I(i)}, \qquad (3)$$

where I ranges over all nonempty subsets $I \subseteq \{1, \ldots, n\}$. Define the algebra $\mathcal{A}_G$ as the quotient $\mathcal{A}_G = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_G$.

A non-negative integer sequence $b = (b_1, \ldots, b_n)$ is a G-parking function if and only if the monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ is nonvanishing in the algebra $\mathcal{A}_G$.

For a monomial ideal $\mathcal{I}$, the set of all monomials that do not belong to $\mathcal{I}$ is a basis of the quotient of the polynomial ring modulo $\mathcal{I}$, called the *standard monomial basis.* Thus the monomials x^b, where b ranges over G-parking functions, form the standard monomial basis of the algebra $\mathcal{A}_G$.

Corollary 8. *$\mathcal{A}_G$ is a finite-dimensional linear space over $\mathbb{K}$. Its dimension is equal to the number of oriented spanning trees of the digraph G:*

$$\dim \mathcal{A}_G = N_G.$$

Power algebra associated to an undirected graph

Let G be an undirected graph on the set of vertices $0, 1, \ldots, n$. In this case the dimension of the algebra $\mathcal{A}_G$ is equal to the number of usual spanning trees of G.

For a nonempty subset I in $\{1, \ldots, n\}$, let $D_I = \sum_{i \in I, j \notin I} a_{ij} = \sum_{i \in I} d_I(i)$ be the total number of edges that join some vertex in I with a vertex outside of I. For any nonempty subset $I \subseteq \{1, \ldots, n\}$, let

$$p_I = \left(\sum_{i \in I} x_i \right)^{D_I}. \tag{4}$$

Let $\mathcal{J}_G = \langle p_I \rangle$ be the ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials p_I for all nonempty subsets I. Define the algebra $\mathcal{B}_G$ as the quotient $\mathcal{B}_G = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_G$.

The algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ are graded. For a graded algebra $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \cdots$, the *Hilbert series* of $\mathcal{A}$ is the formal power series in q given by

$$\operatorname{Hilb} \mathcal{A} = \sum_{k \geq 0} q^k \dim \mathcal{A}^k.$$

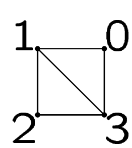
Our main proposition is as follows.

Theorem 9. *The monomials x^b, where b ranges over G-parking functions, form a linear basis of the algebra $\mathcal{B}_G$. Thus the Hilbert series of the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ coincide termwise:* $\operatorname{Hilb} \mathcal{A}_G = \operatorname{Hilb} \mathcal{B}_G$. *In particular, both these algebras are finite-dimensional as linear spaces over $\mathbb{K}$ and*

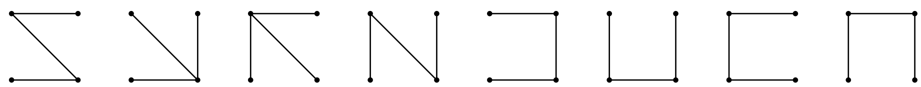
$$\dim \mathcal{A}_G = \dim \mathcal{B}_G = N_G$$

is the number of spanning trees of the graph G.

Example 10. Let $n = 3$ and let G be the graph given by

$G =$  . (5)

The graph G has 8 spanning trees:



The ideals $\mathcal{I}_G$ and $\mathcal{J}_G$ are given by

$$\mathcal{I}_G = \left\langle x_1^3, x_2^2, x_3^3, x_1^2 x_2, x_1^2 x_3^2, x_2 x_3^2, x_1 x_2^0 x_3 \right\rangle,$$
$$\mathcal{J}_G = \left\langle x_1^3, x_2^2, x_3^3, (x_1+x_2)^3, (x_1+x_3)^4, (x_2+x_3)^3, (x_1+x_2+x_3)^2 \right\rangle.$$

The standard monomial basis of the algebra $\mathcal{A}_G$ is $\{1, x_1, x_2, x_3, x_1^2, x_1x_2, x_2x_3, x_3^2\}$. The corresponding G-parking functions are the exponent vectors of the basis elements:

$$(0,0,0),(1,0,0),(0,1,0),(0,0,1),(2,0,0),$$
$$(1,1,0),(0,1,1),(0,0,2).$$

We have $\dim \mathcal{A}_G = \dim \mathcal{B}_G = 8$ is the number of spanning trees of G, and $\mathrm{Hilb}\, \mathcal{A}_G = \mathrm{Hilb}\, \mathcal{B}_G = 1 + 3q + 4q^2$.

We will refine Theorem 9 and interpret dimensions of graded components of the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ in terms of external acticity of spanning trees. Let us fix a linear ordering of all edges of the graph G. For a spanning tree T of G, an edge $e \in G \setminus T$ is called *externally active* if there exists a cycle C in the graph G such that e is the minimal edge of C and $(C \setminus \{e\}) \subset T$. The *external activity* of a spanning tree is the number of externally active edges. Let N_G^k denote the number of spanning trees $T \subset G$ of external activity k. Even though the notion of external activity depends on a particular choice of ordering of edges, the numbers N_G^k are known to be invariant on the choice of ordering.

Let $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ be the k-th graded components of the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$, correspondingly.

Theorem 11. *The dimensions of the k-th graded components $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ are equal to*

$$\dim \mathcal{A}_G^k = \dim \mathcal{B}_G^k = N_G^{|G|-n-k},$$

the number of spanning trees of G of external activity $|G| - n - k$, where $|G|$ denotes the number of edges of G.

Examples: tree ideals and their generalizations
Two algebras of dimension $(n+1)^{n-1}$

Suppose that $G = K_{n+1}$ is the complete graph on $n+1$ vertices. As we have already mentioned, the K_{n+1}-parking functions are the usual parking functions of size n defined earlier.

Let $\mathcal{I}_n = \langle m_I \rangle$ and $\mathcal{J}_n = \langle p_I \rangle$ be the ideals in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials m_I and the polynomials p_I, correspondingly, given by

$$m_I = (x_{i_1} \cdots x_{i_r})^{n-r+1},$$
$$p_I = (x_{i_1} + \cdots + x_{i_r})^{r(n-r+1)},$$

where in both cases $I = \{i_1, \ldots, i_r\}$ runs over all nonempty subsets of $\{1, \ldots, n\}$. Let $\mathcal{A}_n = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_n$ and $\mathcal{B}_n = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_n$.

Corollary 12. *The graded algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ have the same Hilbert series. They are finite-dimensional, as linear spaces over $\mathbb{K}$. Their dimensions are equal to*

$$\dim \mathcal{A}_n = \dim \mathcal{B}_n = (n+1)^{n-1}.$$

The images of the monomials x^b, where b ranges over parking functions of size n, form linear bases in both algebras $\mathcal{A}_n$ and $\mathcal{B}_n$.

An *inversion* in a tree T on the $n+1$ vertices labelled $0, \ldots, n$ is a pair of vertices labelled i and j such that $i > j$ and the vertex i belongs to the shortest path in T that joins the vertex j with the root 0.

Corollary 13. *The dimension* $\dim \mathcal{A}_n^k = \dim \mathcal{B}_n^k$ *of the k-th graded components of the algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ is equal to*

(A) *the number of parking functions b of size n such that $b_1 + \cdots + b_n = k$;*

(B) *the number of trees on $n+1$ vertices with external activity $\binom{n}{2} - k$;*

(C) *the number of trees on $n+1$ vertices with $\binom{n}{2} - k$ inversions.*

It is well known that the numbers (A), (B), and (C) are equal. The *inversion polynomial* is defined as the sum $I_n(q) = \sum_T q^{\#\text{ of inversions in } T}$ over all trees T on $n+1$ labelled vertices. Thus the Hilbert series of the algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ are equal to

$$\mathrm{Hilb}\,\mathcal{A}_n = \mathrm{Hilb}\,\mathcal{B}_n = q^{\binom{n}{2}}\, I_n(q^{-1}).$$

Two algebras of dimension $l\,(l+kn)^{n-1}$

It is possible to extend the previous example as follows. Fix two non-negative integers k and l. Let $G = K_{n+1}^{k,l}$ be the complete graph on the vertices $0, 1, \ldots, n$ with the edges (i,j), $i, j \neq 0$, of multiplicity k and the edges $(0,i)$ of multiplicity l. The $K_{n+1}^{k,l}$-parking functions are the non-negative integer sequences $b = (b_1, \ldots, b_n)$ such that, for $r = 1, \ldots, n$,

$$\#\{i \mid b_i < l + k(r-1)\} \geq r.$$

The definition of these functions can be also formulated as $c_i < l + (i-1)\,k$, where $c_1 \leq \cdots \leq c_n$ is the increasing rearrangement of elements of b. Such functions were studied by Pitman and Stanley and then by Yan. They demonstrated that their number equals $l\,(l+kn)^{n-1}$. One can show, using for example the Matrix-Tree Theorem (1), that the number of spanning trees in the graph $K_{n+1}^{k,l}$ equals $l\,(l+kn)^{n-1}$. Thus Theorem 7 recovers the above formula for the number of $K_{n+1}^{k,l}$-parking functions.

Let $\mathcal{I}_{n,k,l} = \langle m_I \rangle$ and $\mathcal{J}_{n,k,l} = \langle p_I \rangle$ be the ideals in the ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials m_I and the polynomials p_I, correspondingly, given by

$$m_I = (x_{i_1} \cdots x_{i_r})^{l+k(n-r)},$$
$$p_I = (x_{i_1} + \cdots + x_{i_r})^{r(l+k(n-r))},$$

where in both cases $I = \{i_1, \ldots, i_r\}$ runs over all nonempty subsets of $\{1, \ldots, n\}$. Let $\mathcal{A}_{n,k,l} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_{n,k,l}$ and $\mathcal{B}_{n,k,l} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_{n,k,l}$.

Corollary 14. *The graded algebras $\mathcal{A}_{n,k,l}$ and $\mathcal{B}_{n,k,l}$ have the same Hilbert series. They are* *finite-dimensional, as linear spaces over $\mathbb{K}$. Their dimensions are*

$$\dim \mathcal{A}_{n,k,l} = \dim \mathcal{B}_{n,k,l} = l\,(l + kn)^{n-1}.$$

The images of the monomials x^b, where b ranges over $K_{n+1}^{k,l}$-parking functions, form linear bases in both algebras $\mathcal{A}_{n,k,l}$ and $\mathcal{B}_{n,k,l}$.

Monotone monomial ideals and their deformations

A *monotone monomial family* is a collection $\mathcal{M} = \{m_I \mid I \in \Sigma\}$ of monomials in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ labelled by a set Σ of nonempty subsets in $\{1, \ldots, n\}$ that satisfies the following three conditions:

(MM1) For $I \in \Sigma$, m_I is a monomial in the variables x_i, $i \in I$.

(MM2) For $I, J \in \Sigma$ such that $I \subset J$ and $i \in I$, we have $\deg_{x_i}(m_I) \geq \deg_{x_i}(m_J)$.

(MM3) For $I, J \in \Sigma$, $\text{lcm}(m_I, m_J)$ is divisible by m_K for some $K \supseteq I \cup J$ in Σ.

The *monotone monomial ideal* $\mathcal{I} = \langle \mathcal{M} \rangle$ associated with a monotone monomial family

$\mathcal{M}$ is the ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials m_I in $\mathcal{M}$.

It follows from (MM1) and (MM2) that condition (MM3) can be replaced by the condition: For $I, J \in \Sigma$ there is $K \supseteq I \cup J$ in Σ such that m_K is a monomial in the x_i, $i \in I \cup J$. This condition is always satisfied if $I, J \in \Sigma$ implies that $I \cup J \in \Sigma$.

The monomial ideal $\mathcal{I}_G$ constructed in Section for a digraph G is monotone. In this case Σ is the set of all nonempty subsets in $\{1, \ldots, n\}$ and m_I is given by (3).

Let $I = \{i_1, \ldots, i_r\}$. For a monomial $m \in \mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$, an *I-deformation* of m is a homogeneous polynomial $p \in \mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ of degree deg$(p) =$ deg(m) satisfying the *generosity condition*

$$\mathbb{K}[x_{i_1}, \ldots, x_{i_r}] = \langle R_m \rangle \oplus (p), \qquad (6)$$

where $\langle R_m \rangle$ is the linear span of the set R_m of monomials in $\mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ which are not divisible by m, (p) is the ideal in $\mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ generated by p, and "$\oplus$" stands for a direct sum of subspaces. Notice that the generosity condition is satisfied for a Zarisky open set of polynomials in $\mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ of degree deg(m). For example, the polynomial $p = a\,x_1 + b\,x_2$ is a $\{1, 2\}$-deformation of the monomial $m = x_1$ if and only if $a \neq 0$.

The following lemma describes a class of I-deformations of monomials.

Lemma 15. *Let $I = \{i_1, \ldots, i_r\}$, let m be a monomial in $\mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$, and let $\alpha_1, \ldots, \alpha_r \in \mathbb{K} \setminus \{0\}$. Then the polynomial*

$$p = (\alpha_1 x_{i_1} + \cdots + \alpha_r x_{i_r})^{\deg m}$$

is an I-deformation of the monomial m.

A *deformation* of a monotone monomial ideal $\mathcal{I} = \langle m_I \mid I \in \Sigma \rangle$ is an ideal $\mathcal{J} = \langle p_I \mid I \in \Sigma \rangle$ generated by polynomials p_I such that p_I is an I-deformation of m_I for each $I \in \Sigma$. For example, according to Lemma 15, the ideal

$\mathcal{J}_G$ given in Section is a deformation of the monotone monomial ideal $\mathcal{I}_G$.

Theorem 16. *Let $\mathcal{I}$ be a monotone monomial ideal, and R be the standard monomial basis of the algebra $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$, i.e., R is the set of monomials that do not belong to $\mathcal{I}$. Let $\mathcal{J}$ be a deformation of the ideal $\mathcal{I}$, and $\mathcal{B} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}$.*

Then the monomials in R linearly span the algebra $\mathcal{B}$.

Remark that the set of monomials R may or may not be a basis for $\mathcal{B}$.

Corollary 17. *Let $\mathcal{I}$ be a monotone monomial ideal, $\mathcal{J}$ be a deformation of the ideal $\mathcal{I}$, $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$, and $\mathcal{B} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}$. Then we have the following termwise inequalities for the Hilbert series:*

$$\mathrm{Hilb}\,\mathcal{I} \leq \mathrm{Hilb}\,\mathcal{J} \qquad \Leftrightarrow \qquad \mathrm{Hilb}\,\mathcal{A} \geq \mathrm{Hilb}\,\mathcal{B}.$$

In some cases the Hilbert series are actually equal to each other. According to Theorem 9, $\operatorname{Hilb}\mathcal{A}_G = \operatorname{Hilb}\mathcal{B}_G$, for any graph G. However in general the Hilbert series may not be equal to each other. It would be interesting to describe a general class of monotone monomial ideals and their deformations with equal Hilbert series.

There is an obvious correspondence between the generators m_I of a monotone monomial ideal $\mathcal{I}$ and the generators p_I of its deformation $\mathcal{J}$. Notice however that (except for very special cases) the monomial generator m_I does not belong to the boundary of the Newton polytopes of it polynomial deformation p_I. Thus the monomial m_I is not the leading term of the polynomial p_I for any term order. This shows that the above results cannot be tackled by the standard Gröbner bases technique.

Hilbert series and dimensions of monotone monomial ideals

In this section we give formulas for the Hilbert series and dimensions of monotone monomial ideals. Then we prove Theorem 7.

Let $\{m_I \mid I \in \Sigma\}$, $m_I = \prod_{i\in I} x_i^{\nu_I(i)}$, be a monotone monomial family, let $\mathcal{I} = \langle m_I \rangle$ be the corresponding ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, and let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$.

Proposition 18. *The Hilbert series of the algebra $\mathcal{A}$ equals*

$$\operatorname{Hilb} \mathcal{A} = \frac{1 + \sum_{k\geq 1} (-1)^k \sum_{I_1 \subsetneq \cdots \subsetneq I_k} q^{d(I_1,\ldots,I_k)}}{(1-q)^n},$$

where the sum is over all strictly increasing chains in Σ and the number $d(I_1, \ldots, I_k)$ is the degree of the monomial $m_{\{I_1,\ldots,I_k\}}$.

Remark that if $I_1, \ldots, I_k$ is not a chain of subsets then $|\tilde{M}_{I_1} \cap \cdots \cap \tilde{M}_{I_k}|$ may not be a polynomial in the $\nu_I(i)$. It may include expressions like $\min(\nu_I(i), \nu_J(i))$. Thus the inclusion-exclusion principle does not immediately produce a polynomial expression for $\dim \mathcal{A}$. Miraculously, all non-polynomial terms cancel each other.