# Some Quantitative results in the Borcea-Branden theory

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$p$ degree $n$ polynomial : All our polynomials will be over $\mathbb{C}$.

- $\sigma(p)$ : Roots of $p$.
- $\mathcal{K}(p)$ : Convex hull of roots of $p$.
- $\sigma(p')$ : Roots of derivative aka critical points of $p$.

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Enough to show that if $\sigma(p) \subset \mathbb{H}$ (open upper half plane), then so does $\sigma(p')$.

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This also shows: Assume that the roots of $p$ do not all lie on a line. Then,

$$\lambda \in \sigma(p') \cap \partial\mathcal{K}(p) \Longrightarrow \lambda \in \sigma(p).$$

# Other odds and ends

Beyond the Gauss-Lucas theorem, there are other classical results.

**Lemma (Fejer, Toeplitz, 1910's)**

*Fix $\Lambda = \{\lambda_1, \cdots, \lambda_n\} \subset \mathbb{C}$. Let $\mathcal{E}(\Lambda)$ be all points that are critical points of a polynomial with roots in $\Lambda$,*

$$\mathcal{E}(\Lambda) = \{\zeta \in \mathbb{C} \mid \exists p : \sigma(p) \subset \Lambda,\ p'(\zeta) = 0\}.$$

*Then, $\mathcal{E}(\Lambda)$ is everywhere dense in $\mathcal{K}(\Lambda)$.*

## Theorem (Specht, 1959)

*Given a polynomial with roots* $\{\lambda_1, \cdots, \lambda_n\}$, *define the points,*

$$\mu_{ij} = \frac{\lambda_i}{n} + \frac{(n-1)\lambda_j}{n}, \quad 1 \leq i < j \leq n.$$

*Then, any circular region containing exactly one of the* $\lambda_i$ *and none of the* $\mu_{ij}$ *does not contain any critical points. In particular,*

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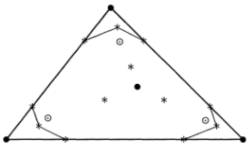

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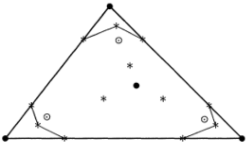

Specht's theorem for a polynomial of degree 4.
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Note: The Gauss-Lucas theorem does not extend to entire functions! Consider

$$f(z) = z\, exp(z^2), \quad f'(z) = (2z^2 + 1)exp(z^2).$$

# Kakeya's problem

**Theorem (Kakeya, 1917)**

*If a degree n polynomial p has two zeroes in* $\mathbb{D}$*, then it has a critical point in* $\mathbb{D}\csc\left(\frac{\pi}{n}\right) \sim \mathbb{D}n/\pi$*.*

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$\phi(n,p)$: The smallest number such that if a degree $n$ polynomial has at least $p$ roots in a disc of radius 1, then it has at least $p-1$ roots in a disc of radius $\phi(n,p,m)$.

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- Gauss-Lucas : $\phi(n,n)=1$.
- Kakeya: $\phi(n,2)=\csc\left(\frac{\pi}{n}\right)$.
- Marden(1939) : $\phi(n,p)\leq\csc\left(\frac{\pi}{2(n-p+1)}\right)$.
- Biernacki(1945) : $\phi(n,n-1)=\sqrt{1+\frac{1}{n}}$ for $n$ odd.

# A quantitative Gauss-Lucas theorem

Recall, $\mathcal{K}(p)$ : Convex hull of roots of $p$.

**Theorem (R, '17)**

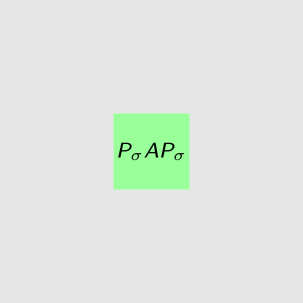
*Let $p$ be a polynomial of degree $n$. Then, for any $c \geq 1/2$, we have that*

$$|\mathcal{K}(p^{(cn)})| \leq 4(c - c^2)\,|\mathcal{K}(p)|.$$

**Theorem (Bourgain-Tzafriri, 91)**

*For any zero diagonal contraction* $A \in M_n(\mathbb{C})$, $\exists\, \sigma \subset [n]$ *such that* $|\sigma| = \Omega(n\epsilon^2)$ *and*

$$||P_\sigma A P_\sigma|| < \epsilon.$$




Illustrating the Bourgain-Tzafriri theorem

# Cauchy-Poincare, R.C.Thompson, Marcus-Spielman-Srivastava

**Theorem (Cauchy-Poincare)**

*$A \in M_n(\mathbb{C})^{sa}$. Then, the eigenvalues of $\chi[A]$ and $\chi[A_i]$ interlace.*

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**Lemma (Markov principle)**

*$p_1, \cdots, p_n$ be same degree, monic, real rooted with common interlacer. Then, $\forall k \ \exists i$,*

$$\lambda_k(p_i) \leq \lambda_k(p_1 + \cdots + p_n).$$

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**Lemma (Obreshkoff)**

*$\{p_i\}_{i=1}^n$ degree $k$ monic real rooted. Common interlacer iff every convex combination real rooted.*

We will use $A_S$ to represent the principal submatrix of $A$ with rows and columns from $S$ *removed*.

**Theorem (MSS, 2014 + R.C.Thompson, 1963)**

*Let $A \in M_n(\mathbb{C})^{sa}$. Then, $\exists i \in [n]$ such that,*

$$\lambda_1\left(\chi[A_i]\right) \leq \lambda_1\left(\sum \chi[A_i]\right) = \lambda_1\left(\chi'[A]\right).$$

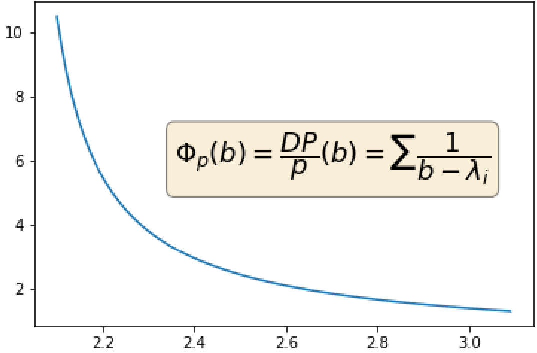
*For any $k \in [n]$, there is a size $k$ subset $S \subset [n]$ such that,*

$$\lambda_1\left(\chi[A_S]\right) \leq \lambda_1\left(\sum_{|S|=k} \chi[A_S]\right) = \lambda_1\left(\chi^{(k)}[A]\right).$$

Note: Same holds with inequalities reversed.

$p$ : Real rooted degree $n$ polynomial. For $b \geq \lambda_1(p)$ and $\varphi > 0$, define

$$\Phi_p(b) := \sum \frac{1}{b - \lambda_i}, \quad \text{smax}_\varphi(p) := \Phi^{-1}(\varphi) = \lambda_1(p' - \varphi p).$$




Log barrier

# Using the log barrier

**Proposition (Marcus, 2014)**

*Let p be real rooted and* $\varphi > 0$*. Then,*

$$\mathrm{smax}_\varphi(p') \leq \mathrm{smax}_\varphi(p) - \frac{1}{\varphi}, \quad \rightsquigarrow \quad \mathrm{smax}_\varphi(p^{(k)}) \leq \mathrm{smax}_\varphi(p) - \frac{k}{\varphi}.$$

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Follows from concavity of $\dfrac{1}{\Phi_p}$ above $\lambda_1(p)$. Optimization yields

**Theorem**

*Let $p$ be a degree $n$ real rooted polynomial with roots in $[-1, 1]$ and with the average of its roots $0$. Then, for any $c \geq \frac{1}{2}$, we have that,*

$$\mathrm{roots}\left(p^{(cn)}\right) \subset [-2\sqrt{c - c^2}, 2\sqrt{c - c^2}].$$

Note : This is a quantitative Gauss-Lucas theorem for real rooted polynomials.

# Majorization for sequences

Given non-increasing real sequences $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ say that $\mathbf{b}$ majorizes $\mathbf{a}$, which we denote by $\mathbf{a} \prec \mathbf{b}$ if,

$$a_1 \leq b_1, \quad a_1 + a_2 \leq b_1 + b_2, \quad \cdots \quad a_1 + \cdots + a_{n-1} \leq b_1 + \cdots + b_{n-1},$$

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*Given $A \in M_n(\mathbb{C})^{sa}$, the diagonal of $A$ is majorized by the eigenvalues of $A$.*

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Another very useful characterization,

**Lemma (Birkhoff)**

$\mathbf{b}$ *majorizes* $\mathbf{a}$ *iff there is a Doubly Stochastic $n \times n$ matrix $A$ such that $A\mathbf{b} = \mathbf{a}$.*

# Majorization for matrices

Given a matrix $A$, we will use $\sigma(A)$ to denote its eigenvalues.

**Theorem (Ky Fan)**

*Given a matrix $A \in M_n(\mathbb{C})$, we have that*

$$\operatorname{Re} \sigma(A) \prec \sigma(\operatorname{Re} A).$$

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Given a matrix $A \in M_n(\mathbb{C})$ let $B$ be a compression of $A$ to a $n-1$ dimensional subspace,

$$B = PA\,|_{P\mathbb{C}^n}, \quad P \text{ projection}, \ \operatorname{rank}(P) = n-1.$$

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**Theorem (Pereira, 2003, Malamud, 2003)**

*Every matrix has a differentiator. As a consequence: given a polynomial $p$ with roots $\{\lambda_1, \cdots, \lambda_n\}$, let $R(p)$ be the polynomial whose roots are $\{Re\lambda_1, \cdots, Re\lambda_n\}$. Then,*

$$\operatorname{Re}\sigma(Dp) \prec \sigma(DR(p)).$$

This has several consequences.

# The De Brujin-Springer and Schoenberg conjectures

## Theorem (Schoenberg's conjecture, Pereira, Malamud)

*Given a polynomial p with average of roots* 0*,*

$$\sum_{\mu \in \sigma(p')} |\mu|^2 \leq \frac{n-2}{n} \sum_{\lambda \in \sigma(p)} |\lambda|^2.$$

*Further, we have equality iff the roots are collinear.*

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Say $\mathbf{a} = (a_1, \cdots, a_m) \prec \mathbf{b} = (b_1, \cdots, b_n)$ if there is a Doubly stochastic $m \times n$ matrix $A$ such that $A\mathbf{b} = \mathbf{a}$.

Doubly stochastic here means : Entries positive, each row sum 1, each column sum $m/n$.

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**Theorem (De Bruijn-Springer conjecture, Pereira, Malamud)**

*Given a polynomial,*

$$\sigma(p') \prec \sigma(p).$$

# Majorization for higher derivatives

**Proposition (Borcea-Branden)**

*Suppose $p$ and $q$ are real rooted and such that $\sigma(q) \prec \sigma(p)$. Then,*

$$\sigma(q') \prec \sigma(p').$$

Iterating the Pereira-Malamud theorem and the above theorem yields

**Proposition**

*Let $p$ be a polynomial of degree $n$ and let $k < n$. Then,*

$$\sigma\left(R(p^{(k)})\right) \prec \sigma((Rp)^{(k)}).$$

For a real rooted polynomial $p$, let

$$\text{span}(p) := \max_{\lambda\in\sigma(p)} \lambda - \min_{\lambda\in\sigma(p)} \lambda.$$

# Shrinking for complex rooted polynomials

## Proposition

*Let p be a polynomial of degree n. Then, for any $k < n$,*

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**Proposition**

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Letting $K(p)$ denote the convex hull of the roots of $p$, we see

$$|\operatorname{Proj}_{\mathbb{R}}\, K(p^{(cn)})| \leq 2\sqrt{c - c^2} |\operatorname{Proj}_{\mathbb{R}}\, K(p)|.$$

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The same works in other directions as well.

**Theorem (R, '17)**

*Let $p$ be a polynomial of degree $n$. Then, for any $c \geq 1/2$, we have that*

$$|\mathcal{K}(p^{(cn)})| \leq 4(c - c^2)\,|\mathcal{K}(p)|.$$

# Part II : On the Kadison-Singer problem

# The Paving problem

**Theorem (MSS, 13)**

*For any* $\epsilon < 1$, *there is* $r = O(\epsilon^{-2})$ *so that for any zero diagonal* **hermitian** *contraction* $A \in M_n(\mathbb{C})^{sa}$, *there are diagonal projections* $Q_1, \cdots, Q_r$ *with* $Q_1 + \cdots + Q_r = I$,

$$\lambda_{max}\left(Q_i A Q_i\right) < \epsilon, \quad 1 \leq i \leq r.$$

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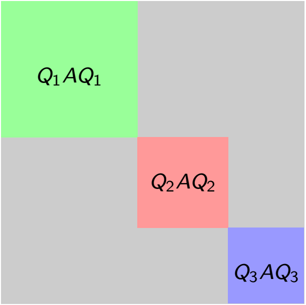


A **Paving** of a matrix $A$

# Weaver's $KS_r$

**Theorem (MSS 13)**

*There is a $\epsilon < 1/2$ and $r \in \mathbb{N}$ such that given vectors $v_1, \cdots, v_m$ in $\mathbb{C}^n$ such that*

$$\sum_{i=1}^{m} v_i v_i^* = I_n, \quad ||v_i||^2 \leq \frac{1}{2},$$

*then there is a partition $X_1 \amalg \cdots \amalg X_r = [n]$ such that*

$$\lambda_{max}\left(\sum_{i \in X_j} v_i v_i^*\right) < \frac{1}{2} + \epsilon, \quad j \in [r]. \qquad \text{(Weaver's } KS_r\text{)}$$

MSS deduced this from

**Theorem (MSS 2013, Cohen 2016)**

*Let $A_1, \cdots, A_m$ be independent random PSD matrices such that*

$$\mathbb{E}[A_1 + \cdots + A_m] = I, \quad \mathbb{E}\,\mathrm{Tr}(A_i) \leq \epsilon, \quad i \in [m].$$

*Then,*

$$\mathbb{P}\left[||A_1 + \cdots + A_m|| < (1 + \sqrt{\epsilon})^2)\right] > 0.$$

Uses Mixed Characteristic polynomials.

# A direct approach to the paving problem

$A \in M_n(\mathbb{C})^{sa}$ hermitian. $\mathcal{X} = X_1 \amalg \cdots \amalg X_r = [n]$, partition. Write,

$$A(\mathcal{X}) = P_{X_1} A P_{X_1} + \cdots + P_{X_r} A P_{X_r}.$$

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**The method of Interlacing polynomials**

$\mathcal{X}$ : Paving. Consider $\chi[A(\mathcal{X})]$. Then,

$$\sum_{\mathcal{X} \in \mathcal{P}_r} \chi[A(\mathcal{X})],$$

is real rooted.

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is real rooted. Futher, $\exists$ paving $\mathcal{X}$ s.t.,

$$\lambda_1 \; A(\mathcal{X}) \leq \lambda_1 \; \sum_{\mathcal{X} \in \mathcal{P}_r} \chi[A(\mathcal{X})].$$

# Expected Characteristic polynomials

**Theorem (R, 16)**

*$A \in M_n(\mathbb{C})$ hermitian. Then There exists a r paving $\mathcal{X}$ s.t.,*

$$\begin{aligned}\lambda_1 A(\mathcal{X}) &\leq \lambda_1 \sum_{\mathcal{X} \in \mathcal{P}_2} \chi[A(\mathcal{X})] \\ &= \lambda_1 \frac{\partial^{(r-1)n}}{\partial z_1^{r-1} \ldots \partial {z_n}^{r-1}} \det[Z - A]^r \mid_{Z=xI} .\end{aligned}$$

# Expected Characteristic polynomials

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$$= \lambda_1 \frac{\partial^{(r-1)n}}{\partial z_1^{r-1} \dots \partial {z_n}^{r-1}} \det[Z - A]^r \mid_{Z=xI} .$$

Recall the definition of real stability (here $\mathbb{H}$ denotes the open upper half plane).

**Definition**

$p \in \mathbb{R}[z_1, \dots, z_n]$ is said to be real stable if it is non-vanishing on $\mathbb{H}^n$.

There is a purely one variable expression for this too. Define for $A \in M_n(\mathbb{C})$ and $r \in \mathbb{R}$,

$$\det_r(A) := \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) r^{c(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}, \quad \chi_r[A] := \det_r(xI - A).$$

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**Proposition (Branden, 2011)**

$\det_r[A]$ *is positive for all PSD* $A \in \mathbb{C}$ *iff* $r \in \mathbb{Z}$.

# Barrier method

Given a real stable polynomial $p(z_1, \ldots, z_n)$, say that $a$ is above the roots, $a \in Ab_p$ if

$$p(a + z) \neq 0 \quad \forall z \in \mathbb{R}^n_+.$$

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We have

$$a \in Ab_p \Longrightarrow a \in Ab_{\partial_i p}.$$

Derivatives shift zero free orthants left. Measure how far $a$ is from the zero set of $p$ using.

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We have the following facts about $\Phi_p^i$. Let $a \in Ab_p$.

1. $\Phi_p^i(a) > 0$.
2. $\partial_j \Phi_p(a) < 0$.
3. $\partial_i^2 \Phi_p(a) > 0$.

# Barrier method : Continued

Our goal is to get an estimate on the max root of

$$\chi_r[A] = \partial_1^{r-1} \dots \partial_n^{r-1} p(Z) \mid_{Z=xI} := \partial_1^{r-1} \dots \partial_n^{r-1} \det[Z - A]^r \mid_{Z=xI} .$$

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**Proposition (R, 17)**

*$p$ real stable of degree $r$ in $z_i$ and $a = (a_1, \dots, a_n) \in Ab_p$ Then*

$$\Phi^j_{\partial_i^{r-1} p}(a - \delta e_i) \leq \Phi^j_p(a), \quad \textit{provided } \delta \leq \frac{r}{\Phi^i_p(a)} - \frac{a_i}{r}.$$

# Optimization

Iterating this, we get

## Proposition

*Let $p(z_1, \cdots, z_n)$ be real stable and of degree at most $r$ in each variable and let $a \in Ab_p$. Then*

$$\left( a - \frac{r}{\Phi_p^1(a)} + \frac{a_1}{r}, \cdots, a - \frac{r}{\Phi_p^n(a)} + \frac{a_n}{r} \right) \in Ab_{\partial_1^{r-1} \cdots \partial_n^{r-1} p}.$$

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Log barriers for $p(Z) = \det[Z - A]^r$. Note $\partial_i \det[Z - A] = \det[(Z - A)_i]$.

**Proposition**

*Let $A$ be a PSD contraction with diagonal entries all at most $\alpha$ and let $a > 1$. Then,*

$$\frac{\partial_i p}{p}(a, \cdots, a) = r e_i^*(aI - A)^{-1} e_i \leq r\left(\frac{\alpha}{b-1} + \frac{1-\alpha}{a}\right).$$

# Optimization

## Proposition (R)

*Let A be a PSD contraction with diagonal entries all at most $\alpha$. Then for $\alpha \leq k/(k+1)$,*

$$\lambda_1 \chi_r[A] \leq \left( \sqrt{\alpha} + \sqrt{\frac{1-\alpha}{k}} \right)^2 .$$

This slightly improves the MSS estimate

$$\lambda_1 \ \chi_r[A] \leq \left( \sqrt{\alpha} + \sqrt{\frac{1}{k}} \right)^2 . \tag{MSS}$$

For the case when $r = 2$, this is weaker than the Bownik-Casazza-Marcus-Speegle estimate of

$$\lambda_1 \ \chi_2[A] \leq \left( \sqrt{\alpha} + \sqrt{\frac{1}{2} - \alpha} \right)^2 . \tag{BCMS}$$

Two basic questions

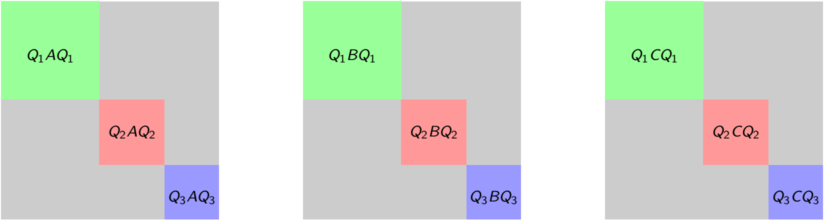
| Question | Conjecture | MSS | Bownik, Casazza, Marcus, Speegle | Branden, Marcus | R. |
|---|---|---|---|---|---|
| Paving diagonal half PSD contractions | 3 | 12 | * | Previous Talk | 4 |
| $\alpha$ such that PSD contractions with diagonals less than $\alpha$ can be 2 paved | 0.5 | 0.08 | 0.25 | Previous Talk | 0.25 |

# Multi-Paving

## Question

Given $A_1, \cdots, A_k$ zero diagonal hermitian contractions, find a paving that works for all the matrices simultaneously. That is, diagonal projections $Q_1, \cdots, Q_r$ with $Q_1 + \cdots + Q_r = I$,

$$\lambda_{max}\left(Q_i A_j Q_i\right) < \epsilon, \quad i \in [r], \quad j \in [k].$$




**Multi Paving** matrices $A$, $B$, $C$

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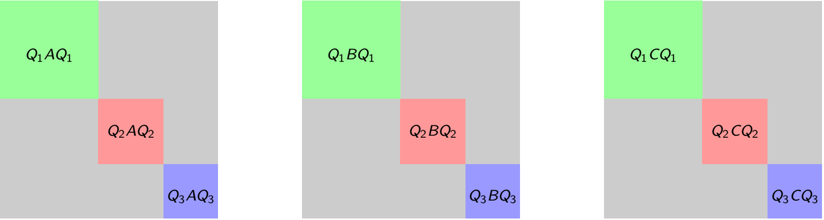


**Multi Paving** matrices $A, B, C$

MSS prooof : $r = O(\epsilon^{-2k})$. Single variable result: $r \geq C\epsilon^{-2}$.

# Multi Paving

**Theorem (R, Srivastava, 17)**

*Given $A_1, \cdots, A_k$ zero diagonal hermitian contractions, there are diagonal projections $Q_1, \cdots, Q_r$ with $Q_1 + \cdots + Q_r = I$, where*

$$r = \frac{18k}{\epsilon^2},$$

*and*

$$||Q_i A_j Q_i|| < \epsilon, \quad i \in [r], \quad j \in [k].$$

Asymptotically optimal in both $k$ and $\epsilon$.

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Asymptotically optimal in both $k$ and $\epsilon$.

**Question**

For each $k$, let $A_1^k, \cdots, A_m^k$ be independent random PSD matrices in such that

$$\mathbb{E}[A_1^k + \cdots + A_m^k] = I, \quad \mathbb{E}\,\mathrm{Tr}(A_i^k) \leq \epsilon, \quad i \in [m].$$

Then is it true that

$$\mathbb{P}\left[||A_1^k + \cdots + A_m^k|| < (\sqrt{k} + \sqrt{\epsilon})^2\right] > 0 \quad \forall k?$$

# Part III : Some Analytic Borcea-Branden theory

## A basic question

Given a polynomial, let $MAP(p)$ denote its multiaffine part.

**Theorem (Borcea, Branden)**

$$p \to MAP(p),$$

*preserves real stability.*

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Several questions about the applications of real stable polynomials can be reduced to the following,

**Question**

Let $p$ be a real stable polynomial. Get estimates for the hyperbolicity cone of $MAP(p)$.

# Lieb-Sokal lemmas

**Proposition**

*Let $p(z_1, \cdots, z_n)$ and $q(z_1, \cdots, z_n)$ be real stable. Then so is*

$$q(\partial_1, \cdots, \partial_n)p.$$

Can one get estimates on the location of zero free orthants for the above polynomial?

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Can one get estimates on the location of zero free orthants for the above polynomial?

In the one variable case : this is just a chain of linear operators.

Log barrier techniques work.

Given a multiaffine $q$, let $\overline{q}$ be the flip of $q$, that is, if we write

$$q = \sum_{S \subset [n]} a_S z^S,$$

then,

$$\overline{q} := \sum_S a_S z^{[n] \setminus S}.$$

**Lemma**

*Let $p, q$ be multiaffine. Then*

$$[q(\partial_1, \dots, \partial_n)p](2z_1, \cdots, 2z_n) = \prod_{i=1}^{n} \left( \frac{\partial}{\partial y_i} + \frac{\partial}{\partial z_i} \right) p(z_1, \cdots, z_n)\overline{q}(y_1, \cdots, y_n) \mid_{y_i = z_i \forall i} .$$

# An analytical Lieb-Sokal lemma

**Theorem (An analytical Lieb-Sokal lemma, R 17)**

*Let* $p, q \in \mathbb{R}[z_1, \cdots, z_n]$ *be multiaffine real stable polynomials and let* $r = q(\partial)p$. *Let* $\mathbf{a}$ *lie above the roots of* $p$ *and* $\mathbf{b}$ *lie above the roots of* $\overline{q}$. *Then,*

$$\mathbf{c} := \mathbf{a} + \mathbf{b} - \left(\frac{1}{\varphi_1}, \cdots, \frac{1}{\varphi_n}\right),$$

*lies above the roots of* $q(\partial)p$ *where,*

$$\varphi_i = \frac{\partial_i\, p}{p}(\mathbf{a}) + \frac{\partial_i\, \overline{q}}{\overline{q}}(\mathbf{b}), \qquad i \in [n].$$

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Remark : One can, using polarizations and some combinatorics remove the condition that the polynomials be multiaffine. One can also allow $q$ to be in the Weyl algebra.

Thanks for listening!