On the conditions for entire functions related to the partial theta function to belong to the Laguerre-Pólya class

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The Laguerre-Pólya class

Definition

Definition 1. A real entire function f is said to be in the *Laguerre-Pólya class*, written $f \in \mathcal{L}-\mathcal{P}$, if it can be expressed in the form

$$f(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right) e^{\frac{x}{x_k}}, \tag{1}$$

where $c, \beta, x_k \in \mathbb{R}$, $x_k \neq 0$, $\alpha \geq 0$, n is a non-negative integer and $\sum_{k=1}^{\infty} \frac{1}{x_k^2} < \infty$. The product on the right-hand side can be finite or empty (in the latter case the product equals 1).

Laguerre-Pólya's Theorem

Theorem

Theorem A *(E.Laguerre and G.Pólya, see, for example, [1, p. 42-46])).*

1. *Let* $(P_n)_{n=1}^{\infty},\ P_n(0) = 1$, *be a sequence of complex polynomials having only real zeros which converges uniformly in the circle* $|z| \leq A, A > 0$. *Then this sequence converges uniformly on compact sets in* $\mathbb{C}$ *to an entire function* f, *and* f *is from the* $\mathcal{L} - \mathcal{P}$ *class.*
2. *And conversely, for any* $f \in \mathcal{L} - \mathcal{P}$ *there is a sequence of complex polynomials with only real zeros which converges uniformly on compact sets of* $\mathbb{C}$ *to* f.

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Multiplier Sequences

Definition

Definition 2. A sequence $(\gamma_k)_{k=0}^{\infty}$ of real numbers is called a multiplier sequence if, whenever the real polynomial $P(x) = \sum_{k=0}^{n} a_k z^k$ has only real zeros, the polynomial $\sum_{k=0}^{n} \gamma_k a_k z^k$ has only real zeros. The class of multiplier sequences is denoted by $\mathcal{MS}$.

Theorem

Theorem B *(G. Pólya and J.Schur). Let* $(\gamma_k)_{k=0}^{\infty}$ *be a given real sequence. The following three statements are equivalent.*

Algebraic and Transcendental Characterizations of Multiplier Sequences

1. $(\gamma_k)_{k=0}^{\infty}$ is a multiplier sequence.
2. For every $n \in \mathbb{N}$ the polynomial $P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k$ has only real zeros of the same sign.
3. The power series $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ converges absolutely in the whole complex plane and the entire function $\Phi(z)$ or the entire function $\Phi(-z)$ admits the representation

$$Ce^{\sigma z} z^m \prod_{k=1}^{\infty} (1 + \frac{z}{x_k}), \tag{2}$$

where $C \in \mathbb{R}, \sigma \geq 0, m \in \mathbb{N} \cup \{0\}, 0 < x_k \leq \infty, \ \sum_{k=1}^{\infty} \frac{1}{x_k} < \infty.$

Multiplier Sequences

Corollary

Corollary *(Theorem B).*
The sequence $(\gamma_0, \gamma_1, \ldots, \gamma_l, 0, 0, \ldots)$ *is a multiplier sequence if and only if the polynomial* $P(z) = \sum_{k=0}^{l} \frac{\gamma_k}{k!} z^k$ *has only real zeros of the same sign.*

Complex Zero Decreasing Sequences

1. For a real polynomial P we will denote by $Z_c(P)$ the number of nonreal zeros of P counting multiplicities.

Definition

Definition 3. A sequence $(\gamma_k)_{k=0}^{\infty}$ of real numbers is said to be a complex zero decreasing sequence if

$$Z_c\left(\sum_{k=0}^{n} \gamma_k a_k z^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_k z^k\right), \tag{3}$$

for any real polynomial $\sum_{k=0}^{n} a_k z^k$. The class of complex zero decreasing sequences is denoted by $\mathcal{CZDS}$.

2. Obviously, $\mathcal{CZDS} \subset \mathcal{MS}$. The existence of nontrivial $\mathcal{CZDS}$ sequences is a consequence of the following remarkable theorem proved by Laguerre and extended by Pólya.

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Complex Zero Decreasing Sequences

Theorem

Theorem C *(E. Laguerre). Suppose f is an entire function from the Laguerre-Pólya class having only negative zeros. Then the sequence* $(f(k))_{k=0}^{\infty}$ *is a complex zero decreasing sequence.*

1. As it follows from the above theorem,

$$\left(a^{-k^2}\right)_{k=0}^{\infty} \in \mathcal{CZDS},\ a \geq 1, \quad \left(\frac{1}{k!}\right)_{k=0}^{\infty} \in \mathcal{CZDS}. \qquad (4)$$

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Quotients of Taylor Coefficients

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with positive coefficients. We will use the following notations:

1. $q_n = q_n(f) := \frac{a_{n-1}^2}{a_{n-2}a_n}, \; n \geq 2.$
2. It is easy to see that

$$a_n = \frac{a_1}{q_2^{n-1} q_3^{n-2} \ldots q_{n-1}^2 q_n} \left(\frac{a_1}{a_0}\right)^{n-1}, \; n \geq 2.$$

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Examples

1. Let $h_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{k! a^{k^2}}$, $a \geq 1$. Then $h_a \in \mathcal{L} - \mathcal{P}$, $q_n(h_a) = \frac{n}{n-1} \cdot a^2$, $q_n(h_a)$ are decreasing in n and $\lim_{n\to\infty} q_n(h_a) = a^2$.
2. The following identity belongs to Gauss:

$$\varphi_a(z) := \sum_{k=0}^{\infty} \frac{z^k}{(a^k - 1)(a^{k-1} - 1) \cdot \ldots \cdot (a-1)} =$$

$$\prod_{k=1}^{\infty} \left(1 + \frac{z}{a^k}\right), \ a > 1.$$

Thus $\varphi_a \in \mathcal{L} - \mathcal{P}$, $q_n(\varphi_a) = \frac{a^n - 1}{a^{n-1} - 1}$, $q_n(\varphi_a)$ are decreasing in n and $\lim_{n\to\infty} q_n(\varphi_a) = a$.

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Sufficient Condition for an Entire Function to Have Only Real Zeros

Theorem

Theorem D *(J. I. Hutchinson, [2, p. 327]) . Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function with positive coefficients. Inequalities* $q_n(f) \geq 4$, $\forall n \geq 2$, *are valid if and only if the following two properties hold:*
(i) The zeros of f *are all real, simple and negative and*
(ii) the zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$, $m < n$, *formed by taking any number of consecutive terms of* f, *are all real and non-positive.*

Easy to check: the constant 4 is the smallest possible in both statements.

Partial Theta Function

Definition

Definition 4. The entire function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, is called the *partial theta function*.

1. Note that $q_n(g_a) = a^2$ for all n.
2. The paper [4] by Olga Katkova, Tetyana Lobova and A.V. gives the answer to the question: for which $a > 1$ the entire function g_a belongs to the Laguerre-Pólya class.

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Conditions for the Partial Theta Function to Belong to the Laguerre-Pólya class

Theorem

Theorem E *(O.M.Katkova, T.Lobova and A.V., [4]). There exists a constant* q_∞ $(q_\infty \approx 3.23363666)$ *such that:*

1. $g_a(z) \in \mathcal{L} - \mathcal{P} \Leftrightarrow \; a^2 \geq q_\infty;$
2. $g_a(z) \in \mathcal{L} - \mathcal{P} \Leftrightarrow$ *there exists* $x_0 \in (-a^3, -a) : \; g_a(x_0) \leq 0;$
3. *for every* $n \geq 2$ *we have* $S_n(z, g_a) \in \mathcal{L} - \mathcal{P} \; \Leftrightarrow$ *there exists* $x_n \in (-a^3, -a) : \; S_n(x_n, g_a) \leq 0;$
4. *for every* $n \geq 2$ *there exists a constant* $c_n > 1$: $S_n(z, g_a) := \sum_{j=0}^{n} z^j a^{-j^2} \in \mathcal{L} - \mathcal{P} \; \Leftrightarrow \; a^2 \geq c_n$, *and*

$$4 = c_2 > c_4 > c_6 > \ldots \quad \text{and} \quad \lim_{n\to\infty} c_{2n} = q_\infty;$$

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Partial Theta Function

Many interesting and important facts about zeros of the partial theta function and its derivative were obtained by Vladimir Kostov in [1], [2], [3] and [4].

The Constant q_∞

Theorem

Theorem F *(V.P.Kostov, B.Shapiro, [5]). Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function with positive coefficients and* $S_n(z) = \sum_{j=0}^{n} a_j z^j$ *be its sections. Suppose that* $\exists N \in \mathbb{N}:$ $\forall n \geq N$ *the sections* $S_n(z) = \sum_{j=0}^{n} a_j z^j$ *belong to the Laguerre-Pólya class. Then* $\liminf_{n\to\infty} q_n(f) \geq q_\infty$.

Entire functions related to the partial theta function

We will investigate a family of entire functions

$$f^{(m,a)}(z) = \sum_{k=0}^{+\infty} \frac{z^k (k!)^m}{a^{k^2}}, \quad a > 1,\ m \geq 0,$$

and their Taylor sections

$$S_n^{(m,a)}(z) = \sum_{k=0}^{n} \frac{z^k (k!)^m}{a^{k^2}}.$$

For every $n \in \mathbb{N}, n \geq 2$, there exists a constant $d_{(n,m)} \geq 1$ such that $S_n^{(m,a)} \in \mathcal{L} - \mathcal{P} \Leftrightarrow a^2 \geq d_{(n,m)}$. There exists a constant $d_{(\infty,m)} > 1$ such that $f^{(m,a)} \in \mathcal{L} - \mathcal{P} \Leftrightarrow a^2 \geq d_{(\infty,m)}$.

Entire functions related to the partial theta function

Theorem

(Anton Bohdanov and A.V., [2]). (1) For every fixed $m \geq 1$ the function $f^{(m,a)}$ belongs to the class $\mathcal{L} - \mathcal{P}$ if and only if there exists $x_0 = x_0(m) \in [-q_2(f^{(m,a)}); -1]$ such that $f^{(m,a)}(x_0) \leq 0$. For every fixed $m \geq 1$ and $n \in \mathbb{N}, n \geq 2$, the polynomial $S_n^{(m,a)}$ has only real zeros if and only if there exists $x_0 = x_0(m,n) \in [-q_2(f^{(m,a)}); -1]$ such that $S_n^{(m,a)}(x_0) \leq 0$.

(2) $3 \cdot 2^m < d_{(3,m)} < d_{(5,m)} < d_{(7,m)} < \ldots < d_{(\infty,m)}$.

(3) $\lim_{n\to\infty} d_{(2n+1,m)} = d_{(\infty,m)}$.

(4) $4 \cdot 2^m = d_{(2,m)} > d_{(4,m)} > d_{(6,m)} > \ldots > d_{(\infty,m)}$.

(5) $\lim_{n\to\infty} d_{(2n,m)} = d_{(\infty,m)}$.

(6) For every $n \in \mathbb{N}, n \geq 2$, the function $d_{(n,m)}$ is the continuous increasing function as a function of m. The function $d_{(\infty,m)}$ is also the continuous increasing function of m.

Entire functions related to the partial theta function

Anton Bohdanov in [1] obtained an analogous result for the case $0 < m < 1$ (the same method but very cumbersome estimates).

Entire functions related to the partial theta function

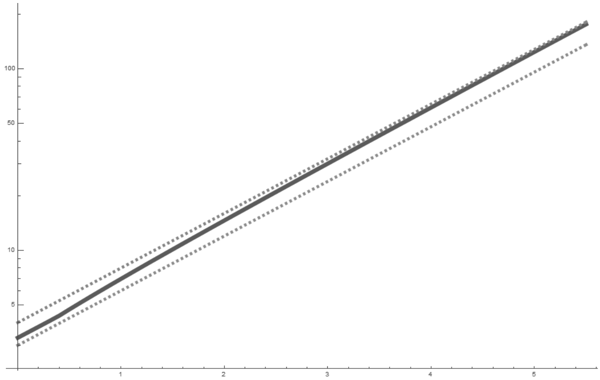

Figure: Behavior of $d_{(3,m)}$(on y-axis with a logarithmic scale) with different values of m (on x-axis, m ranges from 0 to 5.6) shown as a thick line. Dashed ones are the bounds given by theoretical estimates.

Entire functions related to the partial theta function

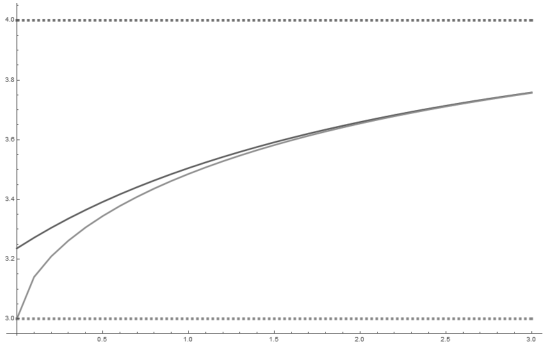

Figure: Behavior of $d_{(3,m)}$(the lower regular line) and $d_{(4,m)}$ (the upper one) with different values of m. Dashed lines are the bounds given by theoretical estimates.

Open problem

For which $a > 0$ the function

$$f^{(m,a)}(z) = \sum_{k=0}^{+\infty} \frac{z^k (k!)^m}{a^{k^2}}, \quad m < 0,$$

(and its Taylor sections) belongs to the Laguerre-Pólya class? It is well-known that for every negative integer m and every $a > 0$ we have $f^{(m,a)} \in \mathcal{L} - \mathcal{P}$. What about other negative m?

Entire functions having the decreasing second quotients of Taylor coefficients

Theorem

(Thu Hien Nguyen and A.V., [1]).

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$*,* $a_k > 0$*, be an entire function. Suppose that* $q_n(f)$ *are decreasing in* n*, i.e.* $q_2 \geq q_3 \geq q_4 \geq \ldots,$ *and* $\lim_{n\to\infty} q_n(f) = b \geq q_\infty$*.*

Then all the zeros of f *are all real and negative, in other words* $f \in \mathcal{L} - \mathcal{P}$*.*

Entire functions having the increasing second quotients of Taylor coefficients

Theorem

(Thu Hien Nguyen and A.V.).
Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$, *be an entire function. Suppose that* $q_n(f)$ *are increasing in n, i.e.* $q_2 \leq q_3 \leq q_4 \leq \ldots,$ *and* $\lim_{n\to\infty} q_n(f) = b < q_\infty$.
Then f has at least two non-real zeros, in other words $f \notin \mathcal{L} - \mathcal{P}$.

Thanks for attention!

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