The Random Normal Matrix Model: Insertion of a Point Charge

Yacin Ameur

Motivation: normal random matrices

Let M a $n \times n$ normal matrix,

$$p_n(\zeta) = \det(\zeta - M).$$

For randomized M the asymptotic distribution of

$$\log |p_n(\zeta)|, \qquad |p_n(\zeta)|, \qquad (n \to \infty)$$

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Variational approach: study conditional distribution, given that there is a charge at $\zeta = 0$.

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The droplet $S = S[Q] = \operatorname{supp}\sigma$. We assume that

$$S \subset \operatorname{Int}\{Q < +\infty\}.$$

Then

$$d\sigma = \Delta Q \cdot \mathbf{1}_S \cdot dA.$$

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We assume

$$0 \in \operatorname{Int} S.$$

Example: cubic potential (Teodorescu et al)

$$Q = |z|^2 - \mathrm{Re}\,(z^3).$$

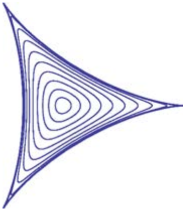

Figure 1: Droplets of some multiples of the cubic potential

Algebraic potentials

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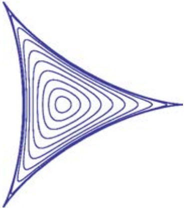

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Algebraic potential:

$$Q = |\zeta|^{2k} - \mathrm{Re} \sum_{1}^{d} t_j \zeta^j.$$

Standard RNM-model

Particles/eigenvalues $\{\zeta_j\}_1^n$ in external field nQ.

Energy:

$$H_n = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_1^n Q(\zeta_j).$$

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Probability law:

$$dP_n(\zeta) = e^{-H_n(\zeta)}\, dA_n(\zeta) / \int_{\mathbb{C}^n} e^{-H_n}\, dA_n.$$

$(dA_n(\zeta_1, \ldots, \zeta_n) = dA(\zeta_1) \cdots dA(\zeta_n))$.

Insertion of charge of strength $c > -1$

n-dependent potential

$$V_n = Q + \frac{2c}{n} \log \frac{1}{|\zeta|} - \frac{1}{n} u(\zeta).$$

The ensemble with potential V_n has *exactly the same droplet*.

n-dependent terms have purely *statistical effect*.

Logarithmic term affects the microscopic behaviour near 0 and near the boundary.

We take $u = 0$ or $u =$ harmonic part of Green's function, to accommodate different kinds of boundary behaviour.

Insertion as a balayage operation

Let $\mathbf{R}_n$ 1-point function for Q, $\tilde{\mathbf{R}}_n$ 1-point for V_n (with $u = 0$).
Thus $\mathbf{R}_n(\mathbb{C}) = \tilde{\mathbf{R}}_n(\mathbb{C}) = n$.

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Theorem

In the sense of measures,

$$\tilde{\mathbf{R}}_n - \mathbf{R}_n \to c(\omega_\infty - \delta_0)$$

where ω_∞ is the harmonic measure of $\mathbb{C} \setminus S$, evaluated at ∞.

If instead $V_n = Q + \frac{2c}{n} G$ then

$$\tilde{\mathbf{R}}_n - \mathbf{R}_n \to c(\omega_0 - \delta_0)$$

where ω_0 the harmonic measure with respect to S, evaluated at 0.

Insertion as balayage: pictures

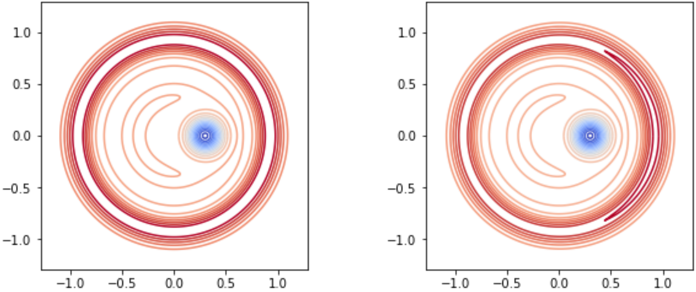

Figure 2: Some level curves of ρ_n with respect to Ginibre potential $V_n = |\zeta|^2 - 2c \log |\zeta - a|/n$ when $n = 40$, $c = 1$, $a = 0.3$ with pure-log normalization and Green's normalization, respectively. (Blue means negative and red means positive.)

Rescaling, Mittag-Leffler limit

Consider

$$Q = |\zeta|^{2k} + \mathrm{Re}\, H(\zeta) + O(|\zeta|^{2k+1}), \quad V_n = Q - \frac{2c}{n} \log |\zeta|,$$

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"Typical" result: $R_n \to R$ where
$R(z) = kE_{1/k,(1+c)/k}(|z|^2)e^{-V_0(z)}$,

$$E_{a,b}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj+b)}.$$

Note. $E_{1,1} = \exp$.

Examples

If $k = 1$, $c = 0$ we get $R = 1$ (infinite Ginibre ensemble).

In general, $R(z) \sim \Delta V_0(z)$ as $z \to \infty$.

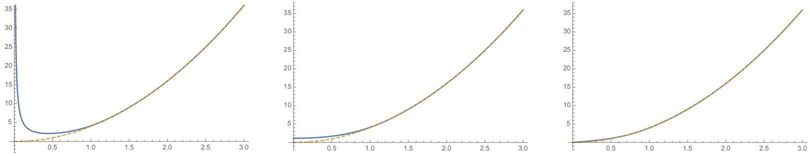

Figure 3: The graph of R restricted to the positive real axis for combined singularities of type $(c, 2)$, with $c = -0.5$, $c = 0$, and $c = 0.5$. The graph of ΔV_0 is drawn with an orange dashed line.

Correlation kernels

Let $\mathbf{K}_n$ be the reproducing kernel of $\mathrm{Pol}(n)$ with respect to $d\mu_n = e^{-nV_n}\, dA$. Then

$$\mathbf{R}_n(\zeta_1, \ldots, \zeta_k) = \det(\mathbf{K}_n(\zeta_i, \zeta_j))_{k\times k}.$$

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Rescaling

$$K_n(z,w) = r_n^2\mathbf{K}_n(r_n z, r_n w)$$

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Cocycle: $c_n(z,w) = g(z)g(w)^{-1}$ where $g:\mathbb{C}\to\mathbb{T}$ is continuous. c_nK_n is another correlation kernel.

Normal families theorem

Taylor's formula: (Q any potential, $0 \in \operatorname{Int} S$)

$$Q(z) = Q_0(z) + h(z) + O(|\zeta|^{2k+1})$$

Q_0 positive definite, homogeneous degree $2k$,

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Microscopic potential

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Theorem

There are cocycles c_n such that

$$c_n(z,w)K_n(z,w) = L_n(z,w)e^{-V_0(z)/2 - V_0(w)/2}(1 + o(1))$$

where L_n are Hermitian-entire and locally bounded on $\mathbb{C}^2$.

Limiting point fields

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$L_0(z, w)$ Bergman kernel of $L^2_a(\mu_0)$.

$L = \lim L_{n_k}$ a subsequential limit.

Theorem

L is the Bergman kernel for some subspace of $L^2_a(\mu_0)$ and $L \leq L_0$ in the sense of positive matrices.

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Define

$$K(z,w) = L(z,w)e^{-V_0(z)/2 - V_0(w)/2}, \quad R(z) = K(z,z).$$

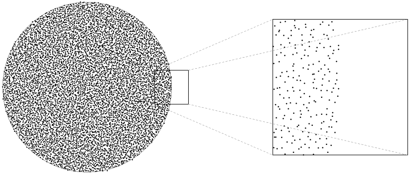
K is the correlation kernel of a *limiting point field* $\{z_j\}_1^\infty$.

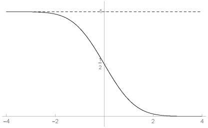
The Ginibre and erfc-limits

Theorem

(i) If 0 *is a regular bulk point then* $L = e^{z\bar{w}}$ *and* $K(z,w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2}$ *is Ginibre kernel,* $R = 1$.

(ii) If 0 *is a regular boundary point then* $L = e^{z\bar{w}} F(z + \bar{w})$ *where* $F(z) = (1/2)\,\mathrm{erfc}(z/\sqrt{2})$.





Fock-Sobolev space

Let $L_0(z, w)$ Bergman kernel for $L^2_a(\mu_0)$. If $Q_0 = |z|^{2k}$ then

$$L_0(z, w) = kE_{1/k,(1+c)/k}(z\bar{w}).$$

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Conjectures:

- At a bulk point each limiting 1-point function L is $L = L_0$.

- At a boundary point, L is the Bergman kernel for a certain closed subspace of $L_a^2(\mu_0)$. (For the erfc-kernel, the space is the closed linear span in Bargmann-Fock space of $z \mapsto e^{\bar{w}z}F(z+\bar{w})$, $F(z) = \mathrm{erfc}(z/\sqrt{2})$.)

A non-symmetric example

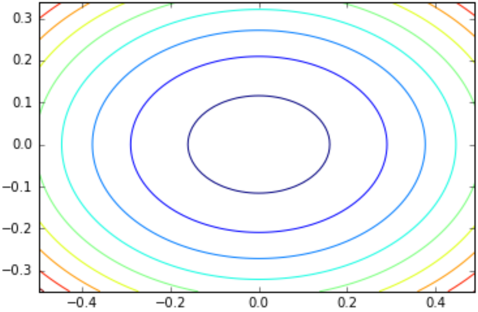

Figure 4: Some level curves of $R_0(z) = L_0(z, z)e^{-Q_0(z)}$ for $Q_0(z)$ proportional to $|z|^4 - |z|^2 \mathrm{Re}(z^2)/2$ and $c = 0$.

Nontriviality in the bulk; homogeneous potentials

Theorem

Each limiting 1-point function satisfies

$$R(z) = \Delta Q_0(z) \cdot (1 + O(e^{-\alpha|z|^{2k}})), \qquad (z \to \infty)$$
$$R(z) = O(|z|^{2c}), \qquad (z \to 0).$$

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$$R(z) = O(|z|^{2c}), \qquad (z \to 0).$$

Theorem

Suppose that $Q = Q_0 + h + O(|\zeta|^{2k+1})$, $(\zeta \to 0)$ *where*

$$h(\zeta) = Re\,(t\zeta^{2k}).$$

Then $L = L_0$ *(and* $R(z) = L(z,z)e^{-V_0(z)}$*.)*

Ward's equation; Zero-one law

Define

$$B(z,w) = \frac{|K(z,w)|^2}{R(z)}, \quad C(z) = \int_{(w)} \frac{B(z,w)}{z-w}.$$

Theorem

If $R > 0$ at some point then $R > 0$ on $\mathbb{C}^*$ and

$$\bar{\partial} C = R - \Delta V_0 - \Delta \log R.$$

Proof follows by rescaling in Ward's identity,

$$\mathbf{E}_n W_n[\psi] = 0,$$

$$W_n[\psi] = \sum \partial\psi(\zeta_j) - n\sum[\psi\partial V_n](\zeta_j) + \frac{1}{2}\sum_{j\neq k} \frac{\psi(\zeta_j) - \psi(\zeta_k)}{\zeta_j - \zeta_k}.$$

Dominant radial case

Suppose $Q_0(z) = Q_0(|z|)$. Wlog

$$Q_0 = |z|^{2k}.$$

A limiting kernel is *symmetric* if

$$L(z, w) = E(z\bar{w}).$$

Theorem

If L is symmetric, then $E = kE_{1/k,(1+c)/k}$.

Problem: How to show that each L is symmetric?

Radial symmetry and orthogonal polynomials

Suppose

$$Q = |\zeta|^{2k} + \operatorname{Re} H(\zeta), \quad H(\zeta) = t_1\zeta + \cdots + t_d\zeta^d.$$

We can write

$$L_n(z,w) = \sum_{j=0}^{n-1} q_{n,j}(z)\bar{q}_{n,j}(w)e^{-nH(r_n z)/2 - n\bar{H}(r_n w)/2}$$

where $q_{n,j}$ are certain orthogonal polynomials. In fact,

$$q_{n,j}(z) = r_n^{1+c} p_{n,j}(r_n z)$$

where $p_{n,j}$ are the OP's w.r.t. $e^{-nV_n}\, dA$.

Lemma

We have $\partial_\theta L_n(z,z) \to 0$ provided that the weighted polynomials $p_{n-d}e^{-nV_n/2}, \ldots, p_{n+d}e^{-nV_n/2}$ are concentrated near the outer boundary of S.

Asymptotics of orthogonal polynomials

Lemma

(HW) If K is a compact subset of $\operatorname{Int}\operatorname{Pc} S$ and if $|j-n| \le d$ then

$$\int_K |p_{n,j}|^2 e^{-nV_n}\, dA = O(n^{-\infty}).$$

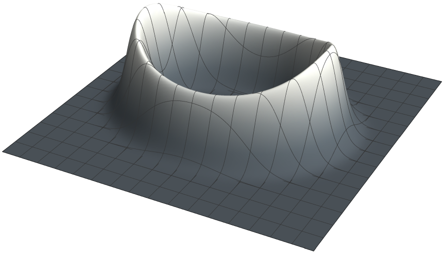

Figure 5: The graph of $|p_n(\zeta)|^2 e^{-nQ(\zeta)}$ with $Q(z) = a|\zeta|^2 + b\mathrm{Re}\,\zeta$ (ellipse ensemble).

Universality for algebraic potentials

Corollary

If $V_n = |\zeta|^{2k} + Re \sum_1^d t_i \zeta^i - (2c/n) \log|\zeta|$ *then the point processes* $\{\zeta_k\}$ *converge to the limiting point field with 1-point function* $R(z) = kE_{1/k,(1+c)/k}(|z|^2)e^{-V_0(z)}$.

Corollary

Let $(\zeta_j)_1^n$ *be random sample associated with* V_n *and let* $p_n(\zeta) = \prod_1^n(\zeta - \zeta_j)$. *Then*

$$(\log|p_n(0)| - \mathbf{E}_n \log|p_n(0)|)/\sqrt{\log n} \Rightarrow N(0, 1/k), \qquad (n \to \infty).$$

Lemniscate ensemble

Potential $Q(\zeta) = |\zeta^k - k^{-1/2}|^2$.

Droplet $|\zeta^k - k^{-1/2}| \leq k^{-1/2}$.

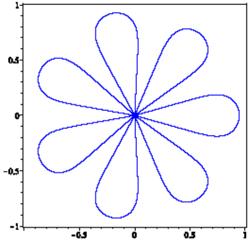

Figure 6: 0 is a boundary point at which $\Delta Q = 0$.

Scaling limit

Rescaling: $z = n^{1/2k}\zeta$. Rescaled droplet is

$$\mathrm{Re}\,(z^k) \geq 0.$$

Here $Q_0 = |\zeta|^{2k}$ and $c = 0$ so the Fock-Sobolev space has r.k.

$$L_0(z, w) = kE_{1/k,1/k}(z\bar{w}).$$

So $L \leq L_0$ for each limiting kernel.

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