Laplace Polynomials, Laplace Continued Fraction and Ramanujan's Identity

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Outline

Introduction

Mills' Ratio: notations

$$R(t) := \frac{\bar{\Phi}(t)}{\varphi(t)},$$

where

$$\bar{\Phi}(t) = \int_t^{\infty} \varphi(x)dx, \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Laplace CF

Laplace CF

We introduce Laplace polynomials, $P_k(t)$ and $Q_k(t)$ that form rational approximation to the continued fraction

$$R(t) = \cfrac{1}{t + \cfrac{1}{t + \cfrac{2}{t + \cfrac{3}{\ddots}}}}.$$

studied by Laplace.

Identities

We compute MGF for these polynomials and prove the following seemingly unrelated identities,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{n}\frac{(n+m)!}{m!j!(m+2n+1-j)!}\,2^{-n}=\sqrt{2\pi}e^{2}\left(\Phi(2)-\Phi(1)\right),$$

$$\sum_{k=0}^{n}(-1)^{k}\frac{1}{2k+1}\binom{n}{k}=\frac{2^{2n}\,(n!)^{2}}{(2n+1)!},$$

and

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{\ddots}}}}=\sqrt{\frac{e\pi}{2}}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)!!}.$$

Mills' Ratio

Mills' Ratio

Mills' ratio is the function

$$R(t) = \frac{\bar{\Phi}(t)}{\varphi(t)}, \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

where

$$\bar{\Phi}(t) = \int_t^{\infty} \varphi(s)\,ds$$

is the tail of the Normal distribution.

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is the tail of the Normal distribution.

Normal Hazard Rate

$$\lambda(t) := 1/R(t) = \frac{\varphi(t)}{\bar{\Phi}(t)}; \quad \mathbb{E}[\xi \mid \xi > t] = \lambda(t).$$

The importance of doing Mills' Ratio

Applications

- Computation of Likelihood Ratio

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- Bounds for functionals of Gaussian processes

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The origins of the project

- Ian Iscoe and Taehan Bae "Correlations under stress"

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- Computation of Likelihood Ratio
- Bounds for functionals of Gaussian processes
- Reliability models

The origins of the project

- Ian Iscoe and Taehan Bae "Correlations under stress"
- Convexity of the function $\lambda(t)$.

History

History

Figure: P.S. Laplace (1749-1827)

Continued fraction

Laplace (1812 or 1795 (?))

$$R(t) \sim \frac{1}{t} - \frac{1}{t^3} + \frac{1 \cdot 3}{t^5} - \frac{1 \cdot 3 \cdot 5}{t^7} + \dots \qquad \text{for } t > 0 \,. \tag{1}$$

based on the continued fraction expansion,

$$R(t) = \cfrac{1}{t + \cfrac{1}{t + \cfrac{2}{t + \cfrac{3}{t + \cfrac{4}{\ddots}}}}} \,. \tag{2}$$

Laplace polynomials

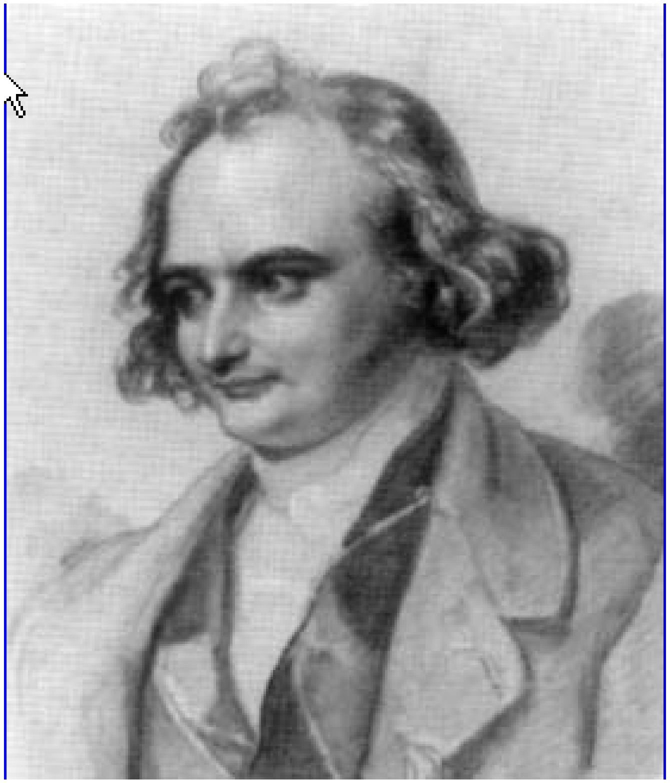
Pour faire usage de cette expression, il faut réduire la fraction continue

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{2q}{1+\ldots}}}$$

en fractions alternativement plus grandes et plus petites que la fraction entière. Les deux premières fractions sont $\frac{1}{1}$, $\frac{1}{1+q}$; les numérateurs des fractions suivantes sont tels que le numérateur de la fraction $i^{\text{ième}}$ est égal au numérateur de la fraction $(i-1)^{\text{ième}}$ plus au numérateur de la fraction $(i-2)^{\text{ième}}$, multiplié par $(i-1)q$; les dénominateurs se forment de la même manière. Ces fractions successives sont

$$\frac{1}{1},\quad \frac{1}{1+q},\quad \frac{1+2q}{1+3q},\quad \frac{1+5q}{1+6q+3q^2},\quad \frac{1+9q+8q^2}{1+10q+15q^2},\quad \ldots.$$

Convergence of continued fraction



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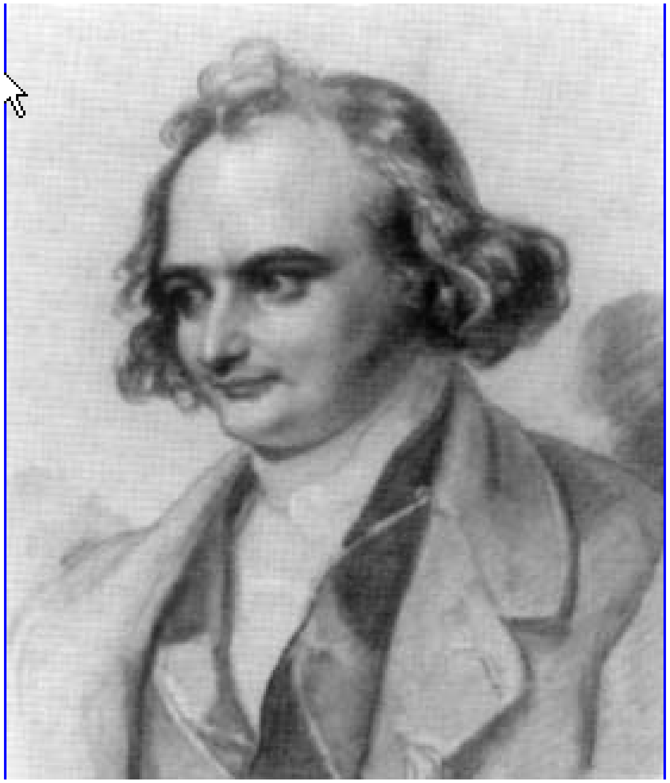

C. G. Jacobi (1804-1851)

C.G. Jacobi: 1834

26.

De fractione continua, in quam integrale $\int_x^\infty e^{-xx}dx$ evolvere licet.

(Auct. *C. G. J. Jacobi*, prof. ord. math. Regiom.)

Integrale propositum, cuius in fractionibus coelestibus aliisque quaestionibus usus est, ill. *Laplace* (M. C. T. IV. L. X.) in fractionem continuam evolutum dedit sequentem, posito $q = \frac{1}{2xx}$,

$$1.\quad \int_x^\infty e^{-xx}\partial x = \frac{e^{-xx}}{2x}\cdot\cfrac{1}{1+\cfrac{q}{1+\cfrac{2q}{1+\cfrac{3q}{1+4q \atop \vdots}}}}$$

Demonstratio tamen viri cum per series divergentes procedat, hodie vix probabitur; quae hoc modo accuratior redditur.

Statuamus

$$2.\quad v = e^{+xx}\int_x^\infty e^{-xx}\partial x,$$

habemus differentiando,

$$3.\quad \frac{\partial v}{\partial x} = +2xv - 1.$$

Qua aequatione iterum n vicibus differentiata, prodit

$$4.\quad \frac{\partial^{n+1}v}{\partial x^{n+1}} = 2x\frac{\partial^n v}{\partial x^n} + 2n\frac{\partial^{n-1}v}{\partial x^{n-1}},$$

sive posito

The Method

Our approach is based on the following observations:

Two important properties

- The function $R(t)$ can be represented as a Laplace transform of a probability density function, $\varphi(x)$,

$$\frac{1}{\sqrt{2\pi}}R(t) = \int_0^\infty e^{-tx}\varphi(x)dx, \quad t \geq 0,$$

- The function $R(t)$ satisfies the differential equation

$$\frac{dR(t)}{dt} = tR(t) - 1.$$

Complete monotonicity of $R(t)$

Proof

- Consider r.v. $\xi \sim \mathcal{N}(0,1)$ and take $\eta = |\,\xi\,|$. Then

$$\mathbb{E}[e^{-t\eta}] = 2e^{t^2/2}\bar{\Phi}(t) = \sqrt{\frac{2}{\pi}}R(t).$$

Corollary

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Corollary

- The function $R(t)$ is completely monotone:

$$(-1)^n \frac{d^n R(t)}{dt^n} \geq 0 \quad \forall t \geq 0.$$

Derivatives $\frac{d^n R(t)}{dt^n}$

$n = 1;\ t > 0$

$$\frac{dR(t)}{dt} = t\,R(t) - 1 \leq 0, \quad R(t) \leq 1/t.$$

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Asymptotics

$$R(t) \sim \frac{1}{t}, \qquad t \to \infty.$$

General relations

k^{th} derivative of $R(t)$

$$\frac{d^k R(t)}{dt^k} = P_k(t)\, R(t) - Q_{k-1}(t)$$

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$$\frac{d^k R(t)}{dt^k} = P_k(t)\, R(t) - Q_{k-1}(t)$$

Inequalities for $R(t)$

$$\frac{Q_{k-2}(t)}{P_{k-1}(t)} \leq R(t) \leq \frac{Q_{k-1}(t)}{P_k(t)}, \qquad k = 2, 4, 6, \ldots$$

Recurrent Equation for Laplace polynomials

Equation for polynomials

$$P_{k+1}(t) = tP_k(t) + P'_k(t), \tag{3}$$

$$Q_k(t) = P_k(t) + Q'_{k-1}(t), \tag{4}$$

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Equation for coefficients

Recurrent Equation for Laplace polynomials

Equation for polynomials

$$P_{k+1}(t) = tP_k(t) + P_k'(t), \tag{3}$$

$$Q_k(t) = P_k(t) + Q_{k-1}'(t), \tag{4}$$

Equation for coefficients

$$p_{k+1,m} = p_{k,m-1} + (m+1) \cdot p_{k,m+1}, \tag{5}$$

$$q_{k,m} = p_{k,m} + (m+1) \cdot q_{k-1,m+1}. \tag{6}$$

Coefficients of Laplace polynomials

k	$P_k(t)$	$Q_{k-1}(t)$
1	t	1
2	t^2+1	t
3	t^3+3t	t^2+2
4	t^4+6t^2+3	t^3+5t
5	t^5+10t^3+15t	t^4+9t^2+8
6	$t^6+15t^4+45t^2+15$	t^5+14t^3+33t
7	$t^7+21t^5+105t^3+105t$	$t^6+20t^4+87t^2+48$

Table: Polynomials $P_k(t)$ and $Q_{k-1}(t)$.

Coefficients $p_{k,m}$

k	m								
	0	1	2	3	4	5	6	7	8
0	1	0	0	...					0
1	0	1	0	...					0
2	1	0	1	0	...				0
3	0	3	0	1	0	...			0
4	3	0	6	0	1	0	...		0
5	0	15	0	10	0	1	0	...	0
6	15	0	45	0	15	0	1	0	0
7	0	105	0	105	0	21	0	1	0
8	105	0	420	0	210	0	28	0	1

Table: Matrix $\mathfrak{p}$.

Formula for $p_{k,m}$

Theorem

If $k - m \equiv 1 \pmod 2$, *the coefficients* $p_{k,m} = 0$. *If* $k \equiv m \pmod 2$

$$p_{k,m} = \frac{k!}{m! \cdot 2^n \; n!}, \quad k - m = 2n, \tag{7}$$

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Corollary

The double generating function

$$\mathcal{P}(r,s) := \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} p_{k,m} \cdot s^m \frac{r^k}{k!}$$

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$$p_{k,m} = \frac{k!}{m! \cdot 2^n \; n!}, \qquad k - m = 2n, \tag{7}$$

Corollary

The double generating function

$$\mathcal{P}(r,s) := \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} p_{k,m} \cdot s^m \frac{r^k}{k!} = \exp\left(rs + \frac{r^2}{2}\right).$$

Calculus of Laplace polynomials

Lemma

$$P_k'(t) = kP_{k-1}(t).$$

$P_k(t)$ solves equation

$$y''(t) + ty(t) - ky(t) = 0.$$

Laplace and Hermite

Hermite polynomials

$$H_k(t) = (-1)^k \frac{1}{\varphi(t)} \frac{d^k \varphi(t)}{dt^k}, \quad k = 0, 1, \dots,$$

$$\mathcal{H}(r, t) := \sum_{k=0}^{\infty} H_k(t) \cdot \frac{r^k}{k!} = \exp\left(rt - \frac{r^2}{2}\right).$$

Denote by $h_{k,m}$ the coefficients of the Hermite polynomials: $H_k(t) = \sum_{m=0}^{k} h_{k,m} t^m$. Then

$$h_{k,m} = (-1)^{(k-m)/2} p_{k,m}, \quad m = 0, 1, \dots, k, \quad k \in \mathbb{Z}_+.$$

Differential representation

Laplace and Hermite cont.

Denote $\psi(t) = 1/\varphi(t) = \sqrt{2\pi}e^{t^2/2}$.

$$\psi(t)P_k(t) = \frac{d^k\psi(t)}{dt^k}, \quad k = 0, 1, \ldots,$$

Matrix $\mathfrak{q}$

k	m								
	0	1	2	3	4	5	6	7	8
0	1	0	0	...					0
1	0	1	0	...					0
2	2	0	1	0	...				0
3	0	5	0	1	0	...			0
4	8	0	9	0	1	0	...		0
5	0	33	0	14	0	1	0	...	0
6	48	0	87	0	20	0	1	0	0
7	0	279	0	185	0	27	0	1	0

Table: Coefficients $q_{k,m}$.

Coefficients of matrix $\mathfrak{q}$

Lemma

The coefficients $q_{k,m}$ satisfy the relations

$$q_{k,m} = 0 \quad \text{for } k - m \equiv 1 \pmod 2,$$

$$m! \cdot q_{k,m} = \sum_{j=0}^{n} (m+j)! \cdot p_{k-j,m+j}, \quad \text{for } k \equiv m \pmod 2. \tag{8}$$

Coefficients of matrix $\mathfrak{q}$

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The coefficients $q_{k,m}$ satisfy the relations

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$$m! \cdot q_{k,m} = \sum_{j=0}^{n} (m+j)! \cdot p_{k-j,m+j}, \quad \text{for } k \equiv m \pmod 2. \tag{8}$$

Proof

Let $\hat{q}_{k,m} = m! \cdot q_{k,m}$. Then from (6) we obtain

$$\hat{q}_{k,m} = m! \cdot p_{k,m} + \hat{q}_{k-1,m+1}. \tag{9}$$

Coefficients $q_{k,m}$

Theorem

Define $n := (k-m)/2$. Then

$$q_{k,m} = 0 \quad \text{for } k - m \equiv 1 \pmod 2,$$

$$m! \cdot q_{k,m} = (k-n)! \cdot 2^{-n} \sum_{i=0}^{n} \binom{k+1}{i}, \quad \text{for } k \equiv m \pmod 2.$$

Proof of theorem

Proof

$$S := m! \cdot q_{k,m} = (k-n)! \cdot \sum_{i=0}^{n} \binom{k-n+i}{i} 2^{-i};$$

Cauchy integral representation

$$\binom{k}{n} = \frac{1}{2\pi i} \oint_{\gamma} \frac{(1+z)^k}{z^{n+1}}\, dz,$$

$$S = 2 \cdot \frac{1}{2\pi i} \oint_{\gamma} \frac{(1+z)^{m+n}}{z-1} dz + 2^{-n} \cdot \frac{1}{2\pi i} \oint_{\gamma} \frac{(1+z)^{m+2n+1}}{z^{n+1}(1-z)} dz.$$

Double generating function of $\mathfrak{q}$

Definition of g.f.

$$\mathcal{Q}(r,s) := \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q_{k,m} s^m \frac{r^{k+1}}{(k+1)!}.$$

Double generating function of $\mathfrak{q}$

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$$\mathcal{Q}(r,s) := \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q_{k,m} s^m \frac{r^{k+1}}{(k+1)!}.$$

Master equation for $\mathcal{Q}$

$$\mathcal{Q}(r,s) = \sqrt{2\pi}\, e^{(r+s)^2/2} \cdot (\Phi(s+r) - \Phi(s)).$$

Derivation of $\mathcal{Q}$

$$\mathcal{Q}(s,t) = \sum_{k=0}^{\infty} Q_k(t) \frac{s^{k+1}}{(k+1)!} \quad \text{and} \quad \mathcal{P}(s,t) = \sum_{k=0}^{\infty} P_k(t) \frac{s^k}{k!} .$$

Taylor series for function $R(t)$

$$R(s+t) = \sqrt{2\pi} e^{(s+t)^2/2} \bar{\Phi}(s+t)$$

$$R(s+t) = \sum_{k=0}^{\infty} \frac{d^k R(t)}{dt^k} \frac{s^k}{k!} .$$

Derivation of $\mathcal{Q}$ (contd.)

$$\frac{d^k R(t)}{dt^k} = P_k(t)R(t) - Q_{k-1}(t).$$

$$\begin{aligned}
\sqrt{2\pi}\, e^{(s+t)^2/2} \cdot \bar{\Phi}(s+t) &= \sum_{k=0}^{\infty} \frac{d^k R(t)}{dt^k} \frac{s^k}{k!} \\
&= R(t) \sum_{k=0}^{\infty} P_k(t) \frac{s^k}{k!} - \sum_{k=0}^{\infty} Q_{k-1}(t) \frac{s^k}{k!} \\
&= R(t) \cdot e^{st+s^2/2} - \mathcal{Q}(s,t) \\
&= \sqrt{2\pi}\, e^{t^2/2} \cdot \bar{\Phi}(t) \cdot e^{st+s^2/2} - \mathcal{Q}(s,t).
\end{aligned}$$

Derivation of Master Equation

Equation

$$R(s+t) + \mathcal{Q}(s,t) = \mathcal{P}(s,t)R(t).$$

New identity (?? 2014)

$\mathcal{Q}(1,1)$

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q_{k,m}\frac{1}{(k+1)!} = \mathcal{Q}(1,1).$$

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$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q_{k,m}\frac{1}{(k+1)!} = \mathcal{Q}(1,1).$$

Triple Binomial Sum

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{n} \frac{(n+m)!}{m!j!(m+2n+1-j)!}\, 2^{-n} = \sqrt{2\pi}e^{2}\left(\Phi(2)-\Phi(1)\right).$$

Riordan (1957)

$\mathcal{Q}(r, 0)$

$$\mathcal{Q}(r, 0) = e^{r^2/2} \cdot \left(1 - 2\bar{\Phi}(r)\right).$$

Riordan (1957)

$\mathcal{Q}(r, 0)$

$$\mathcal{Q}(r,0) = e^{r^2/2} \cdot \left(1 - 2\bar{\Phi}(r)\right) .$$

Famous identity

$$\sum_{k=0}^{n} (-1)^k \frac{1}{2k+1} \binom{n}{k} = \frac{2^{2n} (n!)^2}{(2n+1)!} .$$

Ramanujan

Identity

$$\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{\ddots}}}}} = \frac{1}{\sqrt{\frac{e\pi}{2}} - \sum_{n=0}^{\infty} \frac{1}{(2n+1)!!}} - 1.$$

Ramanujan (contd)

Definition

$$\mathcal{S}(s) := \cfrac{1}{s + \cfrac{2}{s + \cfrac{3}{\ddots}}}.$$

Ramanujan (contd)

Definition

$$\mathcal{S}(s) := \cfrac{1}{s + \cfrac{2}{s + \cfrac{3}{\ddots}}}.$$

Proposition (Functional form)

$$\mathcal{S}(s) = \frac{1}{e^{s^2/2}\sqrt{\frac{\pi}{2}} - \sum_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+1)!!}} - s.$$

Ramanujan (contd)

$$R(s) + \mathcal{Q}(s,0) = e^{s^2/2} \cdot R(0)$$

$$\mathcal{Q}(s,0) = \sum_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+1)!!}.$$

$$\mathcal{S}(s) = -s + \frac{1}{R(s)}.$$

$$\frac{1}{s+\mathcal{S}(s)} = e^{s^2/2}\sqrt{\frac{\pi}{2}} - \sum_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+1)!!}.$$

Ramanujan (contd)

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Ram. Identity:

Substitute $s = 1$.

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