Capacity Preserving Operators

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Our Goal

Definition

For $p \in \mathbb{R}[x] \equiv \mathbb{R}[x_1, ..., x_n]$, we say p is *real stable* whenever $p(x) \neq 0$ for $x \in \mathcal{H}_+^n$.

Main goal: obtain bounds on combinatorial info via real stable polynomials which encode that info.

- Matching polynomial – matchings of a graph
- Product of linear forms – permanent of a matrix

objects $\rightarrow$ multivariate polynomials $\rightarrow$ apply operators $\rightarrow$ information

Can we use and/or emulate the Borcea-Brändén characterization to transfer quantitative information about *coefficients/evaluations*?

Two Motivating Examples

(BB) Multivariate matching polynomial $= \text{MAP}(\prod_{(i,j)\in E}(1 - x_i x_j))$

- $(1 - x_i x_j)$ is real stable, products are real stable.
- MAP = "Multi-Affine Part" preserves real-stability.
- Plug in x for all variables $\to$ univariate matching poly is real-rooted.
- What about bounds on coefficients?

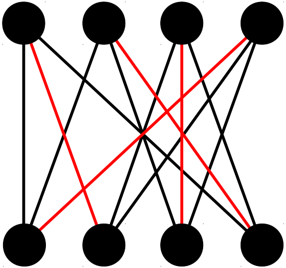
(Gurvits) Doubly stochastic matrix $M \to \prod_{r\in\text{rows}} r \cdot x$

- $p_M(x) := \prod_i \sum_j m_{ij} x_j$ is real stable.
- (coefficient of $x_1 x_2 \cdots x_n$) $= \partial_{x_1} \cdots \partial_{x_n} p$ is the permanent of M.
- We can obtain a bound on the permanent by analyzing ∂_{x_k}.

Both cases: want to obtain bounds on how certain linear operators affect the coefficients of a real stable polynomial.

An Explicit Example: Schrijver's Inequality

Let G be a d-regular bipartite graph with $2n$ total vertices.



Bipartite adjacency matrix, M:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

perfect matchings = permanent

$$p_M = (x_1 + x_2 + x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4)(x_1 + x_2 + x_3)$$

- $\text{pm}(G) = \text{per}(M) = \partial_{x_1} \cdots \partial_{x_n} p_M$
- Schrijver: $\text{pm}(G) \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)^n$
- # k-edge matchings $\sim \sum_{S \in \binom{[n]}{k}} \partial_x^S p_M(1) \geq$?

Gurvits' Method

Throughout: x is a vector, $x > 0$ is element-wise, $x^\alpha := \prod_{k=1}^n x_k^{\alpha_k}$, etc.

Definition (Gurvits)

For $p \in \mathbb{R}_+[x]$ and $\alpha \in \mathbb{R}_+^n$, we define $\mathrm{Cap}_\alpha(p) := \inf_{x>0} \frac{p(x)}{x^\alpha}$.

Theorem (Gurvits)

Let $p \in \mathbb{R}_+[x] \equiv \mathbb{R}_+[x_1, ..., x_n]$ be n-homogeneous and real stable. Then:

$$\mathrm{Cap}_{(1^{n-1})}(\partial_{x_k} p|_{x_k=0}) \geq \left(\frac{n-1}{n}\right)^{n-1} \mathrm{Cap}_{(1^n)}(p)$$

- Gives a simple proof of the van der Waerden lower bound for the permanent of a doubly stochastic matrix ($\mathrm{per}(M) \geq \frac{n!}{n^n}$)
- Essentially implies Schrijver's perfect matching inequality
- Can be interpreted as a capacity preservation result for $\partial_{x_k}|_{x_k=0}$

Can we generalize this result to other operators?

General Form of the Method

Fix $p \in \mathbb{R}_+^\lambda[x]$ (degree at most λ_k in x_k) and linear $T : \mathbb{R}_+^\lambda[x] \to \mathbb{R}_+^\gamma[x]$.

$$\text{Cap}_\beta(T[p]) \geq c_{T,\alpha,\beta,\lambda} \cdot \text{Cap}_\alpha(p)$$

What we need to happen:

- Series of linear operators which lead to a desired quantity.
- Capacity of starting polynomial is easy to compute.
- If T is a functional and $\beta = \varnothing$, then $T[p] = \text{Cap}_\beta(T[p])$.

Bounds are achieved when p is real stable and T preserves real stability: can theoretically lower-bound any quantity which is derivable in this way.

First Idea: Inner Product Bounds

Certain differential operators can be interpreted via (real) inner products.

- E.g., $\text{per}(M) = q(\partial_x) p_M(x)|_{x=0}$ for $q = x_1 \cdots x_n$.
- Can we obtain/utilize bounds on inner products of polynomials?

Definition

For $p, q \in \mathbb{R}^\lambda[x]$, define $\langle p, q \rangle^\lambda := \sum_{0 \leq \mu \leq \lambda} \binom{\lambda}{\mu}^{-1} p_\mu q_\mu$.

Observation: $\text{per}(M) = \partial_{x_1} \cdots \partial_{x_n} p_M = \langle x_1 \cdots x_n, p_M \rangle^\lambda \cdot \prod_k \lambda_k$

Why this inner product?

- Practical – inductive structure leads to the bounds we want
- Useful – amenable to BB-style ideas (similar to apolarity form)
- Natural – unique SO_2^n-invariant bilinear form (up to degree)

First Idea: Inner Product Bounds

Theorem (Anari-Gharan, 2017)

For real stable multiaffine $p, q \in \mathbb{R}_+[x]$ and $\alpha \in \mathbb{R}_+^n$, we have:

$$\langle p, q \rangle^{(1^n)} \geq \alpha^\alpha (1-\alpha)^{1-\alpha} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_\alpha(q)$$

Proof: Strongly Rayleigh inequalities.

Theorem (Anari-Gharan, 2017)

For real stable $p, q \in \mathbb{R}_+[x]$ and $\alpha \in \mathbb{R}_+^n$, we have:

$$q(\partial_x)p(x)|_{x=0} \geq e^{-\alpha} \alpha^\alpha \operatorname{Cap}_\alpha(p) \operatorname{Cap}_\alpha(q)$$

Already: $\operatorname{per}(M) \geq e^{-(1^n)}(1^n)^{(1^n)} \operatorname{Cap}_{(1^n)}(p_M) = e^{-n} \operatorname{Cap}_{(1^n)}(p_M)$

Lemma (Gurvits)

If M is doubly stochastic, then $\operatorname{Cap}_{(1^n)}(p_M) = 1$.

First Idea: Inner Product Bounds

Can we do better if we know the degree of the polynomial?

Theorem

For real stable $p, q \in \mathbb{R}_+^\lambda[x]$ and $\alpha \in \mathbb{R}_+^n$, we have:

$$\langle p, q \rangle^\lambda \geq \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_\alpha(q)$$

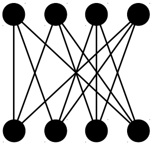
Proof: Capacity and $\langle \cdot, \cdot \rangle$ play nice with polarization; follows from the prior multiaffine result.

So: $\operatorname{per}(M) = \langle x_1 \cdots x_n, p_M \rangle^\lambda \cdot \prod_k \lambda_k \geq \left(\frac{\lambda - 1}{\lambda}\right)^{\lambda - 1} \operatorname{Cap}_{(1^n)}(p_M)$

- Limits to the e^{-n} bound as $\lambda \to \infty$.
- Looks similar to Gurvits' theorem, but not quite as strong/general.
- Easy to achieve Schrijver's inequality as a corollary.

Proof of Schrijver's Inequality

G is a d-regular bipartite graph on $2n$ vertices, with incidence matrix M.



$$\mathrm{M} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$p_M = (x_1 + x_2 + x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4)(x_1 + x_2 + x_3)$$

Recall: $\mathrm{pm}(G) = \mathrm{per}(M) \geq \left(\frac{\lambda - 1}{\lambda}\right)^{\lambda - 1} \mathrm{Cap}_{(1^n)}(p_M)$

- d-regularity implies $\frac{1}{d}M$ is doubly stochastic
- Lemma implies $\mathrm{Cap}_{(1^n)}(p_M) = d^n \cdot \mathrm{Cap}_{(1^n)}(p_{\frac{1}{d}M}) = d^n$
- d-regularity implies p_M is of degree $\lambda = (d, d, ..., d)$
- $\left(\frac{\lambda - 1}{\lambda}\right)^{\lambda - 1} = \prod_{k=1}^n \left(\frac{d-1}{d}\right)^{d-1} = \left(\frac{d-1}{d}\right)^{n(d-1)}$

Therefore: $\mathrm{pm}(G) \geq \left(\frac{d-1}{d}\right)^{n(d-1)} \cdot d^n = \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n$

Other Bounds on Matchings

What about non-bipartite G? Via the matching polynomial?

Unfortunate problem: matching polynomial does not have non-negative coefficients, and this is essentially unavoidable for non-bipartite G.

What about counting k-matchings for bipartite G?

Theorem (Csikvári, 2014)

Let G be a d-regular bipartite graph with $2n$ vertices. Then:

$$\mu_k(G) \geq \binom{n}{k} d^k \left(\frac{nd-k}{nd}\right)^{nd-k} \left(\frac{n}{n-k}\right)^{n-k}$$

- Reduces to Schrijver's inequality for $k = n$ (here $0^0 = 1$).
- Implies Friedland's lower matching conjecture.
- Actually able to bound k-matchings for biregular bipartite graphs.
- Can prove these bounds using capacity-preservers.

The Symbol for Capacity

BB: stability properties shared between operator and its symbol.

Recall: $\langle p, q \rangle^\lambda = \sum_{\mu \leq \lambda} \binom{\lambda}{\mu}^{-1} p_\mu q_\mu$

Definition

Given linear $T : \mathbb{R}^\lambda[x] \to \mathbb{R}^\gamma[x]$, we define $\text{Symb}(T) \in \mathbb{R}^{(\lambda,\gamma)}[z, x]$ via:

$$T[p](x) = \langle \text{Symb}(T)(z, x), p(z) \rangle^\lambda$$

Lemma

For a given linear operator $T : \mathbb{R}^\lambda[x] \to \mathbb{R}^\gamma[x]$, *we have:*

$$\text{Symb}(T)(z, x) = T\left[(1 + xz)^\lambda\right] = \sum_{\mu \leq \lambda} \binom{\lambda}{\mu} z^\mu T(x^\mu)$$

Is there a similar operator-symbol correspondence for capacity?

From Inner Products to Operators

Theorem

For real stable $p, q \in \mathbb{R}_+^\lambda[x]$ and $\alpha \in \mathbb{R}_+^n$, we have:

$$\langle p, q \rangle^\lambda \geq \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_\alpha(q)$$

For T and p with desired properties, and fixed $x > 0$:

$$\begin{aligned} T[p](x) &= \langle \operatorname{Symb}(T)(z, x), p(z) \rangle^\lambda \\ &\geq \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_\alpha(\operatorname{Symb}(T)(\cdot, x)) \end{aligned}$$

Divide by x^β and take $\inf_{x>0}$ on both sides (recall $\operatorname{Cap}_\beta(p) := \inf_{x>0} \frac{p(x)}{x^\beta}$):

$$\operatorname{Cap}_\beta(T[p]) \geq \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_{(\alpha, \beta)}(\operatorname{Symb}(T))$$

Capacity Preserving Operators

Theorem

Let $T : \mathbb{R}_+^\lambda[x] \to \mathbb{R}_+^\gamma[x]$ *be such that* $\text{Symb}(T)(z,x) \in \mathbb{R}_+^{(\lambda,\gamma)}[z,x]$ *is real stable in* z *for every* $x > 0$. *For any real stable* $p \in \mathbb{R}_+^\lambda[x]$*:*

$$\text{Cap}_\beta(T[p]) \geq \left[\frac{\alpha^\alpha(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda}\text{Cap}_{(\alpha,\beta)}(\text{Symb}(T))\right]\text{Cap}_\alpha(p)$$

Moreover, this bound is tight for any fixed α, β, *and* T.

Tightness is demonstrated by considering $p(x) = (xy+1)^\lambda$ for fixed $y > 0$.

Corollary

The above theorem holds for any operator preserving real stability and non-negative coefficients, which has image of dimension greater than 2.

Application: Gurvits' Theorem

Theorem (Gurvits)

Let $p \in \mathbb{R}_+[x] \equiv \mathbb{R}_+[x_1, ..., x_n]$ be n-homogeneous and real stable. Then:

$$\text{Cap}_{(1^{n-1})}(\partial_{x_k} p|_{x_k=0}) \geq \left(\frac{n-1}{n}\right)^{n-1} \text{Cap}_{(1^n)}(p)$$

Recall: $\text{Cap}_\beta(T(p)) \geq \left[\frac{\alpha^\alpha(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \text{Cap}_{(\alpha,\beta)}(\text{Symb}(T))\right] \text{Cap}_\alpha(p)$

- $\lambda = (n, ..., n), \alpha = (1^n), \beta = (1^{n-1}) \to \frac{\alpha^\alpha(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} = \left(\frac{(n-1)^{n-1}}{n^n}\right)^n$
- $\text{Symb}(\partial_{x_k}|_{x_k=0}) = \partial_{x_k}(xz+1)^\lambda\big|_{x_k=0} = \lambda_k z_k (xz+1)^{\lambda'}$
- $\text{Cap}_{(1^n,1^{n-1})}(\lambda_k z_k (xz+1)^{\lambda'}) = n\left(\frac{n^n}{(n-1)^{n-1}}\right)^{n-1}$

Therefore: $\frac{\alpha^\alpha(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \text{Cap}_{(\alpha,\beta)}(\text{Symb}(T)) = \left(\frac{n-1}{n}\right)^{n-1}$

Application: Csikvári's Theorem

Theorem (Csikvári, 2014)

Let G be a d-regular bipartite graph with $2n$ vertices. Then:

$$\mu_k(G) \geq \binom{n}{k} d^k \left(\frac{nd-k}{nd}\right)^{nd-k} \left(\frac{n}{n-k}\right)^{n-k}$$

Recall: $\text{Cap}_\beta(T(p)) \geq \left[\frac{\alpha^\alpha(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \text{Cap}_{(\alpha,\beta)}(\text{Symb}(T))\right] \text{Cap}_\alpha(p)$

- M is bipartite adjacency matrix, p_M is associated product of linears
- d-regularity implies $\mu_k(G) = d^{k-n} \sum_{S \in \binom{[n]}{k}} \partial_x^S p_M(1) =: d^{k-n} T(p_M)$
- d-regularity implies $\text{Cap}_{(1^n)}(p_M) = d^n$
- d-regularity implies $\lambda = (d, ..., d)$
- $\alpha = (1^n)$ implies $\frac{\alpha^\alpha(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} = \frac{(d-1)^{nd-n}}{d^{nd}}$
- $\beta = \varnothing$ implies $\text{Cap}_\beta(T(p_M)) = T(p_M)$

Application: Csikvári's Theorem (continued)

- $\text{Symb}(T) = \sum_{S \in \binom{[n]}{k}} \partial_x^S (xz+1)^\lambda \Big|_{x=1} = \sum_{S \in \binom{[n]}{k}} d^k z^S (z+1)^{\lambda - S}$

Lemma

If $p \in \mathbb{R}_+[x] \equiv \mathbb{R}_+[x_1, ..., x_n]$ is symmetric, then:

$$\text{Cap}_{(t,...,t)}(p) = \text{Cap}_{nt}(p(x_0, ..., x_0))$$

- $\text{Symb}(T)$ is symmetric:

$$\text{Cap}_{(1^n)} \left[\sum_{S \in \binom{[n]}{k}} d^k z^S (z+1)^{\lambda - S} \right] = \text{Cap}_n \left[\binom{n}{k} d^k z_0^k (z_0 + 1)^{dn-k} \right]$$

- Easier: $\text{Cap}_n \left[\binom{n}{k} d^k z_0^k (z_0+1)^{dn-k} \right] = \binom{n}{k} d^k \frac{(nd-k)^{nd-k}}{(n-k)^{n-k}(nd-n)^{nd-n}}$

Further Questions

Applications of capacity-preservers, beyond differential operators?

Can we get similar bounds based only on the *total* degree of a given homogeneous polynomial?

SO_n-invariant inner product: $\langle p, q \rangle_{SO_n}^d := \sum_\mu \binom{d}{\mu}^{-1} p_\mu q_\mu$

Conjecture (Gurvits, 2009)

For real stable d-homogeneous polynomials $p, q \in \mathbb{R}_+[x]$, we have:

$$\langle p, q \rangle_{SO_n}^d \geq n^{-d} \, \mathrm{Cap}_\alpha(p) \, \mathrm{Cap}_\alpha(q)$$

What about similar results for polynomials which take matrices as input?

- Some bound on Frobenius inner product? Some other inner product?
- Possibly related to SO_n inner product above.