On the asymptotic zero distribution of orthogonal polynomials on the unit circle with variable Verblunsky coefficients

František Štampach

joint with Rostyslav Kozhan

Institute Mittag-Leffler

Hausdorff geometry of polynomials and polynomial sequences

June 1, 2018

Contents

Notation

- For $t > 0$ and a doubly indexed sequence $X_{n,N}$ and $t > 0$ we write

$$\lim_{n/N \to t} X_{n,N} = X$$

Notation

- For $t > 0$ and a doubly indexed sequence $X_{n,N}$ and $t > 0$ we write

$$\lim_{n/N \to t} X_{n,N} = X$$

which means that

$$\lim_{j \to \infty} X_{n_j, N_j} = X,$$

for any $\{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ and $\{N_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ such that

$$n_j, N_j \to \infty \quad \text{and} \quad n_j/N_j \to t, \quad \text{as } j \to \infty.$$

Notation

- For $t>0$ and a doubly indexed sequence $X_{n,N}$ and $t>0$ we write

$$\lim_{n/N\to t} X_{n,N} = X$$

which means that

$$\lim_{j\to\infty} X_{n_j,N_j} = X,$$

for any $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ and $\{N_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ such that

$$n_j, N_j \to \infty \quad \text{and} \quad n_j/N_j \to t, \quad \text{as } j \to \infty.$$

(The meaning of the used topology will be always clear from the context.)

The variable coefficient OPRL

- Kuijlaars & Van Assche, JAT99.

The variable coefficient OPRL

- Kuijlaars & Van Assche, JAT99.
- Fix $t > 0$. Let for all $N \in \mathbb{N}$, sequences

$$\{a_{n,N}\}_{n=0}^{\infty} \subset (0, \infty) \quad \text{and} \quad \{b_{n,N}\}_{n=0}^{\infty} \subset \mathbb{R}$$

  are given.

The variable coefficient OPRL

- Kuijlaars & Van Assche, JAT99.
- Fix $t > 0$. Let for all $N \in \mathbb{N}$, sequences

$$\{a_{n,N}\}_{n=0}^{\infty} \subset (0, \infty) \quad \text{and} \quad \{b_{n,N}\}_{n=0}^{\infty} \subset \mathbb{R}$$

are given. Suppose further that there exist functions $a, b \in C((0, t])$ such that

$$\lim_{n/N \to s} a_{n,N} = a(s) \quad \text{and} \quad \lim_{n/N \to s} b_{n,N} = b(s),$$

for all $s \in (0, t]$.

The variable coefficient OPRL

- Kuijlaars & Van Assche, JAT99.
- Fix $t > 0$. Let for all $N \in \mathbb{N}$, sequences

$$\{a_{n,N}\}_{n=0}^{\infty} \subset (0, \infty) \quad \text{and} \quad \{b_{n,N}\}_{n=0}^{\infty} \subset \mathbb{R}$$

  are given. Suppose further that there exist functions $a, b \in C((0,t])$ such that

$$\lim_{n/N \to s} a_{n,N} = a(s) \quad \text{and} \quad \lim_{n/N \to s} b_{n,N} = b(s),$$

  for all $s \in (0, t]$.
- As a particular case, one can take

$$a_{n,N} := a\left(\frac{n+1}{N}t\right) \quad \text{and} \quad b_{n,N} := b\left(\frac{n+1}{N}t\right).$$

The variable coefficient OPRL

- Kuijlaars & Van Assche, JAT99.
- Fix $t > 0$. Let for all $N \in \mathbb{N}$, sequences

$$\{a_{n,N}\}_{n=0}^{\infty} \subset (0, \infty) \quad \text{and} \quad \{b_{n,N}\}_{n=0}^{\infty} \subset \mathbb{R}$$

are given. Suppose further that there exist functions $a, b \in C((0, t])$ such that

$$\lim_{n/N \to s} a_{n,N} = a(s) \quad \text{and} \quad \lim_{n/N \to s} b_{n,N} = b(s),$$

for all $s \in (0, t]$.

- As a particular case, one can take

$$a_{n,N} := a\left(\frac{n+1}{N} t\right) \quad \text{and} \quad b_{n,N} := b\left(\frac{n+1}{N} t\right).$$

- Consider a family of OPRL $p_{n,N}$ determined by the recurrence

$$p_{n+1,N}(x) = (x - b_{n,N})\, p_{n,N}(x) - a_{n-1,N}^2\, p_{n-1,N}(x)$$

with initial conditions $p_{-1,N}(x) = 0$ and $p_{0,N}(x) = 1$.

The variable coefficient OPRL

- Denote

$$\nu_{n,N} := \frac{1}{n}\sum_{k=1}^{n}\delta_{x_k},$$

where $x_k = x_{k,n,N}$ are roots of $p_{n,N}$.

The variable coefficient OPRL

- Denote

$$\nu_{n,N} := \frac{1}{n}\sum_{k=1}^{n}\delta_{x_k},$$

  where $x_k = x_{k,n,N}$ are roots of $p_{n,N}$.
- **Question:** What is the asymptotic distribution of roots of $p_{n,N}$ if $n/N \to t$?

The variable coefficient OPRL

- Denote

$$\nu_{n,N} := \frac{1}{n}\sum_{k=1}^{n}\delta_{x_k},$$

  where $x_k = x_{k,n,N}$ are roots of $p_{n,N}$.
- **Question:** What is the asymptotic distribution of roots of $p_{n,N}$ if $n/N \to t$?

Theorem (Kuijlaars & Van Assche)

One has

$$\lim_{n/N\to t}\nu_{n,N} = \frac{1}{t}\int_0^t \omega_{[b(s)-2a(s),b(s)+2a(s)]}\mathrm{d}s,$$

where

$$\frac{\mathrm{d}\omega_{[\alpha,\beta]}}{\mathrm{d}x}(x) = \frac{1}{\pi\sqrt{(x-\alpha)(\beta-x)}}, \quad \text{for } \alpha < x < \beta,$$

and $\omega_{[\alpha,\beta]} = \delta_\alpha$, if $\alpha = \beta$.

The variable coefficient OPRL

- Denote

$$\nu_{n,N} := \frac{1}{n}\sum_{k=1}^{n}\delta_{x_k},$$

  where $x_k = x_{k,n,N}$ are roots of $p_{n,N}$.
- **Question:** What is the asymptotic distribution of roots of $p_{n,N}$ if $n/N \to t$?

Theorem (Kuijlaars & Van Assche)

One has

$$\lim_{n/N\to t}\nu_{n,N} = \frac{1}{t}\int_0^t \omega_{[b(s)-2a(s),b(s)+2a(s)]}\mathrm{d}s,$$

where

$$\frac{\mathrm{d}\omega_{[\alpha,\beta]}}{\mathrm{d}x}(x) = \frac{1}{\pi\sqrt{(x-\alpha)(\beta-x)}}, \quad \text{for } \alpha < x < \beta,$$

and $\omega_{[\alpha,\beta]} = \delta_\alpha$, if $\alpha = \beta$.

- **Remark:** The measure $\omega_{[\alpha,\beta]}$ is the equilibrium measure of $[\alpha, \beta]$ (log. pot. theory).

Locally Toeplitz matrices

- Equivalently, the theorem gives the asymptotic eigenvalue distribution of Jacobi matrices

$$J_{n,N} = \begin{pmatrix} b_{0,N} & a_{0,N} & & & & \\ a_{0,N} & b_{1,N} & a_{1,N} & & & \\ & a_{1,N} & b_{2,N} & a_{2,N} & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-3,N} & b_{n-2,N} & a_{n-2,N} \\ & & & & a_{n-2,N} & b_{n-1,N} \end{pmatrix},$$

as $n/N \to t$.

Locally Toeplitz matrices

- Equivalently, the theorem gives the asymptotic eigenvalue distribution of Jacobi matrices

$$J_{n,N} = \begin{pmatrix} b_{0,N} & a_{0,N} & & & & \\ a_{0,N} & b_{1,N} & a_{1,N} & & & \\ & a_{1,N} & b_{2,N} & a_{2,N} & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-3,N} & b_{n-2,N} & a_{n-2,N} \\ & & & & a_{n-2,N} & b_{n-1,N} \end{pmatrix},$$

  as $n/N \to t$.
- The concept of **locally Toeplitz matrices** (Tilli):

$$\begin{aligned} \text{locally Toeplitz structure} \quad &\longleftrightarrow \quad a_{i,j} = a_{i+1,j+1} + o(1), \quad \text{as } n/N \to t, \\ \text{Toeplitz structure} \quad &\longleftrightarrow \quad a_{i,j} = a_{i+1,j+1}. \end{aligned}$$

Locally Toeplitz matrices

- Equivalently, the theorem gives the asymptotic eigenvalue distribution of Jacobi matrices

$$J_{n,N} = \begin{pmatrix} b_{0,N} & a_{0,N} & & & & \\ a_{0,N} & b_{1,N} & a_{1,N} & & & \\ & a_{1,N} & b_{2,N} & a_{2,N} & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-3,N} & b_{n-2,N} & a_{n-2,N} \\ & & & & a_{n-2,N} & b_{n-1,N} \end{pmatrix},$$

  as $n/N \to t$.
- The concept of **locally Toeplitz matrices** (Tilli):

$$\begin{aligned} \text{locally Toeplitz structure} \quad &\longleftrightarrow \quad a_{i,j} = a_{i+1,j+1} + o(1), \quad \text{as } n/N \to t, \\ \text{Toeplitz structure} \quad &\longleftrightarrow \quad a_{i,j} = a_{i+1,j+1}. \end{aligned}$$

- Tilli deduced the asymptotic eigenvalue distribution for Hermitian locally Toeplitz matrices in LAA98 - a generalization of the result from JAT99.

Locally Toeplitz matrices

- Equivalently, the theorem gives the asymptotic eigenvalue distribution of Jacobi matrices

$$J_{n,N} = \begin{pmatrix} b_{0,N} & a_{0,N} & & & & \\ a_{0,N} & b_{1,N} & a_{1,N} & & & \\ & a_{1,N} & b_{2,N} & a_{2,N} & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-3,N} & b_{n-2,N} & a_{n-2,N} \\ & & & & a_{n-2,N} & b_{n-1,N} \end{pmatrix},$$

  as $n/N \to t$.
- The concept of **locally Toeplitz matrices** (Tilli):

$$\begin{aligned} \text{locally Toeplitz structure} \quad &\longleftrightarrow \quad a_{i,j} = a_{i+1,j+1} + o(1), \quad \text{as } n/N \to t, \\ \text{Toeplitz structure} \quad &\longleftrightarrow \quad a_{i,j} = a_{i+1,j+1}. \end{aligned}$$

- Tilli deduced the asymptotic eigenvalue distribution for Hermitian locally Toeplitz matrices in LAA98 - a generalization of the result from JAT99.
- Tilli's motivation stems from a discretization of a 1D Sturm–Liouville operator.

Kac–Murdock–Szegő matrices

- The topic of JAT99 and LAA98 appeared much earlier in a paper by M. Kac, W. L. Murdock, and G. Szegő **in 1953**.

Kac–Murdock–Szegő matrices

- The topic of JAT99 and LAA98 appeared much earlier in a paper by M. Kac, W. L. Murdock, and G. Szegő **in 1953**.
- They introduced the $n \times n$ matrices with entries:

$$(T_n(a))_{i,j} = a_{j-i}\left(\frac{i+j}{2n}\right), \qquad \text{(the "KMS matrix")}$$

where $a_k \in C([0,1])$ are given.

Kac–Murdock–Szegő matrices

- The topic of JAT99 and LAA98 appeared much earlier in a paper by M. Kac, W. L. Murdock, and G. Szegő **in 1953**.
- They introduced the $n \times n$ matrices with entries:

$$(T_n(a))_{i,j} = a_{j-i}\left(\frac{i+j}{2n}\right), \qquad \text{(the “KMS matrix”)}$$

$$\left(\dot{T}_n(a)\right)_{i,j} = a_{j-i}\left(\frac{\min(i,j)}{n}\right) \quad \text{and} \quad \left(\ddot{T}_n(a)\right)_{i,j} = a_{j-i}\left(\frac{\max(i,j)}{n}\right)$$

where $a_k \in C([0,1])$ are given.

Kac–Murdock–Szegő matrices

- The topic of JAT99 and LAA98 appeared much earlier in a paper by M. Kac, W. L. Murdock, and G. Szegő **in 1953**.
- They introduced the $n \times n$ matrices with entries:

$$(T_n(a))_{i,j} = a_{j-i}\left(\frac{i+j}{2n}\right), \qquad \text{(the “KMS matrix”)}$$

$$\left(\dot{T}_n(a)\right)_{i,j} = a_{j-i}\left(\frac{\min(i,j)}{n}\right) \quad \text{and} \quad \left(\ddot{T}_n(a)\right)_{i,j} = a_{j-i}\left(\frac{\max(i,j)}{n}\right)$$

  where $a_k \in C([0,1])$ are given.
- Under the following assumptions:

i) $a_{-k} = \overline{a_k}$ ii) $\sum_{k\in\mathbb{Z}} \max\{|a_k(x)| \mid x \in [0,1]\} < \infty$,

all three matrices have the same asymptotic eigenvalue distribution.

Kac–Murdock–Szegő matrices

- The symbol:

$$a(x,t) := \sum_{k\in\mathbb{Z}} a_k(x)e^{\mathrm{i}kt}.$$

Kac–Murdock–Szegő matrices

- The symbol:

$$a(x,t) := \sum_{k\in\mathbb{Z}} a_k(x) e^{\mathrm{i}kt}.$$

Theorem (Kac, Murdock, Szegő)

For all $\phi \in C(\mathbb{R})$, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(\lambda_k(a)) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \phi(a(x,t)) \mathrm{d}t \mathrm{d}x,$$

where $\lambda_k(a)$ are eigenvalues of $T_n(a)$ (or $\dot{T}_n(a)$ or $\ddot{T}_n(a)$).

Kac–Murdock–Szegő matrices

- The symbol:

$$a(x,t) := \sum_{k\in\mathbb{Z}} a_k(x)e^{\mathrm{i}kt}.$$

Theorem (Kac, Murdock, Szegő)

For all $\phi \in C(\mathbb{R})$, one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \phi(\lambda_k(a)) = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \phi(a(x,t))\mathrm{d}t\mathrm{d}x,$$

where $\lambda_k(a)$ are eigenvalues of $T_n(a)$ (or $\dot{T}_n(a)$ or $\ddot{T}_n(a)$).

- The tridiagonal case corresponds to the symbol: $a(x,t) = a(x)e^{-\mathrm{i}t} + b(x) + a(x)e^{\mathrm{i}t}$.

Kac–Murdock–Szegő matrices

- The symbol:

$$a(x,t) := \sum_{k\in\mathbb{Z}} a_k(x)e^{\mathrm{i}kt}.$$

Theorem (Kac, Murdock, Szegő)

For all $\phi \in C(\mathbb{R})$, one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \phi(\lambda_k(a)) = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \phi(a(x,t))\mathrm{d}t\mathrm{d}x,$$

where $\lambda_k(a)$ are eigenvalues of $T_n(a)$ (or $\dot{T}_n(a)$ or $\ddot{T}_n(a)$).

- The tridiagonal case corresponds to the symbol: $a(x,t) = a(x)e^{-\mathrm{i}t} + b(x) + a(x)e^{\mathrm{i}t}$.
- By making use of the substitution

$$t = \arccos\left(\frac{\xi - b(x)}{2a(x)}\right), \; t \in [0,\pi],$$

in the integral on the RHS, one obtains the asymptotic zero distribution of variable OPRL.

Contents

OPUC - Szegő recursion

- **Goal of the talk**: Asymptotic zero distribution of OPUC with variable Verblunsky coefficients.

OPUC - Szegő recursion

- **Goal of the talk**: Asymptotic zero distribution of OPUC with variable Verblunsky coefficients.
- Notation:

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\} \quad \text{and} \quad \mathbb{T} := \partial\mathbb{D}.$$

OPUC - Szegő recursion

- **Goal of the talk**: Asymptotic zero distribution of OPUC with variable Verblunsky coefficients.
- Notation:

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\} \quad \text{and} \quad \mathbb{T} := \partial\mathbb{D}.$$

- Recall that OPUC is a family of monic polynomials $\{\Phi_n\}_{n=0}^{\infty}$ given by the *Szegő recursion*:

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n\Phi_n^*(z), \quad n \in \mathbb{N}_0,$$

and $\Phi_0(z) = 1$, where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\overline{z})}$ and $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$ are the *Verblunsky coefficients*.

OPUC - Szegő recursion

- **Goal of the talk**: Asymptotic zero distribution of OPUC with variable Verblunsky coefficients.
- Notation:
$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\} \quad \text{and} \quad \mathbb{T} := \partial\mathbb{D}.$$
- Recall that OPUC is a family of monic polynomials $\{\Phi_n\}_{n=0}^{\infty}$ given by the *Szegő recursion*:
$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z), \quad n \in \mathbb{N}_0,$$
and $\Phi_0(z) = 1$, where $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\overline{z})}$ and $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$ are the *Verblunsky coefficients*.
- There is a $1-1$ correspondence between probability measures on $\mathbb{T}$ with infinite support and the sequence $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$.

OPUC - CMV matrix

- The probability measure μ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(\mathbb{N}_0)$ is given by the *CMV matrix*

$$\mathcal{C} := \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \overline{\alpha_4}\rho_3 & -\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

where $\rho_n = \sqrt{1 - |\alpha_n|^2}$.

OPUC - CMV matrix

- The probability measure μ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(\mathbb{N}_0)$ is given by the *CMV matrix*

$$\mathcal{C} := \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \overline{\alpha_4}\rho_3 & -\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

  where $\rho_n = \sqrt{1 - |\alpha_n|^2}$.
- The CMV matrix is a universal model for any unitary operator with simple spectrum.

OPUC - CMV matrix

- The probability measure μ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(\mathbb{N}_0)$ is given by the *CMV matrix*

$$\mathcal{C} := \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \overline{\alpha_4}\rho_3 & -\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

  where $\rho_n = \sqrt{1 - |\alpha_n|^2}$.
- The CMV matrix is a universal model for any unitary operator with simple spectrum.
- Denote $\mathcal{C}_n \in \mathbb{C}^{n,n}$ the cut-off CMV matrix, i.e., the $n \times n$ matrix obtained by truncation of $\mathcal{C}$ from the upper-left corner.

OPUC - CMV matrix

- The probability measure μ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(\mathbb{N}_0)$ is given by the *CMV matrix*

$$\mathcal{C} := \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \overline{\alpha_4}\rho_3 & -\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

  where $\rho_n = \sqrt{1 - |\alpha_n|^2}$.
- The CMV matrix is a universal model for any unitary operator with simple spectrum.
- Denote $\mathcal{C}_n \in \mathbb{C}^{n,n}$ the cut-off CMV matrix, i.e., the $n \times n$ matrix obtained by truncation of $\mathcal{C}$ from the upper-left corner.
- Note that

$$\mathcal{C}_n = \mathcal{C}_n(\alpha_0, \dots, \alpha_{n-1}).$$

OPUC - CMV matrix

- The probability measure μ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(\mathbb{N}_0)$ is given by the *CMV matrix*

$$\mathcal{C} := \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \overline{\alpha_4}\rho_3 & -\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

  where $\rho_n = \sqrt{1 - |\alpha_n|^2}$.
- The CMV matrix is a universal model for any unitary operator with simple spectrum.
- Denote $\mathcal{C}_n \in \mathbb{C}^{n,n}$ the cut-off CMV matrix, i.e., the $n \times n$ matrix obtained by truncation of $\mathcal{C}$ from the upper-left corner.
- Note that

$$\mathcal{C}_n = \mathcal{C}_n(\alpha_0, \dots, \alpha_{n-1}).$$

- It holds

$$\Phi_n(z) = \det(z - \mathcal{C}_n).$$

OPUC - CMV matrix

- The probability measure μ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(\mathbb{N}_0)$ is given by the *CMV matrix*

$$\mathcal{C} := \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \overline{\alpha_4}\rho_3 & -\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

  where $\rho_n = \sqrt{1 - |\alpha_n|^2}$.
- The CMV matrix is a universal model for any unitary operator with simple spectrum.
- Denote $\mathcal{C}_n \in \mathbb{C}^{n,n}$ the cut-off CMV matrix, i.e., the $n \times n$ matrix obtained by truncation of $\mathcal{C}$ from the upper-left corner.
- Note that

$$\mathcal{C}_n = \mathcal{C}_n(\alpha_0, \dots, \alpha_{n-1}).$$

- It holds

$$\Phi_n(z) = \det(z - \mathcal{C}_n).$$

- The zeros of Φ_n are located in $\mathbb{D}$.

OPUC - CMV matrix

- The probability measure μ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^{\infty}$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(\mathbb{N}_0)$ is given by the *CMV matrix*

$$\mathcal{C} := \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \overline{\alpha_4}\rho_3 & -\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

  where $\rho_n = \sqrt{1 - |\alpha_n|^2}$.
- The CMV matrix is a universal model for any unitary operator with simple spectrum.
- Denote $\mathcal{C}_n \in \mathbb{C}^{n,n}$ the cut-off CMV matrix, i.e., the $n \times n$ matrix obtained by truncation of $\mathcal{C}$ from the upper-left corner.
- Note that

$$\mathcal{C}_n = \mathcal{C}_n(\alpha_0, \dots, \alpha_{n-1}).$$

- It holds

$$\Phi_n(z) = \det(z - \mathcal{C}_n).$$

- The zeros of Φ_n are located in $\mathbb{D}$. $\Rightarrow$ $\mathcal{C}_n$ is not unitary.

POPUC

- If α_{n-1} is replaced by a parameter $\beta \in \mathbb{T}$, we arrive at the so-called para-orthogonal polynomials (=POPUC)

$$\Phi_n^{(\beta)}(z) := \det\left(z - \mathcal{C}_n^{(\beta)}\right),$$

where

$$\mathcal{C}_n^{(\beta)} = \mathcal{C}_n(\alpha_0, \ldots, \alpha_{n-2}, \beta).$$

POPUC

- If α_{n-1} is replaced by a parameter $\beta \in \mathbb{T}$, we arrive at the so-called para-orthogonal polynomials (=POPUC)

$$\Phi_n^{(\beta)}(z) := \det\left(z - \mathcal{C}_n^{(\beta)}\right),$$

  where

$$\mathcal{C}_n^{(\beta)} = \mathcal{C}_n(\alpha_0, \dots, \alpha_{n-2}, \beta).$$

- Matrix $\mathcal{C}_n^{(\beta)}$ is unitary.

POPUC

- If α_{n-1} is replaced by a parameter $\beta \in \mathbb{T}$, we arrive at the so-called para-orthogonal polynomials (=POPUC)

$$\Phi_n^{(\beta)}(z) := \det\left(z - \mathcal{C}_n^{(\beta)}\right),$$

  where

$$\mathcal{C}_n^{(\beta)} = \mathcal{C}_n(\alpha_0, \dots, \alpha_{n-2}, \beta).$$

- Matrix $\mathcal{C}_n^{(\beta)}$ is unitary.
- The zeros of $\Phi_n^{(\beta)}$ are located in $\mathbb{T}$ and are simple.

POPUC

- If α_{n-1} is replaced by a parameter $\beta \in \mathbb{T}$, we arrive at the so-called para-orthogonal polynomials (=POPUC)

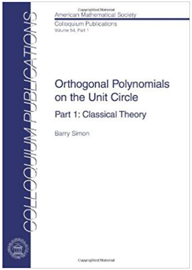
$$\Phi_n^{(\beta)}(z) := \det\left(z - \mathcal{C}_n^{(\beta)}\right),$$

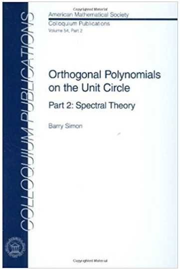
where

$$\mathcal{C}_n^{(\beta)} = \mathcal{C}_n(\alpha_0, \dots, \alpha_{n-2}, \beta).$$

- Matrix $\mathcal{C}_n^{(\beta)}$ is unitary.
- The zeros of $\Phi_n^{(\beta)}$ are located in $\mathbb{T}$ and are simple.

Two books on OPUC by B. Simon:





Contents

Variable (P)OPUC

For $\{\alpha_{n,N} \in \overline{\mathbb{D}} \mid n, N \in \mathbb{N}_0\}$ and $\beta \in \mathbb{T}$, we introduce

$$\Phi_{n,N}(z) := \det\left(z - \mathcal{C}_n(\alpha_{0,N}, \alpha_{1,N}, \dots, \alpha_{n-1,N})\right)$$

Variable (P)OPUC

For $\{\alpha_{n,N} \in \overline{\mathbb{D}} \mid n, N \in \mathbb{N}_0\}$ and $\beta \in \mathbb{T}$, we introduce

$$\Phi_{n,N}(z) := \det\left(z - \mathcal{C}_n(\alpha_{0,N}, \alpha_{1,N}, \dots, \alpha_{n-1,N})\right)$$

and

$$\Phi_{n,N}^{(\beta)}(z) := \det\left(z - \mathcal{C}_n(\alpha_{0,N}, \alpha_{1,N}, \dots, \alpha_{n-2,N}, \beta)\right).$$

Variable (P)OPUC

For $\{\alpha_{n,N} \in \overline{\mathbb{D}} \mid n, N \in \mathbb{N}_0\}$ and $\beta \in \mathbb{T}$, we introduce

$$\Phi_{n,N}(z) := \det\left(z - \mathcal{C}_n(\alpha_{0,N}, \alpha_{1,N}, \dots, \alpha_{n-1,N})\right)$$

and

$$\Phi_{n,N}^{(\beta)}(z) := \det\left(z - \mathcal{C}_n(\alpha_{0,N}, \alpha_{1,N}, \dots, \alpha_{n-2,N}, \beta)\right).$$

Program:

Variable (P)OPUC

For $\{\alpha_{n,N} \in \overline{\mathbb{D}} \mid n, N \in \mathbb{N}_0\}$ and $\beta \in \mathbb{T}$, we introduce

$$\Phi_{n,N}(z) := \det\left(z - \mathcal{C}_n(\alpha_{0,N}, \alpha_{1,N}, \ldots, \alpha_{n-1,N})\right)$$

and

$$\Phi_{n,N}^{(\beta)}(z) := \det\left(z - \mathcal{C}_n(\alpha_{0,N}, \alpha_{1,N}, \ldots, \alpha_{n-2,N}, \beta)\right).$$

Program:

1. First we deduce the asymptotic zero distribution for $\Phi_{n,N}^{(\beta)}$, as $n/N \to t$, (2 proofs).

Variable (P)OPUC

For $\{\alpha_{n,N} \in \overline{\mathbb{D}} \mid n, N \in \mathbb{N}_0\}$ and $\beta \in \mathbb{T}$, we introduce

$$\Phi_{n,N}(z) := \det\left(z - \mathcal{C}_n(\alpha_{0,N}, \alpha_{1,N}, \dots, \alpha_{n-1,N})\right)$$

and

$$\Phi_{n,N}^{(\beta)}(z) := \det\left(z - \mathcal{C}_n(\alpha_{0,N}, \alpha_{1,N}, \dots, \alpha_{n-2,N}, \beta)\right).$$

Program:

1. First we deduce the asymptotic zero distribution for $\Phi_{n,N}^{(\beta)}$, as $n/N \to t$, (2 proofs).
2. By making use of the result for $\Phi_{n,N}^{(\beta)}$ we obtain the asymptotic zero distribution for $\Phi_{n,N}$ (under an additional assumption).

Intermezzo: The equilibrium measure of a circular arc

- Define the probability measure on $\mathbb{T}$ by

$$\mathrm{d}\nu_a\left(e^{\mathrm{i}\theta}\right) := \frac{1}{2\pi}\frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_a/2) - \cos^2(\theta/2)}}\mathrm{d}\theta, \quad \theta \in (\theta_a, 2\pi - \theta_a),$$

where $\theta_a = 2\arcsin a$ and $0 \leq a < 1$.

Intermezzo: The equilibrium measure of a circular arc

- Define the probability measure on $\mathbb{T}$ by

$$\mathrm{d}\nu_a\left(e^{\mathrm{i}\theta}\right) := \frac{1}{2\pi}\frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_a/2)-\cos^2(\theta/2)}}\mathrm{d}\theta, \quad \theta \in (\theta_a, 2\pi-\theta_a),$$

  where $\theta_a = 2\arcsin a$ and $0 \leq a < 1$.
- For $a = 1$, we put $\nu_1 := \delta_{-1}$.

Intermezzo: The equilibrium measure of a circular arc

- Define the probability measure on $\mathbb{T}$ by

$$d\nu_a\left(e^{i\theta}\right) := \frac{1}{2\pi}\frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_a/2)-\cos^2(\theta/2)}}d\theta, \quad \theta \in (\theta_a, 2\pi-\theta_a),$$

  where $\theta_a = 2\arcsin a$ and $0 \leq a < 1$.
- For $a = 1$, we put $\nu_1 := \delta_{-1}$.
- Then ν_a the equilibrium measure of the circular arc $\Gamma_a := \{e^{i\theta} \mid (\theta_a, 2\pi-\theta_a)\} \subset \mathbb{T}$.

Intermezzo: The equilibrium measure of a circular arc

- Define the probability measure on $\mathbb{T}$ by

$$\mathrm{d}\nu_a\left(e^{\mathrm{i}\theta}\right) := \frac{1}{2\pi}\frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_a/2)-\cos^2(\theta/2)}}\mathrm{d}\theta, \quad \theta \in (\theta_a, 2\pi-\theta_a),$$

  where $\theta_a = 2\arcsin a$ and $0 \leq a < 1$.
- For $a = 1$, we put $\nu_1 := \delta_{-1}$.
- Then ν_a the equilibrium measure of the circular arc $\Gamma_a := \{e^{\mathrm{i}\theta} \mid (\theta_a, 2\pi-\theta_a)\} \subset \mathbb{T}$.
- The logarithmic potential of ν_a is

$$U_{\nu_a}(z) = \int_{\Gamma_a} \log\left|z-e^{\mathrm{i}\theta}\right| \mathrm{d}\nu_a\left(e^{\mathrm{i}\theta}\right) = \log|G_a(z)|, \quad z \in \mathbb{C}\setminus\mathbb{T},$$

Intermezzo: The equilibrium measure of a circular arc

- Define the probability measure on $\mathbb{T}$ by

$$\mathrm{d}\nu_a\left(e^{\mathrm{i}\theta}\right) := \frac{1}{2\pi}\frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_a/2) - \cos^2(\theta/2)}}\mathrm{d}\theta, \quad \theta \in (\theta_a, 2\pi - \theta_a),$$

  where $\theta_a = 2\arcsin a$ and $0 \leq a < 1$.
- For $a = 1$, we put $\nu_1 := \delta_{-1}$.
- Then ν_a the equilibrium measure of the circular arc $\Gamma_a := \{e^{\mathrm{i}\theta} \mid (\theta_a, 2\pi - \theta_a)\} \subset \mathbb{T}$.
- The logarithmic potential of ν_a is

$$U_{\nu_a}(z) = \int_{\Gamma_a} \log\left|z - e^{\mathrm{i}\theta}\right| \mathrm{d}\nu_a\left(e^{\mathrm{i}\theta}\right) = \log|G_a(z)|, \quad z \in \mathbb{C}\setminus\mathbb{T},$$

  where, for $0 < a \leq 1$,

$$G_a(z) = \frac{1}{2}\left(z + 1 + \sqrt{\left(z - e^{\mathrm{i}\theta_a}\right)\left(z - e^{-\mathrm{i}\theta_a}\right)}\right),$$

  and, for $a = 0$,

$$G_0(z) = \begin{cases} 1, & \text{if } |z| < 1, \\ z, & \text{if } |z| > 1. \end{cases}$$

Asymptotic zero distribution of variable POPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

Asymptotic zero distribution of variable POPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \nu_{n,N}^{(\beta)} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s,$$

where the measure $\nu_{|\alpha(s)|}$ is the equilibrium measure of a circular arc.

Asymptotic zero distribution of variable POPUC

Theorem:

Let $t > 0$ and $\alpha : [0,t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0,t].$$

Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \nu_{n,N}^{(\beta)} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s,$$

where the measure $\nu_{|\alpha(s)|}$ is the equilibrium measure of a circular arc.

We indicate two ways of proving the theorem:

Asymptotic zero distribution of variable POPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \nu_{n,N}^{(\beta)} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s,$$

where the measure $\nu_{|\alpha(s)|}$ is the equilibrium measure of a circular arc.

We indicate two ways of proving the theorem:

1. based on the ratio asymptotics and a potential-theoretic argument;

Asymptotic zero distribution of variable POPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \nu_{n,N}^{(\beta)} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s,$$

where the measure $\nu_{|\alpha(s)|}$ is the equilibrium measure of a circular arc.

We indicate two ways of proving the theorem:

1. based on the ratio asymptotics and a potential-theoretic argument;
2. a simple moment based proof.

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi_{n+1,N}^{(\beta)}(z)}{\Phi_{n,N}^{(\beta)}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi^{(\beta)}_{n+1,N}(z)}{\Phi^{(\beta)}_{n,N}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu^{(\beta)}_{n,N}}(z) =$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi_{n+1,N}^{(\beta)}(z)}{\Phi_{n,N}^{(\beta)}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu_{n,N}^{(\beta)}}(z) = \frac{1}{n} \log |\Phi_{n,N}^{(\beta)}(z)| =$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi^{(\beta)}_{n+1,N}(z)}{\Phi^{(\beta)}_{n,N}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu^{(\beta)}_{n,N}}(z) = \frac{1}{n} \log |\Phi^{(\beta)}_{n,N}(z)| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi^{(\beta)}_{k+1,N}(z)}{\Phi^{(\beta)}_{k,N}(z)} \right| =$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi^{(\beta)}_{n+1,N}(z)}{\Phi^{(\beta)}_{n,N}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu^{(\beta)}_{n,N}}(z) = \frac{1}{n} \log |\Phi^{(\beta)}_{n,N}(z)| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi^{(\beta)}_{k+1,N}(z)}{\Phi^{(\beta)}_{k,N}(z)} \right| = \int_0^1 \log \left| \frac{\Phi^{(\beta)}_{[ns]+1,N}(z)}{\Phi^{(\beta)}_{[ns],N}(z)} \right| \mathrm{d}s.$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi^{(\beta)}_{n+1,N}(z)}{\Phi^{(\beta)}_{n,N}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu^{(\beta)}_{n,N}}(z) = \frac{1}{n} \log |\Phi^{(\beta)}_{n,N}(z)| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi^{(\beta)}_{k+1,N}(z)}{\Phi^{(\beta)}_{k,N}(z)} \right| = \int_0^1 \log \left| \frac{\Phi^{(\beta)}_{[ns]+1,N}(z)}{\Phi^{(\beta)}_{[ns],N}(z)} \right| \mathrm{d}s.$$

Since $[ns]/N \to st$, as $n/N \to t$, one has

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi^{(\beta)}_{n+1,N}(z)}{\Phi^{(\beta)}_{n,N}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu^{(\beta)}_{n,N}}(z) = \frac{1}{n} \log |\Phi^{(\beta)}_{n,N}(z)| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi^{(\beta)}_{k+1,N}(z)}{\Phi^{(\beta)}_{k,N}(z)} \right| = \int_0^1 \log \left| \frac{\Phi^{(\beta)}_{[ns]+1,N}(z)}{\Phi^{(\beta)}_{[ns],N}(z)} \right| \mathrm{d}s.$$

Since $[ns]/N \to st$, as $n/N \to t$, one has

$$\lim_{n/N \to t} U_{\nu^{(\beta)}_{n,N}}(z) =$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi^{(\beta)}_{n+1,N}(z)}{\Phi^{(\beta)}_{n,N}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu^{(\beta)}_{n,N}}(z) = \frac{1}{n} \log \left|\Phi^{(\beta)}_{n,N}(z)\right| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi^{(\beta)}_{k+1,N}(z)}{\Phi^{(\beta)}_{k,N}(z)} \right| = \int_0^1 \log \left| \frac{\Phi^{(\beta)}_{[ns]+1,N}(z)}{\Phi^{(\beta)}_{[ns],N}(z)} \right| \mathrm{d}s.$$

Since $[ns]/N \to st$, as $n/N \to t$, one has

$$\lim_{n/N \to t} U_{\nu^{(\beta)}_{n,N}}(z) = \int_0^1 \log \left| G_{|\alpha(st)|}(z) \right| \mathrm{d}s =$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi^{(\beta)}_{n+1,N}(z)}{\Phi^{(\beta)}_{n,N}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu^{(\beta)}_{n,N}}(z) = \frac{1}{n} \log \left|\Phi^{(\beta)}_{n,N}(z)\right| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi^{(\beta)}_{k+1,N}(z)}{\Phi^{(\beta)}_{k,N}(z)} \right| = \int_0^1 \log \left| \frac{\Phi^{(\beta)}_{[ns]+1,N}(z)}{\Phi^{(\beta)}_{[ns],N}(z)} \right| \mathrm{d}s.$$

Since $[ns]/N \to st$, as $n/N \to t$, one has

$$\lim_{n/N \to t} U_{\nu^{(\beta)}_{n,N}}(z) = \int_0^1 \log \left| G_{|\alpha(st)|}(z) \right| \mathrm{d}s = \frac{1}{t} \int_0^t \log \left| G_{|\alpha(s)|}(z) \right| \mathrm{d}s =$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:

Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi_{n+1,N}^{(\beta)}(z)}{\Phi_{n,N}^{(\beta)}(z)} = G_{|\alpha|}(z),$$

uniformly in z in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu_{n,N}^{(\beta)}}(z) = \frac{1}{n} \log |\Phi_{n,N}^{(\beta)}(z)| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi_{k+1,N}^{(\beta)}(z)}{\Phi_{k,N}^{(\beta)}(z)} \right| = \int_0^1 \log \left| \frac{\Phi_{[ns]+1,N}^{(\beta)}(z)}{\Phi_{[ns],N}^{(\beta)}(z)} \right| \mathrm{d}s.$$

Since $[ns]/N \to st$, as $n/N \to t$, one has

$$\lim_{n/N \to t} U_{\nu_{n,N}^{(\beta)}}(z) = \int_0^1 \log \left| G_{|\alpha(st)|}(z) \right| \mathrm{d}s = \frac{1}{t} \int_0^t \log \left| G_{|\alpha(s)|}(z) \right| \mathrm{d}s = \frac{1}{t} \int_0^t U_{\nu_{|\alpha(s)|}}(z) \mathrm{d}s.$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

- In summary, we have proved that

$$\lim_{n/N\to t} U_{\nu_{n,N}^{(\beta)}}(z) = U_{\sigma_t}(z), \quad \forall z \in \mathbb{C} \setminus \mathbb{T},$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

- In summary, we have proved that

$$\lim_{n/N\to t} U_{\nu_{n,N}^{(\beta)}}(z) = U_{\sigma_t}(z), \quad \forall z \in \mathbb{C} \setminus \mathbb{T},$$

where

$$\sigma_t := \frac{1}{t}\int_0^t \nu_{|\alpha(s)|}\mathrm{d}s.$$

Proof 1: the ratio asymptotics and a potential-theoretic argument

- In summary, we have proved that

$$\lim_{n/N\to t} U_{\nu_{n,N}^{(\beta)}}(z) = U_{\sigma_t}(z), \quad \forall z \in \mathbb{C} \setminus \mathbb{T},$$

where

$$\sigma_t := \frac{1}{t}\int_0^t \nu_{|\alpha(s)|} \mathrm{d}s.$$

- Hence

$$\lim_{n/N\to t} \nu_{n,N}^{(\beta)} = \sigma_t$$

by the application of Widom's lemma.

Proof 2: a simple moment based proof

The **idea** is to show that

$$\lim_{n/N\to t}\frac{1}{n}\operatorname{Tr}\left(\mathcal{C}_{n,N}^{(\beta)}\right)^{k}=\int_{0}^{2\pi}e^{\mathrm{i}k\theta}\sigma_t\left(e^{\mathrm{i}\theta}\right),\quad\forall k\in\mathbb{Z}.$$

Proof 2: a simple moment based proof

The **idea** is to show that

$$\lim_{n/N\to t} \frac{1}{n} \operatorname{Tr} \left(\mathcal{C}_{n,N}^{(\beta)}\right)^{k} = \int_0^{2\pi} e^{\mathrm{i}k\theta} \sigma_t \left(e^{\mathrm{i}\theta}\right), \quad \forall k \in \mathbb{Z}.$$

Then *all integer moments* of $\nu_{n,N}^{(\beta)}$ converge to the moments of σ_t and the Stone–Weierstrass theorem implies the statement.

Proof 2: a simple moment based proof

The **idea** is to show that

$$\lim_{n/N\to t} \frac{1}{n} \operatorname{Tr}\left(\mathcal{C}_{n,N}^{(\beta)}\right)^{k} = \int_{0}^{2\pi} e^{\mathrm{i}k\theta}\sigma_t\left(e^{\mathrm{i}\theta}\right), \quad \forall k \in \mathbb{Z}.$$

Then *all integer moments* of $\nu_{n,N}^{(\beta)}$ converge to the moments of σ_t and the Stone–Weierstrass theorem implies the statement.

Steps:

Proof 2: a simple moment based proof

The **idea** is to show that

$$\lim_{n/N\to t} \frac{1}{n} \operatorname{Tr}\left(\mathcal{C}_{n,N}^{(\beta)}\right)^k = \int_0^{2\pi} e^{\mathrm{i}k\theta}\sigma_t\left(e^{\mathrm{i}\theta}\right), \quad \forall k \in \mathbb{Z}.$$

Then *all integer moments* of $\nu_{n,N}^{(\beta)}$ converge to the moments of σ_t and the Stone–Weierstrass theorem implies the statement.

Steps:

1. It suffices to show the formula for $k \in \mathbb{N}_0$ because $\mathcal{C}_{n,N}^{(\beta)}$ is unitary.

Proof 2: a simple moment based proof

The **idea** is to show that

$$\lim_{n/N\to t} \frac{1}{n} \operatorname{Tr}\left(\mathcal{C}_{n,N}^{(\beta)}\right)^k = \int_0^{2\pi} e^{\mathrm{i}k\theta} \sigma_t\left(e^{\mathrm{i}\theta}\right), \quad \forall k \in \mathbb{Z}.$$

Then *all integer moments* of $\nu_{n,N}^{(\beta)}$ converge to the moments of σ_t and the Stone–Weierstrass theorem implies the statement.

Steps:

1. It suffices to show the formula for $k \in \mathbb{N}_0$ because $\mathcal{C}_{n,N}^{(\beta)}$ is unitary.
2. $$\lim_{n\to\infty} \frac{1}{n} \operatorname{Tr}(\mathcal{C}_n(\alpha))^k = \int_0^{2\pi} e^{\mathrm{i}k\theta} d\nu_{|\alpha|}(e^{\mathrm{i}\theta}) \quad \text{(known from OPUC theory)}$$

Proof 2: a simple moment based proof

The **idea** is to show that

$$\lim_{n/N\to t}\frac{1}{n}\operatorname{Tr}\left(\mathcal{C}_{n,N}^{(\beta)}\right)^k=\int_0^{2\pi}e^{\mathrm{i}k\theta}\sigma_t\left(e^{\mathrm{i}\theta}\right),\quad \forall k\in\mathbb{Z}.$$

Then *all integer moments* of $\nu_{n,N}^{(\beta)}$ converge to the moments of σ_t and the Stone–Weierstrass theorem implies the statement.

Steps:

1. It suffices to show the formula for $k\in\mathbb{N}_0$ because $\mathcal{C}_{n,N}^{(\beta)}$ is unitary.
2. $$\lim_{n\to\infty}\frac{1}{n}\operatorname{Tr}(\mathcal{C}_n(\alpha))^k=\int_0^{2\pi}e^{\mathrm{i}k\theta}d\nu_{|\alpha|}(e^{\mathrm{i}\theta})\qquad\text{(known from OPUC theory)}$$
3. $$\lim_{n/N\to t}\frac{1}{n}\left(\operatorname{Tr}\left(\mathcal{C}_{n,N}\right)^k-\int_0^1\operatorname{Tr}\left(\mathcal{C}_n(\alpha(st))\right)^k\mathrm{d}s\right)=0,\qquad\text{(slightly combinatorial arguments)}$$

Proof 2: a simple moment based proof

The **idea** is to show that

$$\lim_{n/N\to t}\frac{1}{n}\operatorname{Tr}\left(\mathcal{C}_{n,N}^{(\beta)}\right)^k=\int_0^{2\pi}e^{\mathrm{i}k\theta}\sigma_t\left(e^{\mathrm{i}\theta}\right),\quad \forall k\in\mathbb{Z}.$$

Then *all integer moments* of $\nu_{n,N}^{(\beta)}$ converge to the moments of σ_t and the Stone–Weierstrass theorem implies the statement.

Steps:

1. It suffices to show the formula for $k\in\mathbb{N}_0$ because $\mathcal{C}_{n,N}^{(\beta)}$ is unitary.
2. $$\lim_{n\to\infty}\frac{1}{n}\operatorname{Tr}(\mathcal{C}_n(\alpha))^k=\int_0^{2\pi}e^{\mathrm{i}k\theta}d\nu_{|\alpha|}(e^{\mathrm{i}\theta})$$ (known from OPUC theory)
3. $$\lim_{n/N\to t}\frac{1}{n}\left(\operatorname{Tr}\left(\mathcal{C}_{n,N}\right)^k-\int_0^1\operatorname{Tr}\left(\mathcal{C}_n(\alpha(st))\right)^k\mathrm{d}s\right)=0,$$ (slightly combinatorial arguments)
4. $$\lim_{n/N\to t}\frac{1}{n}\left(\operatorname{Tr}\left(\mathcal{C}_{n,N}^{(\beta)}\right)^k-\operatorname{Tr}\left(\mathcal{C}_{n,N}\right)^k\right)=0$$ (“perturbation” arguments)

Contents

POPUC $\rightarrow$ OPUC

- The previous arguments cannot be used to deduce the asymptotic zero distribution for variable OPUC.

POPUC $\rightarrow$ OPUC

- The previous arguments cannot be used to deduce the asymptotic zero distribution for variable OPUC.
- The ratio asymptotic formula

$$\lim_{n/N \to t} \frac{\Phi_{n+1,N}(z)}{\Phi_{n,N}(z)} = G_{|\alpha|}(z)$$

is valid **only for** $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, in general! As a consequence, we do not know the logarithmic potential of the limiting measure (if exists) on a sufficiently "large" set to recover the limiting measure.

POPUC $\to$ OPUC

- The previous arguments cannot be used to deduce the asymptotic zero distribution for variable OPUC.
- The ratio asymptotic formula

$$\lim_{n/N\to t}\frac{\Phi_{n+1,N}(z)}{\Phi_{n,N}(z)}=G_{|\alpha|}(z)$$

  is valid **only for** $z\in\mathbb{C}\setminus\overline{\mathbb{D}}$, in general! As a consequence, we do not know the logarithmic potential of the limiting measure (if exists) on a sufficiently "large" set to recover the limiting measure.
- Since $C_{n,N}$ is not unitary, we know the key formula from the moment based proof:

$$\lim_{n/N\to t}\frac{1}{n}\operatorname{Tr}\left(\mathcal{C}_{n,N}\right)^{k}=\int_{0}^{2\pi}e^{\mathrm{i}k\theta}\sigma_{t}\left(e^{\mathrm{i}\theta}\right),$$

  for positive integers k **only!** This is also an insufficient information for recovering the limiting measure.

Balayage

- Recall that for any probability measure μ with support in $\overline{\mathbb{D}}$, there exists a unique probability measure $\mathcal{P}(\mu)$ supported on $\mathbb{T}$ such that

$$\int_{\overline{\mathbb{D}}} z^k \mathrm{d}\mu(z) = \int_0^{2\pi} e^{ik\theta} \mathrm{d}\mathcal{P}(\mu)(e^{\mathrm{i}\theta}), \quad \forall k \in \mathbb{N}_0.$$

Balayage

- Recall that for any probability measure μ with support in $\overline{\mathbb{D}}$, there exists a unique probability measure $\mathcal{P}(\mu)$ supported on $\mathbb{T}$ such that

$$\int_{\overline{\mathbb{D}}} z^k \mathrm{d}\mu(z) = \int_0^{2\pi} e^{ik\theta} \mathrm{d}\mathcal{P}(\mu)(e^{\mathrm{i}\theta}), \quad \forall k \in \mathbb{N}_0.$$

Proposition:

Let $t > 0$ and $\alpha : [0,t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0,t].$$

Then

$$\lim_{n/N \to t} \mathcal{P}(\nu_{n,N}) = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s.$$

Asymptotic eigenvalue distribution for variable OPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

If $\alpha(t) \neq 0$, then

$$\lim_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s.$$

Asymptotic eigenvalue distribution for variable OPUC

Theorem:

Let $t > 0$ and $\alpha : [0,t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0,t].$$

If $\alpha(t) \neq 0$, then

$$\lim_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s.$$

Proof: 2 facts about OPUC:

Asymptotic eigenvalue distribution for variable OPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

If $\alpha(t) \neq 0$, then

$$\lim_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s.$$

Proof: 2 facts about OPUC:

1. If for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \sharp\{k \in \{1, \ldots, n\} \mid |z_{k,n}| \geq 1 - \varepsilon\} = 1,$$

where $z_{1,n}, z_{2,n}, \ldots, z_{n,n}$ are zeros of Φ_n,

Asymptotic eigenvalue distribution for variable OPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

If $\alpha(t) \neq 0$, then

$$\lim_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s.$$

Proof: 2 facts about OPUC:

1. If for any $\varepsilon > 0$,

   $$\lim_{n \to \infty} \frac{1}{n} \sharp\{k \in \{1, \dots, n\} \mid |z_{k,n}| \geq 1 - \varepsilon\} = 1,$$

   where $z_{1,n}, z_{2,n}, \dots, z_{n,n}$ are zeros of Φ_n, then $\lim_{n \to \infty} (\nu_n - \mathcal{P}(\nu_n)) = 0$.

Asymptotic eigenvalue distribution for variable OPUC

Theorem:

Let $t > 0$ and $\alpha : [0,t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0,t].$$

If $\alpha(t) \neq 0$, then

$$\lim_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s.$$

Proof: 2 facts about OPUC:

1. If for any $\varepsilon > 0$,
$$\lim_{n\to\infty} \frac{1}{n} \sharp\{k \in \{1,\dots,n\} \mid |z_{k,n}| \geq 1-\varepsilon\} = 1,$$
where $z_{1,n}, z_{2,n}, \dots, z_{n,n}$ are zeros of Φ_n, then $\lim_{n\to\infty} (\nu_n - \mathcal{P}(\nu_n)) = 0.$

2. It holds
$$-\overline{\alpha_{n-1}} = \Phi_n(0).$$

Asymptotic eigenvalue distribution for variable OPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

If $\alpha(t) \neq 0$, then

$$\lim_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \mathrm{d}s.$$

Proof: 2 facts about OPUC:

1. If for any $\varepsilon > 0$,
$$\lim_{n\to\infty} \frac{1}{n} \sharp\{k \in \{1, \dots, n\} \mid |z_{k,n}| \geq 1 - \varepsilon\} = 1,$$
where $z_{1,n}, z_{2,n}, \dots, z_{n,n}$ are zeros of Φ_n, then $\lim_{n\to\infty} (\nu_n - \mathcal{P}(\nu_n)) = 0$.
2. It holds
$$-\overline{\alpha_{n-1}} = \Phi_n(0).$$

- It suffices to show that the zeros of $\Phi_{n,N}$ cluster "mostly" on $\mathbb{T}$, as $n/N \to t$.

Proof cont.

- Let $\{n_j\}$ and $\{N_j\}$ such that $n_j, N_j \to \infty$ and $n_j/N_j \to t$, then

$$-\overline{\alpha_{n_j-1,N_j}} = \Phi_{n_j,N_j}(0) = \prod_{k=1}^{n} \left(-z_{k,n_j,N_j}\right).$$

Proof cont.

- Let $\{n_j\}$ and $\{N_j\}$ such that $n_j, N_j \to \infty$ and $n_j/N_j \to t$, then

$$-\overline{\alpha_{n_j-1,N_j}} = \Phi_{n_j,N_j}(0) = \prod_{k=1}^{n} \left(-z_{k,n_j,N_j}\right).$$

- Then for any $\varepsilon \in (0,1)$,

$$\frac{1}{n_j} \log |\alpha_{n_j-1,N_j}| = \frac{1}{n_j} \sum_{j=1}^{n_j} \log |z_{k,n_j,N_j}| \leq \log(1-\varepsilon) \frac{\sharp\{k : |z_{k,n_j,N_j}| \leq 1-\varepsilon\}}{n_j}$$

Proof cont.

- Let $\{n_j\}$ and $\{N_j\}$ such that $n_j, N_j \to \infty$ and $n_j/N_j \to t$, then

$$-\overline{\alpha_{n_j-1,N_j}} = \Phi_{n_j,N_j}(0) = \prod_{k=1}^{n} \left(-z_{k,n_j,N_j}\right).$$

- Then for any $\varepsilon \in (0,1)$,

$$\underbrace{\frac{1}{n_j}\log \underbrace{|\alpha_{n_j-1,N_j}|}_{\substack{\downarrow \\ \alpha(t)\neq 0}}}_{\substack{\downarrow \\ 0}} = \frac{1}{n_j}\sum_{j=1}^{n_j}\log|z_{k,n_j,N_j}| \leq \log(1-\varepsilon)\frac{\sharp\{k : |z_{k,n_j,N_j}| \leq 1-\varepsilon\}}{n_j}$$

Proof cont.

- Let $\{n_j\}$ and $\{N_j\}$ such that $n_j, N_j \to \infty$ and $n_j/N_j \to t$, then

$$-\overline{\alpha_{n_j-1,N_j}} = \Phi_{n_j,N_j}(0) = \prod_{k=1}^{n} \left(-z_{k,n_j,N_j}\right).$$

- Then for any $\varepsilon \in (0,1)$,

$$\underbrace{\frac{1}{n_j} \log \underbrace{|\alpha_{n_j-1,N_j}|}_{\substack{\downarrow \\ \alpha(t) \neq 0}}}_{\substack{\downarrow \\ 0}} = \frac{1}{n_j} \sum_{j=1}^{n_j} \log |z_{k,n_j,N_j}| \leq \underbrace{\log(1-\varepsilon)}_{<0} \frac{\sharp\{k : |z_{k,n_j,N_j}| \leq 1-\varepsilon\}}{n_j}$$

Proof cont.

- Let $\{n_j\}$ and $\{N_j\}$ such that $n_j, N_j \to \infty$ and $n_j/N_j \to t$, then

$$-\overline{\alpha_{n_j-1,N_j}} = \Phi_{n_j,N_j}(0) = \prod_{k=1}^{n} \left(-z_{k,n_j,N_j}\right).$$

- Then for any $\varepsilon \in (0,1)$,

$$\underbrace{\frac{1}{n_j} \log \underbrace{|\alpha_{n_j-1,N_j}|}_{\substack{\downarrow \\ \alpha(t) \neq 0}}}_{\substack{\downarrow \\ 0}} = \frac{1}{n_j} \sum_{j=1}^{n_j} \log |z_{k,n_j,N_j}| \leq \underbrace{\log(1-\varepsilon)}_{<0} \underbrace{\frac{\sharp\{k : |z_{k,n_j,N_j}| \leq 1-\varepsilon\}}{n_j}}_{\substack{\downarrow \\ 0}}$$

Proof cont.

- Let $\{n_j\}$ and $\{N_j\}$ such that $n_j, N_j \to \infty$ and $n_j/N_j \to t$, then

$$-\overline{\alpha_{n_j-1,N_j}} = \Phi_{n_j,N_j}(0) = \prod_{k=1}^{n} \left(-z_{k,n_j,N_j}\right).$$

- Then for any $\varepsilon \in (0,1)$,

$$\underbrace{\frac{1}{n_j}\log\underbrace{|\alpha_{n_j-1,N_j}|}_{\substack{\downarrow \\ \alpha(t)\neq 0}}}_{\substack{\downarrow \\ 0}} = \frac{1}{n_j}\sum_{j=1}^{n_j}\log|z_{k,n_j,N_j}| \leq \underbrace{\log(1-\varepsilon)}_{<0}\underbrace{\frac{\sharp\{k : |z_{k,n_j,N_j}| \leq 1-\varepsilon\}}{n_j}}_{\substack{\downarrow \\ 0}}$$

- Hence

$$\lim_{n/N\to t}\frac{\sharp\{k : |z_{k,n,N}| > 1-\varepsilon\}}{n} = 1,$$

and the previous theorem implies the result.

The case $\alpha(t) = 0$

- It is not clear what can happen when $\alpha(t) = 0$ at the moment.

The case $\alpha(t) = 0$

- It is not clear what can happen when $\alpha(t) = 0$ at the moment.
- Various situations can occur. Note that, as a special case, the have (standard) OPUC with

$$\alpha_{n,N} = \alpha_n \to 0, \qquad \text{as } n \to \infty.$$

In particular, it is known that the limiting measure need not be supported on $\mathbb{T}$ (or need no exists).

The case $\alpha(t) = 0$

- It is not clear what can happen when $\alpha(t) = 0$ at the moment.
- Various situations can occur. Note that, as a special case, the have (standard) OPUC with

$$\alpha_{n,N} = \alpha_n \to 0, \qquad \text{as } n \to \infty.$$

  In particular, it is known that the limiting measure need not be supported on $\mathbb{T}$ (or need no exists).
- Therefore let us consider the less general setting:

$$\alpha_{n,N} := \alpha\left(\frac{nt}{N}\right),$$

  where $\alpha \in C([0,t])$.

The case $\alpha(t) = 0$

- It is not clear what can happen when $\alpha(t) = 0$ at the moment.
- Various situations can occur. Note that, as a special case, the have (standard) OPUC with

$$\alpha_{n,N} = \alpha_n \to 0, \quad \text{as } n \to \infty.$$

  In particular, it is known that the limiting measure need not be supported on $\mathbb{T}$ (or need no exists).
- Therefore let us consider the less general setting:

$$\boxed{\alpha_{n,N} := \alpha\left(\frac{nt}{N}\right),}$$

  where $\alpha \in C([0,t])$.
- From now until the end of the talk, we investigate the asymptotic distribution of zeros of the polynomials

$$\Phi_N(z) := \Phi_{N,N}(z),$$

  for $N \to \infty$. (The notation is a bit confusing here!)

The case $\alpha(t) = 0$: polynomial vs. exponential decay

- For a given $\alpha \in C([0, t])$, we would like to understand the situation when

$$\lim_{N\to\infty} \left|\alpha\left(\frac{N-1}{N}t\right)\right|^{1/N} = A \in [0, 1].$$

The case $\alpha(t) = 0$: polynomial vs. exponential decay

- For a given $\alpha \in C([0, t])$, we would like to understand the situation when

$$\lim_{N\to\infty} \left|\alpha\left(\frac{N-1}{N}t\right)\right|^{1/N} = A \in [0, 1].$$

Example (Polynomial decay)

If $\alpha \in C([0, t])$ decays at t as

$$\alpha(s) = a(s-t)^m + o\left((s-t)^m\right), \quad \text{as } s \to t-,$$

for some $m \in \mathbb{N}$ and $a \neq 0$. Then $A = 1$.

The case $\alpha(t) = 0$: polynomial vs. exponential decay

- For a given $\alpha \in C([0,t])$, we would like to understand the situation when

$$\lim_{N\to\infty} \left|\alpha\left(\frac{N-1}{N}t\right)\right|^{1/N} = A \in [0,1].$$

Example (Polynomial decay)

If $\alpha \in C([0,t])$ decays at t as

$$\alpha(s) = a(s-t)^m + o\left((s-t)^m\right), \quad \text{as } s \to t-,$$

for some $m \in \mathbb{N}$ and $a \neq 0$. Then $A = 1$.

Example (Exponential decay)

If $\alpha \in C([0,t])$ decays at t as

$$\alpha(s) = A^{\frac{t}{t-s}}\left(1 + o(1)\right), \quad \text{as } s \to t-,$$

for some $A \in (0,1)$. Then $A < 1$.

The polynomial decay and an open problem

Theorem:

Let $\alpha \in C([0,t])$ be such that $A = 1$. Then

$$\lim_{N\to\infty} \nu_N = \frac{1}{t}\int_0^t \nu_{|\alpha(s)|}\mathrm{d}s.$$

The polynomial decay and an open problem

Theorem:

Let $\alpha \in C([0,t])$ be such that $A = 1$. Then

$$\lim_{N\to\infty} \nu_N = \frac{1}{t}\int_0^t \nu_{|\alpha(s)|}\mathrm{d}s.$$

Proof: based on a modification of the previous arguments.

The polynomial decay and an open problem

Theorem:

Let $\alpha \in C([0,t])$ be such that $A = 1$. Then

$$\lim_{N\to\infty} \nu_N = \frac{1}{t}\int_0^t \nu_{|\alpha(s)|}\mathrm{d}s.$$

Proof: based on a modification of the previous arguments.

OPEN PROBLEM:

What happens when $A < 1$?

Thank you!