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1 Loci of complex polynomials

Denote by $\mathcal{C}$ the complex plane and let $\mathcal{C}^* := \mathcal{C} \cup \{\infty\}$. Denote by $\mathcal{P}_n$ the set of all complex polynomials

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0; \quad a_n \neq 0. \tag{1}$$

To every polynomial $p \in \mathcal{P}_n$, we correspond a multiaffine, symmetric polynomial in n complex variables:

$$P(z_1, \ldots, z_n) := \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} S_k(z_1, \ldots, z_n), \tag{2}$$

where

$$S_k(z_1, \ldots, z_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} z_{i_1} \cdots z_{i_k}, \quad k = 1, 2, \ldots, n$$

are the elementary symmetric polynomials of degree k, with $S_0(z_1, \ldots, z_n) := 1$. Obviously, $p(z) = P(z, \ldots, z)$. We say that $P(z_1, \ldots, z_n)$ is the *symmetrization of* $p(z)$.

The n-tuple $\{z_1, \ldots, z_n\}$ is called a *solution* of $p(z)$ if $P(z_1, \ldots, z_n) = 0$, i. e., is a solution of $P = 0$.

Definition 1. *A closed subset Ω of $\mathcal{C}^*$ is called a* locus holder *of $p(z) \in \mathcal{P}_n$ if Ω contains at least one point from every solution of $p(z)$. A minimal by inclusion locus holder Ω is called a* locus *of $p(z)$.*

It was shown in [3] that every locus holder contains a locus. If α is a zero of $p(z)$ and Ω is a locus of $p(z)$, then $\alpha \in \Omega$, since $\{\alpha, \alpha, \ldots, \alpha\}$ is a solution of $p(z)$. A restatement of the classical theorem of Grace, see [2, p. 107], says that every circular domain containing the zeros of $p(z)$ is a locus holder of $p(z)$. In fact, every locus of $p(z)$ allows one to formulate an extreme version of Grace's theorem, see [4].

Obviously, if $p(z) = (z - a)^n$, then $p(z)$ has only one locus and this locus is the point a. Such a polynomial is called *trivial*.

If a polynomial is not trivial, then it has unlimited number of different loci and every locus has interior. It is proved, see [4], that every locus of a non trivial polynomial is equal to the closure of its interior. Hence, every bounded locus is measurable and every polynomial has a locus wits minimal area.

Conjecture 1. *Every polynomial has an unique locus with minimal measure.*

Conjecture 2. *Let $D(p)$ be the smallest closed disk containing all zeros of $p(z)$. A locus of $p(z)$ with minimal area is contained in $D(p)$.*

Conjecture 3. *Every polynomial with only real zeros has an unique locus, which is symmetric in respect to the real axes. This locus is with minimal measure.*

Conjecture 4. *Let $p(z) \in \mathcal{P}_n$ and Ω be a bounded locus of $p(z)$ with boundary Γ. If $z_1 \in \Gamma$ is a zero of $p^{(s)}(z)$, then z_1 is also a zero of $p(z)$ of multiplicity $s+1$.*

A popular conjecture, see [1] and [5], states a triviality criterion:

Conjecture 5 (Casas - Alvero)**.** *An algebraic polynomial $p(z)$ of degree $n \geq 1$ is trivial if and only if $p(z)$ shares a zero with each of its derivatives $p^{(s)}(z)$; $s = 1, 2, \dots, n-1$.*

We propose a triviality criterion in terms of the loci.

Conjecture 6. *If a locus of the derivative of the polynomial $p(z)$ is a locus holder of $p(z)$, then $p(z)$ is trivial.*

It is very likely that Conjecture 5 follows from Conjecture 6. It is interesting that Conjecture 5 is motivated by problems in the Number theory and it is proved in cases connected with the properties of the number n - the degree of the polynomial. For example, Conjecture 5 is true if n is prime.

A solution of $p(z) \in \mathcal{P}_n$ is called **bi-solution** if it is of the form

$$\{\underbrace{z, z, \dots, z}_{s}, \underbrace{\zeta, \zeta, \dots, \zeta}_{n-s}\}; \quad s = 1, 2, \dots, n-1.$$

Conjecture 7. *Every locus of $p(z) \in \mathcal{P}_n$ is determined of its bi-solutions.*

If Conjecture 7 is true, an effective algorithm for calculating the loci of polynomials may be created.

Conjecture 8 (Onion Conjecture)**.** *Let $p(z)$ be a complex polynomial of degree $n \geq 2$ and $\Omega^{(s)}$ be the locus of its derivative $p^{(s)}(z)$; $s = 0, 1, 2 \dots, n-1$ with minimal area. Then the inclusions*

$$\Omega^{(n-1)} \subset \Omega^{(n-2)} \subset \cdots \subset \Omega' \subset \Omega$$

hold.

References

[1] Casas-Alvero, E.: Higher order polar germs, *J. Algebra*, **240** 1, (2001), 326 - 337.

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[4] Sendov, Bl. and Sendov, H.S., Loci of complex polynomials, part II: polar derivatives, *Math. Proc. Camb. Phil. Soc.*, 159 253–273 (2015).

[5] Yakubovich, S.: Towards Casas-Alvero conjecture,arXiv: 1504.00274v2 [math.CA] 14 Aug 2015