Non-Real Zero Decreasing Operators Related to Orthogonal Polynomials

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Jet Lag

14:30 in Stockholm $\leftrightarrow$ 04:30 in Alaska.

Apologies in advance for errors and omissions!

For $f : \mathbb{C} \to \mathbb{C}$ with $f \not\equiv 0$, denote the number (counted with multiplicity) of real and non-real zeros of f by $Z_R(f)$ and $Z_C(f)$, respectively.

Example 1

If $f(z) = z^3(z^2+1)e^z$, then $Z_R(f) = 3$ and $Z_C(f) = 2$.

For the identically zero function, define $Z_C(0) = 0 = Z_R(0)$.

Introduction

Consider an operator $L : \mathbb{R}[x] \to \mathbb{R}[x]$. If L has the property that

$$Z_C(L(p)) \leq Z_C(p) \tag{1}$$

for every real polynomial p, then L is called a *Complex Zero Decreasing Operator*, or CZDO.

Example 2

The differentiation operator $D = d/dx$ is a CZDO. This is a consequence of Rolle's Theorem from elementary calculus and the Fundamental Theorem of Algebra.

Introduction

Let $B = \{b_k\}_{k=0}^{\infty}$ be a basis for $\mathbb{R}[x]$. If L is a linear CZDO and there are real constants $\{\gamma_k\}_{k=0}^{\infty}$ for which

$$L(b_k(x)) = \gamma_k b_k(x) \qquad (k = 0, 1, 2, \dots), \tag{2}$$

then, the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called a *Complex Zero Decreasing Sequence for the basis B*, or a B-CZDS.

Example 3

For any real number r, the geometric sequence $\{r^k\}_{k=0}^{\infty}$ is a CZDS for the standard basis $\{x^k\}_{k=0}^{\infty}$.

Example 4

For the standard basis, the sequence $\{k\}_{k=0}^{\infty}$ corresponds to the differential operator xD, which is a CZDO. It is, therefore, a CZDS for the standard basis.

Theorem 5

(Laguerre's Theorem [10, p. 6], [5, p. 23]) Let $p(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n*. If* α *lies outside the interval* $(-n, 0)$*, then*

$$Z_C\left(\sum_{k=0}^{n}(k+\alpha)a_k x^k\right) \leq Z_C\left(\sum_{k=0}^{n} a_k x^k\right).$$

In particular, if $\alpha \geq 0$*, then the sequence* $\{k+\alpha\}_{k=0}^{\infty}$ *is a CZDS for the standard basis.*

Note:

If $p(x) = \sum_{k=0}^{n} a_k x^k$, then $xp'(x) + \alpha p(x) = \sum_{k=0}^{n}(k+\alpha)a_k x^k$.

Theorem 6

(Laguerre's Theorem; Differential Operator Version) Let $p(x)$ be an arbitrary real polynomial of degree n. If α lies outside the interval $(-n, 0)$, then

$$Z_C\left(xp'(x) + \alpha p(x)\right) \leq Z_C\left(p(x)\right).$$

In particular, if $\alpha \geq 0$, then the differential operator $xD + \alpha I$ is a CZDO.

A generalization of Laguerre's theorem from 2007:

Theorem 7

([11, p. 57, Proposition 68]) Suppose $p(x)$ is an arbitrary real polynomial of degree n. If α, β, c, d are real numbers such that $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + cn \geq 0$, then

$$Z_C\big(-\beta p''(x) + (cx+d)p'(x) + \alpha p(x)\big) \leq Z_C\big(p(x)\big).$$

In particular, if α, β, and c are all non-negative, then $-\beta D^2 + (cx+d)D + \alpha I$ is a CZDO.

Introduction

Motivation: The set of Hermite polynomials H satisfy the differential equation ([13, p. 188])

$$nH_n(x) = -\frac{1}{2}H_n''(x) + xH_n'(x) \qquad (n = 0, 1, 2, \dots).$$

Theorem 8

([11, p. 87, Theorem 101]) Let $p(x) = \sum_{k=0}^{n} a_k H_k(x)$ be an arbitrary real polynomial of degree n. If α lies outside the interval $(-n, 0)$, then

$$Z_C\left(\sum_{k=0}^{n}(k+\alpha)a_k H_k(x)\right) \le Z_C\left(\sum_{k=0}^{n} a_k H_k(x)\right).$$

In particular, if $\alpha \ge 0$, then the sequence $\{k+\alpha\}_{k=0}^{\infty}$ is an H-CZDS.

Introduction

A complete characterization of CZDS is not currently known for any basis (an open problem!). The CZDS which can be interpolated by polynomials are characterized for two bases.

Theorem 9

(Craven-Csordas [4, p. 13]) Let $h(x)$ be a real polynomial. Then $\{h(k)\}_{k=0}^{\infty}$ is a CZDS for the standard basis if and only if either

1. *$h(0) \neq 0$ and $h(x)$ has only real negative zeros, or*
2. *$h(0) = 0$ and $h(x)$ has the form*

$$h(x) = x(x-1)(x-2)\cdots(x-m+1)\prod_{k=1}^{p}(x-b_k) \qquad (3)$$

where $m \geq 1$ and $p \geq 0$ are integers and $b_k < m$ for all k.

This theorem is valid *mutatis mutandis* if 'CZDS for the standard basis' is replaced by '*H*-CZDS' ([11, p. 95, Theorem 111]).

Theorem 10

Let p and q be real polynomials, each with degree at least one, and let $\alpha \geq 0$. Then

$$Z_R(f) \geq Z_R(p) + Z_R(q) - 1$$

where

$$f(x) = q(x)p'(x) + \alpha q'(x)p(x).$$

Note: Laguerre's Theorem is included as a special case by taking $q(x) = x$.

Proof: $Z_R(qp' + \alpha q'p) \geq Z_R(p) + Z_R(q) - 1$

When $\alpha = 0$, we have

$$Z_R(f) = Z_R(qp') = Z_R(q) + Z_R(p') \geq Z_R(q) + Z_R(p) - 1,$$

where the last inequality is a consequence of Rolle's theorem. We will now suppose $\alpha > 0$ for the remainder of the proof.

Proof: $Z_R(qp' + \alpha q'p) \geq Z_R(p) + Z_R(q) - 1$

Suppose x_0 is a zero of $p \cdot q$ and write

$$p(x) = (x - x_0)^m h_1(x) \qquad (h_1(x_0) \neq 0),$$

$$q(x) = (x - x_0)^w h_2(x) \qquad (h_2(x_0) \neq 0).$$

Then

$$f(x) = (x - x_0)^{m+w-1} h_3(x)$$

where

$$h_3(x_0) = (m + \alpha w) h_1(x_0) h_2(x_0) \neq 0.$$

That is to say, if x_0 is a zero of $p \cdot q$ of multiplicity $m + w$, then x_0 is a zero of f of multiplicity $m + w - 1$.

Proof: $Z_R(qp' + \alpha q'p) \geq Z_R(p) + Z_R(q) - 1$

To complete the proof, we show that f must vanish between consecutive real zeros of $p \cdot q$. Define

$$g(x) = \begin{cases} \big[q(x)\big]^{\alpha} & \text{if } q(x) \geq 0 \\ -\big[-q(x)\big]^{\alpha} & \text{if } q(x) < 0, \end{cases}$$

so that

$$\big|q(x)\big|^{1-\alpha} \frac{d}{dx}\left[g(x)p(x)\right] = q(x)p'(x) + \alpha q'(x)p(x) \qquad (q(x) \neq 0)$$

Let $x_1 < x_2$ be consecutive zeros of $p \cdot q$. Then they are also consecutive zeros of $g \cdot p$, which is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By Rolle's theorem, $(g \cdot p)'$, and therefore $q(x)p'(x) + \alpha q'(x)p(x)$, has a zero in the interval (x_1, x_2) and the the proof is finished.

Remark

Theorem 10 is best possible in the sense that the conclusion does not necessarily hold for any $\alpha < 0$.

Example 11

If $\alpha < 0$, $p(x) = x^n(x^2 + \alpha)$, and $q(x) = x$, then

$$f(x) = x^n((\alpha + n + 2)x^2 + \alpha(\alpha + n)).$$

Choosing $n = \max\{m \in \mathbb{Z} | m \geq 0 \text{ and } \alpha + m < 0\}$ yields

$$Z_R(f) = n < n + 2 = Z_R(p) + Z_R(q) - 1.$$

A generalization of Laguerre's Theorem: non-real zeros

Theorem 12

Let p and q be real polynomials and $\alpha \geq 0$. Then

$$Z_C(qp' + \alpha q' p) \leq Z_C(p) + Z_C(q).$$

In particular, if q has only real zeros, then $q(x)D + \alpha q'(x)I$ is a CZDO.

Proof: $Z_C(qp' + \alpha q' p) \le Z_C(p) + Z_C(q)$.

The result is trivial when $qp' + \alpha q' p \equiv 0$.

If either p or q is a non-zero constant function, then the result follows from Rolle's theorem. We now assume that p and q each have degree at least one.

Proof: $Z_C(qp' + \alpha q'p) \leq Z_C(p) + Z_C(q)$.

Suppose

$$p(x) = \sum_{k=0}^{n} a_k x^k \qquad \text{and} \qquad q(x) = \sum_{k=0}^{m} b_k x^k.$$

Then the leading term of

$$f(x) = q(x)p'(x) + \alpha q'(x)p(x)$$

is $(n + \alpha m)a_n b_m x^{n+m-1}$, so f has degree $n + m - 1$. Applying Theorem 10, we have

$$\begin{aligned} Z_C(f) &= n + m - 1 - Z_R(f) \\ &\leq n + m - 1 - (Z_R(p) + Z_R(q) - 1) \\ &= n + m - 1 - (n - Z_C(p) + m - Z_C(q) - 1) \\ &= Z_C(p) + Z_C(q). \end{aligned}$$

Therefore, $Z_C(q(x)p'(x) + \alpha q'(x)p(x)) \leq Z_C(p(x)) + Z_C(q(x))$.

Further extension

Remark 13

The two theorems in this section can be extended to any number of constants and functions. For example, using the same techniques as above, one can show that

$$Z_C(pqr' + \alpha p'qr + \beta pq'r) \leq Z_C(p) + Z_C(q) + Z_C(r),$$

where α and β are non-negative real numbers and p, q, and r are polynomials.

The Jacobi Polynomials

The Jacobi polynomials with parameters $\alpha > -1$ and $\beta > -1$:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!} \frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right].$$

These polynomials satisfy the differential equation [13, p. 258]

$$\left((x^2-1)D^2 + [(2+\alpha+\beta)x+\alpha-\beta]D\right) P_n^{(\alpha,\beta)}(x) \\ = n(n+1+\alpha+\beta)P_n^{(\alpha,\beta)}(x).$$

Proposition 14

The sequence $\{k(k+1+\alpha+\beta)\}_{k=0}^{\infty}$ is a $P^{(\alpha,\beta)}$-CZDS.

Proof: $\{k(k+1+\alpha+\beta)\}_{k=0}^{\infty}$ is a $P^{(\alpha,\beta)}$-CZDS

Define the linear operator $L : \mathbb{R}[x] \to \mathbb{R}[x]$ by

$$L\left(P_k^{(\alpha,\beta)}(x)\right) = k(k+1+\alpha+\beta)P_k^{(\alpha,\beta)}(x) \qquad (k=0,1,2,\dots).$$

From the differential equation,

$$L = \left((x^2-1)D + [(2+\alpha+\beta)x+\alpha-\beta]I\right)D.$$

In Remark 13, take $p(x) = x-1$, $q(x) = x+1$ and replace α and β by $\alpha+1$ and $\beta+1$, respectively. Then

$$(x^2-1)D + [(2+\alpha+\beta)x+\alpha-\beta]I \qquad (\alpha,\beta > -1)$$

is a complex zero decreasing operator. Thus, L is the composition of two CZDO (recall that D is a CZDO) and the proof is complete.

Operator Identities

Example 15

As a consequence of the product rule for differentiation, $(Dx)p(x) = xp'(x) + p(x)$, and thus we obtain the equality

$$Dx = xD + I. \tag{4}$$

Proposition 16

Suppose $\{g_k(x)\}_{k=0}^{m}$ *is a sequence of polynomials satisfying deg*$(g_k) \leq k$ *for all* k. *Then*

$$D^n \sum_{k=0}^{m} g_k(x) D^k = \left[\sum_{j=0}^{m} \sum_{k=j}^{m} \binom{n}{k-j} g_k^{(k-j)}(x) D^j \right] D^n.$$

Sketch of Proof: Applying Leibniz' formula for the nth derivative of a product and re-write the double sum.

The Jacobi polynomials with $\alpha = \lambda = \beta$ are called the ultraspherical polynomials. To ease notation, define

$$P_n^{(\lambda)}(x) = P_n^{(\lambda,\lambda)}(x) \qquad (\lambda > -1; n = 0, 1, 2, \dots).$$

These satisfy

$$\left[(x^2 - 1)D^2 + (1 + \lambda)2xD\right] P_n^{(\lambda)}(x) = n(n + 1 + 2\lambda)P_n^{(\lambda)}(x). \quad (5)$$

Notation: For any $a \in \mathbb{R}$, define

$$\Phi_a = (x^2 - 1)D + (1 + a)2xI. \quad (6)$$

Note: By Theorem 12, Φ_a is a CZDO whenever $a > -1$.

Lemma 17

Suppose $\lambda > -1$. Then, for all non-negative integers n,

$$D^n(\Phi_\lambda D - n(n+1+2\lambda)I) = \left(\Phi_{\lambda+n}\right) D^{n+1},$$

where Φ_a is defined in equation (6).

Proof.

This is an immediate application of Proposition 16. □

Ultraspherical Polynomials

For a collection of operators $L_1, L_2, \ldots, L_n$ on $\mathbb{R}[x]$, we define

$$\left(\prod_{k=1}^{n} L_k\right) p = (L_1 L_2 \cdots L_n)p = L_1(L_2(\cdots(L_n(p)))) \qquad (p \in \mathbb{R}[x]).$$

Proposition 18

Let w be a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$. Then

$$\prod_{k=0}^{w-1} (\Phi_\lambda D - k(k+1+2\lambda)I)^{m_k} = \left[\prod_{k=0}^{w-1} \left[(\Phi_{\lambda+k} D)^{m_k - 1} \Phi_{\lambda+k}\right]\right] D^w,$$

where Φ_a is defined by equation (6).

Sketch of Proof: Use induction and Lemma 17.

Theorem 19

If $\lambda > -1$, w is a positive integer, and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$, then the sequence

$$\left\{ \prod_{k=0}^{w-1} (n(n+1+2\lambda) - k(k+1+2\lambda))^{m_k} \right\}_{n=0}^{\infty} \tag{7}$$

is a $P^{(\lambda)}$-CZDS, where $P^{(\lambda)}$ is the set of ultraspherical polynomials.

Sketch of Proof: The operator is a composition of CZDO:

$$L = \prod_{k=0}^{w-1} ((x^2-1)D^2 + (1+\lambda)2xD - k(k+1+2\lambda)I)^{m_k}$$

$$= \prod_{k=0}^{w-1} (\Phi_\lambda D - k(k+1+2\lambda)I)^{m_k} = \left[\prod_{k=0}^{w-1} \left[(\Phi_{\lambda+k} D)^{m_k - 1} \Phi_{\lambda+k} \right] \right] D^w.$$

The Legendre polynomials

$$P_n(x) = P_n^{(0)}(x) = P_n^{(0,0)}(x) \qquad (n = 0, 1, 2, \dots).$$

Corollary 20

If w is a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$, then the sequence

$$\left\{ \prod_{k=0}^{w-1} (n(n+1) - k(k+1))^{m_k} \right\}_{n=0}^{\infty} \tag{8}$$

is a CZDS for the Legendre basis.

Proof.

Apply Theorem 19 with $\lambda = 0$. □

Chebyshev Polynomials

The Chebyshev polynomials $\mathcal{T} = \{T_n(x)\}$ and $\mathcal{U} = \{U_n(x)\}$ of the first and second kind, respectively, are given by

$$T_n(x) := \frac{n!}{(\frac{1}{2})_n} P_n^{(-1/2)}(x) \qquad (n = 0, 1, 2, \dots),$$

$$U_n(x) := \frac{(n+1)!}{(\frac{3}{2})_n} P_n^{(1/2)}(x) \qquad (n = 0, 1, 2, \dots),$$

where $(a)_n := a(a+1)\cdots(a+n-1)$ is the rising factorial.

Chebyshev Polynomials

In [11, Lemma 156] it is shown that a sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS for a simple set $Q = \{q_k(x)\}_{k=0}^{\infty}$ if and only if it is a $\widehat{Q}$-CZDS, where $\widehat{Q}$ consists of the polynomials

$$\widehat{q}_n(x) = c_n q_n(\alpha x + \beta) \qquad (\beta \in \mathbb{R}; \alpha, c_n \in \mathbb{R} \setminus \{0\}).$$

Combining this with Theorem 19, we have

Corollary 21

If w is a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$, then

1. *the sequence $\left\{\prod_{k=0}^{w-1}(n^2 - k^2)^{m_k}\right\}_{n=0}^{\infty}$ is a $\mathcal{T}$-CZDS, and*
2. *the sequence $\left\{\prod_{k=0}^{w-1}(n(n+2) - k(k+2))^{m_k}\right\}_{n=0}^{\infty}$ is a $\mathcal{U}$-CZDS.*

Proof.

Apply Theorem 19 with $\lambda = -1/2$ and again with $\lambda = 1/2$. □

Generating a Basis B and a class of B-CZDS

Given a basis B and a sequence $\{\gamma_k\}_{k=0}^{\infty}$, a typical strategy in showing that $\{\gamma_k\}_{k=0}^{\infty}$ is a B-CZDS is to find a differential operator representation for the diagonal operator which is a CZDO.

We can also begin with a known CZDO and use it to generate a basis B and a corresponding B-CZDS.

We will focus on bases which are *simple sets*, i.e., those for which $\deg(b_k) = k$ for all k.

Generating a Basis B and a class of B-CZDS

Theorem 22

Let $\alpha \geq 0$ *and let*

$$q(x) = c_0 + c_1 x + \cdots + c_r x^r \qquad (r \geq 1, c_r \neq 0)$$

be a real polynomial with only real zeros. Then there is a simple set of polynomials $B = \{b_n(x)\}_{n=0}^{\infty}$ *which satisfy*

$$q(x) b_n^{(r)}(x) + \alpha q'(x) b_n^{(r-1)}(x) = \gamma_n b_n(x) \qquad (n = 0, 1, 2, \ldots),$$

where

$$\gamma_n = c_r (n + (\alpha - 1) r + 1) \prod_{k=0}^{r-2} (n - k) \qquad (n = 0, 1, 2, \ldots).$$

Consequently, the sequence $\{\gamma_n\}_{n=0}^{\infty}$ *is a* B*-CZDS.*

Generating a Basis B and a class of B-CZDS

Example: Choose $q(x) = (x+1)^3$ and $\alpha = 1$, then $\gamma_n = (n+1)n(n-1)$, and we seek a simple set $B = \{b_n(x)\}_{n=0}^{\infty}$ which solves the differential equation

$$(n+1)n(n-1)b_n(x) = (x+1)^3 b_n'''(x) + 3(x+1)^2 b_n''(x).$$

Any such set B must have the form

$$b_0(x) = r,$$

$$b_1(x) = sx + t,$$

$$b_n(x) = c_n(x+1)^n \quad (n = 2, 3, 4, \dots)$$

where $t \in \mathbb{R}$ and $r, s, c_2, c_3, \dots$ are any (fixed) non-zero real numbers. Thus, the sequence

$$\{(n+1)n(n-1)\}_{n=0}^{\infty}$$

is a B-CZDS for any such basis B.

Extension to Certain Transcendental Entire Functions

A real entire function φ is said to belong to the *Laguerre-Pólya class*, denoted $\varphi \in \mathcal{L} - \mathcal{P}$, if it can be written in the form

$$\varphi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k} \qquad (9)$$

where $b, c, x_k \in \mathbb{R}$, m is a non-negative integer, $a \geq 0$, $0 \leq \omega \leq \infty$, and $\sum_{k=1}^{\omega} x_k^{-2} < \infty$.

Alternatively, $\varphi \in \mathcal{L} - \mathcal{P}$ if and only if φ is the uniform limit on compact subsets of $\mathbb{C}$ of real polynomials having only real zeros (See, for example, [8, Ch. VIII] or [10, Satz 9.2]).

Theorem 23

Suppose φ belongs to the class $\mathcal{L}-\mathcal{P}$, p and q are real polynomials, and $\alpha \geq 0$. Then

$$Z_C(\varphi q p' + \alpha(\varphi q)' p) \leq Z_C(p) + Z_C(q).$$

Proof.

Suppose $\{f_k\}_{k=0}^{\infty}$ is a sequence of real polynomials with only real zeros which converge uniformly on compact subsets of $\mathbb{C}$ to φ. By Theorem 12,

$$Z_C(f_k q p' + \alpha(f_k q)' p) \leq Z_C(p) + Z_C(q) \qquad (k = 0, 1, 2, \dots).$$

Since $f_k q p' + \alpha(f_k q)' p$ converges uniformly on compact subsets of $\mathbb{C}$ to $\alpha(\varphi q)' p + \varphi q p'$, Hurwitz' theorem gives the desired result. □

A new proof of a known result using Theorem 23.

Corollary 24

([11, p. 55, Lemma 67]) Suppose that $p(x)$ is a real polynomial of degree n. If c, d, β are real numbers such that $c \geq 0$ and $\beta \geq 0$, then

$$Z_C\big((cx+d)p(x) - \beta p'(x)\big) \leq Z_C\big(p(x)\big).$$

Proof.

If $\beta = 0$, the result clearly holds. If $\beta > 0$ we may appeal to corollary 23 with $\alpha = \beta^{-1}$, $q(x) = 1$, and

$$\varphi(x) = -\exp\left(-\frac{c}{2}x^2 - d\,x\right) \qquad (c \geq 0, d \in \mathbb{R})$$

to obtain the desired result. □

The generalized Laguerre polynomials are given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \qquad (\alpha > -1; n = 0, 1, 2, \dots).$$

They satisfy the differential equation (see, e.g., [13, p. 204])

$$-x \frac{d^2}{dx^2} L_n^{(\alpha)}(x) + (x - (\alpha+1)) \frac{d}{dx} L_n^{(\alpha)}(x) = n L_n^{(\alpha)}(x). \qquad (10)$$

Notation: For any $a \in \mathbb{R}$, define

$$\Psi_a = -xD + (x - (a+1))I. \qquad (11)$$

Lemma 25

Suppose $\alpha \in \mathbb{R}$*. Then, for all non-negative integers n,*

$$D^n(\Psi_\alpha D - nI) = \Psi_{\alpha+n} D^{n+1}.$$

Proof.

This is an immediate application of Proposition 16. □

Proposition 26

Let w be a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$. Then

$$\prod_{k=0}^{w-1} (\Psi_\alpha D - kI)^{m_k} = \left[\prod_{k=0}^{w-1} \left[(\Psi_{\alpha+k} D)^{m_k - 1} \Psi_{\alpha+k} \right] \right] D^w,$$

where Ψ_a is defined in equation (11).

Sketch of Proof: Use induction and Lemma 25.

Lemma 27

For any $a > -1$, the operator $\Psi_a = -xD + (x-(a+1))I$ is a CZDO.

Proof.

Suppose $a > -1$ and set $c = a+1$. By Theorem 23,

$$Z_C\left(c\frac{d}{dx}(-x\exp(-x/c))\,p(x) + (-x\exp(-x/c))p'(x)\right) \leq Z_C(p).$$

The quantity on the left simplifies to

$$Z_C\left((-xp'(x) + (x-c)p(x))\exp(-x/c)\right).$$

Thus, $Z_C(\Psi_a p(x)) = Z_C(-xp'(x) + (x-c)p(x)) \leq Z_C(p(x))$. □

Theorem 28

Fix $\alpha > -1$. If w is a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$, then the sequence

$$\left\{ \prod_{k=0}^{w-1} (n-k)^{m_k} \right\}_{n=0}^{\infty} \tag{12}$$

is an $L^{(\alpha)}$-CZDS.

Sketch of Proof: The operator is a composition of CZDO:

$$\prod_{k=0}^{w-1} (\Psi_\alpha D - kI)^{m_k} = \left[\prod_{k=0}^{w-1} \left[(\Psi_{\alpha+k} D)^{m_k - 1} \Psi_{\alpha+b} \right] \right] D^w.$$

Open Questions

The "falling-factorial sequence"

$$\{k(k-1)\cdots(k-(m-1))\}_{k=0}^{\infty}$$

is a CZDS for the standard basis. By Corollary 21, any sequence of the form

$$\{k^2(k^2-1)\cdots(k^2-(m-1)^2)\}_{k=0}^{\infty}$$

is a T-CZDS. Can an analog of Theorem 9 be obtained for the Chebyshev basis?

Problem 29

Find a complete characterization of polynomials h for which $\{h(k)\}_{k=0}^{\infty}$ is a T-CZDS, where T denotes the Chebyshev basis.

We note that the characterization will be different from that of the standard basis, since the sequence $\{k\}_{k=0}^{\infty}$ is not a T-CZDS.

Open Questions

The results on ultraspherical and Laguerre CZDS also have a falling factorial nature which leads us to consider the more general problem.

Problem 30

For any basis B, find a complete characterization of polynomials h for which $\{h(k)\}_{k=0}^{\infty}$ is a B-CZDS.

This problem has been solved for the standard basis, the Hermite basis, and any affine transformation of these two bases. Problem 30 remains unsolved for any other choice of the basis B.

Open Questions

No complete characterization of CZDS for the standard basis is known. In particular, it is not known whether every rapidly decreasing sequence is a CZDS for the standard basis (see [5, Problem 4.8]). There is a connection with other bases:

Theorem 31

[11, Theorem 159] Let $B = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of polynomials. If the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a B-CZDS, then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS for the standard basis.

This prompts us to state a weaker version of Problem 4.8(a) of [5].

Problem 32

Is there a simple set B for which $\{\exp(-k^3)\}_{k=0}^{\infty}$ is a B-CZDS?

Our methods of simultaneously generating a basis and CZDS may apply (with a different operator).

References I

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