Linear finite difference operators with constant coefficients and distribution of zeros of polynomials.

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Problem

Let $\Omega \subseteq \mathbb{C}$*.*
Denote by $\pi(\Omega)$ *the class of all univariate polynomials (complex or real) all whose zeros lie in* Ω*.*
Problem. *To describe linear transformations*
$T : \pi(\Omega) \to \pi(\Omega) \cup \{0\}$.

Hyperbolic Polynomials

Definition

Definition 1. $P \in \mathbb{R}[x]$ is *hyperbolic* if all its roots are real or if $P \equiv 0$.

The class of all hyperbolic polynomials is denoted by $\mathcal{HP}$.

Hyperbolicity Preserving

1. 1870's –*C. Hermite* initiated systematic study of linear operators preserving the class of real polynomials with only real zeros. *E. Laguerre* continued this investigation.
2. 1914 – *G. Pólya and J. Schur* completely described the operators acting diagonally on the standard monomial basis $1, x, x^2, \ldots$ of $\mathbb{R}[x]$ and preserving $\mathcal{HP}$.
3. The study was continued by *N. Obreschkov, S. Karlin, B. Levin, G. Csordas, T. Craven, R. Varga, S. Fisk, E. Saff* etc.
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The Laguerre-Pólya class

Definition

Definition 2. A real entire function f is said to be in the *Laguerre-Pólya class*, written $f \in \mathcal{L} - \mathcal{P}$, if it can be expressed in the form

$$f(z) = cz^n e^{-az^2+bz} \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k}\right) e^{\frac{z}{x_k}}, \qquad (1)$$

where $c, b, x_k \in \mathbb{R}$, $x_k \neq 0$, $a \geq 0$, n is a non-negative integer and $\sum_{k=1}^{\infty} \frac{1}{x_k^2} < \infty$. The product on the right-hand side can be finite or empty (in the latter case the product equals 1).

1. *These and only these functions are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros.*

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Linear finite difference operator with constant coefficients

Definition

Definition 3. $T : \mathbb{C}[x] \to \mathbb{C}[x]$ *is a linear finite difference operator of the form*

$$T(P)(x) = \sum_{j=l}^{m} a_j P(x - j\lambda), \tag{2}$$

where $l, m \in \mathbb{Z}, l < m$, $a_j \in \mathbb{C}, l \leq j \leq m, a_l \neq 0, a_m \neq 0$, $\lambda \in \mathbb{C} \setminus \{0\}$.

1. **Main Problem.** To describe all finite difference operators with constant coefficients preserving $\mathcal{HP}$.

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Theorem

Theorem *(Brändén, Krasikov, Shapiro). A linear finite difference operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$ of the form $T(P)(x) = q_0(x)P(x) + q_1(x)P(x-1) + \ldots + q_k(x)P(x-k)$ is hyperbolicity preserving if and only if $q_i(x) \neq 0$ for at most one i, and $q_i(x)$ is hyperbolic for such an i.*

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Non-real λ

Theorem

Theorem *(G. Pólya, 1926).*
$f \in \mathcal{L}-\mathcal{P} \;\Rightarrow\; \forall h \in \mathbb{R}:\; f(x+ih)+f(x-ih) \in \mathcal{L}-\mathcal{P}$.
Theorem *(N.G. de Bruijn, 1950).*
$f \in \mathcal{L}-\mathcal{P} \;\Rightarrow\; \forall h, \alpha \in \mathbb{R}:\; e^{i\alpha} f(x+ih)+e^{-i\alpha} f(x-ih) \in \mathcal{L}-\mathcal{P}$.

Shift operator

Definition

Definition 4.
Let $\lambda \in \mathbb{C}$. The shift operator is
$S_\lambda(P)(x) := P(x - \lambda) : \mathbb{C}[x] \to \mathbb{C}[x]$.

1. $S_\lambda^j(P)(x) = P(x - j\lambda)$
2. $T(P)(x) = \sum_{j=l}^{m} a_j P(x - j\lambda) = \sum_{j=l}^{m} a_j S_\lambda^j(P)(x)$.
3. *Generating Function* of T: $Q(t) = \sum_{j=l}^{m} a_j t^j$

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Description of linear finite difference operators preserving $\mathcal{HP}$

Theorem

Theorem 1 *(-, Tyaglov, Vishnyakova). A linear operator* $T(P)(x) = \sum_{j=l}^{m} a_j P(x - j\lambda)$ *preserves the set of hyperbolic polynomials if and only if the following conditions are satisfied:*

1. $Re\,\lambda = 0$;

2. $l = -m$;

3. All the roots of the generating rational function $Q(t) := \sum_{j=l}^{m} a_j t^j$ *belong to the unit circle* $\{z : |z| = 1\}$*;*

4. $a_m \cdot a_{-m} > 0$.

Example

1. $T(P)(x) = e^{i\alpha} P(x + ih) + e^{-i\alpha} P(x - ih)$
2. De Bruijn's Theorem : $T(P) \in \mathcal{HP}$ whenever $P \in \mathcal{HP}$
3. $T(P)(x) = e^{i\alpha} S_{ih}(P) + e^{-i\alpha} S_{-ih}(P)$
4. $Q(t) = e^{i\alpha} t + e^{-i\alpha} t^{-1} = e^{i\alpha} t^{-1}(t^2 + e^{-2i\alpha})$

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Remark

1. Theorem 1 states that $Q(t)$ is of the form $Q(t) = a_{-m}t^{-m} + a_{-m+1}t^{-m+1} + \ldots + a_m t^m$
2. Since all the roots of $Q(t)$ belong to the unit circle $Q(t) = a_m t^{-m} \prod_{k=1}^{2m}(t + e^{i\theta_k})$
3. Applying Vieta's Theorem we can represent $Q(t)$ in the form $Q(t) = (a_m \cdot a_{-m})^{1/2} \prod_{k=1}^{2m} \left(e^{-i\theta_k/2}\sqrt{t} + e^{i\theta_k} \frac{1}{\sqrt{t}} \right)$
4. Thus $T = C \cdot \prod_{k=1}^{2m} \left(e^{-i\theta_k/2} S_{ih/2} + e^{i\theta_k/2} S_{-ih/2} \right)$, where $C \in \mathbb{R}$

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Equivalent form of Theorem 1

1. Theorem 1 is equivalent to the statement

Theorem

Theorem 1'. $T(\mathcal{HP}) \subseteq \mathcal{HP} \Leftrightarrow T = C \cdot \prod_{k=1}^{2m} \left(e^{i\alpha} S_{ih} + e^{-i\alpha} S_{-ih}\right)$, *where* $\alpha, h, C \in \mathbb{R}$

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2. The sufficiency in Theorem 1 follows from de Bruijn's Theorem.
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Re $\lambda = 0$ and all the non-zero roots of the generating function $Q(t)$ belong to the unit circle

1. $\forall n \in \mathbb{N} : L_n(x) = x^n \in \mathcal{HP} \Rightarrow$
2. $T(L_n)(x) = \sum_{j=l}^{m} a_j (x - j\lambda)^n = x^n \sum_{j=l}^{m} a_j (1 - \frac{j\lambda}{x})^n \in \mathcal{HP}$.
3. Thus all the zeros of the rational function $\sum_{j=l}^{m} a_j (1 - \frac{j\lambda}{x})^n$ are real.
4. Put $x = \frac{n}{y}$, $y \in \mathbb{R} \setminus \{0\}$.
5. For every $n \in \mathbb{N}$ all the zeros of $f_n(y) = \sum_{j=l}^{m} a_j (1 - \frac{j\lambda y}{n})^n$ are real.
6. The sequence $f_n(y)$ converges uniformly on the compact sets to the entire function $f(y) := \sum_{j=l}^{m} a_j e^{-j\lambda y}$ as $n \to \infty$.
7. We conclude that all the zeros of the entire function $f(y) = Q(e^{-\lambda y}) = \sum_{j=l}^{m} a_j e^{-j\lambda y}$ are real.
8. This is possible only if $\lambda = i\beta, \beta \in \mathbb{R}$, and thus for a root z_0 of Q we have $|z_0| = |e^{-i\beta y_0}| = 1$.

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6. The sequence $f_n(y)$ converges uniformly on the compact sets to the entire function $f(y) := \sum_{j=l}^{m} a_j e^{-j\lambda y}$ as $n \to \infty$.
7. We conclude that all the zeros of the entire function $f(y) = Q(e^{-\lambda y}) = \sum_{j=l}^{m} a_j e^{-j\lambda y}$ are real.
8. This is possible only if $\lambda = i\beta, \beta \in \mathbb{R}$, and thus for a root z_0 of Q we have $|z_0| = |e^{-i\beta y_0}| = 1$.

Re $\lambda = 0$ and all the non-zero roots of the generating function $Q(t)$ belong to the unit circle

1. $\forall n \in \mathbb{N} : L_n(x) = x^n \in \mathcal{HP} \Rightarrow$
2. $T(L_n)(x) = \sum_{j=l}^{m} a_j (x - j\lambda)^n = x^n \sum_{j=l}^{m} a_j (1 - \frac{j\lambda}{x})^n \in \mathcal{HP}$.
3. Thus all the zeros of the rational function $\sum_{j=l}^{m} a_j (1 - \frac{j\lambda}{x})^n$ are real.
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Lemma on polynomials having all their roots on a horizontal straight line

1. $Q(t) := \sum_{j=l}^{m} a_j t^j = a_m t^l \prod_{k=1}^{m-l}(t - e^{i\theta_k})$
2. $\Rightarrow T = a_m S_{i\beta}^l \prod_{k=1}^{m-l}(S_{i\beta} - e^{i\theta_k} I)$.

Theorem

Lemma *(KTV). Let $T = S_{i\beta} - e^{i\theta} \cdot I$, where $\beta, \theta \in \mathbb{R}$.*
Suppose $P \in \mathbb{C}[z]$, and all the zeros of P lie on a straight line $\{z : \ Im\, z = c\}$.
Then all the zeros of the polynomial $T(P)$ lie on the straight line $\{z : \ Im\, z = c + \beta/2\}$.

3. It follows from Lemma that for any $P \in \mathcal{HP}$ all the zeros of $T(P)$ belong to the straight line $\{z : \ Im\, z = (m+l) \cdot \beta/2\}$. So, $T(P) \in \mathcal{HP} \Leftrightarrow m + l = 0$.

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Strip preserving

Theorem

Theorem 2 *(KTV). Let* $b > 0$ *be a given number.*
A linear operator
$T(P)(x) = \sum_{j=l}^{m} a_j P(x - j\lambda)$
preserves the set of complex polynomials having all their zeros in the strip $\{z : \ |Im\ z| \leq b\}$
if and only if the following conditions are satisfied:
1. $Re\ \lambda = 0$;
2. $l = -m$;
3. All the roots of the generating rational function $Q(t) := \sum_{j=l}^{m} a_j t^j$ *belong to the unit circle* $\{z : |z| = 1\}$.

Linear operator

$$T_{\theta,h}(P)(x) = \frac{e^{i\theta}P(x+ih) - e^{-i\theta}P(x-ih)}{i}, \quad h > 0, \theta \in \mathbb{R}$$

The fact that every linear finite difference operator with constant coefficients preserving $\mathcal{HP}$ is a composition of linear operators of the form $e^{i\alpha}f(x+ih) + e^{-i\alpha}f(x-ih), \quad h, \ \alpha \in \mathbb{R}$, motivates us to study such kind of operators more in detail.
Further it will be more convenient for us to put $\alpha = \theta - \pi/2$ and to study the operator

$$T_{\theta,h}(P)(x) = \frac{e^{i\theta}P(x+ih) - e^{-i\theta}P(x-ih)}{i}, \quad h > 0, \theta \in \mathbb{R}. \quad (3)$$

Simplicity of roots of $T_{\theta,h}(f)$ for $f \in \mathcal{L} - \mathcal{P}$

1. It is easy to prove that for any hyperbolic polynomial P all the roots of $T_{\theta,h}(P)$ are simple.

Theorem

Theorem 3 *(KTV).* $\forall h > 0,\ \theta \in \mathbb{R}$ *and* $\forall f \in \mathcal{L} - \mathcal{P}$ *all the zeros of* $T_{\theta,h}(f)$ *are simple.*
Corollary. *Let T be a linear finite difference operator with constant coefficients preserving hyperbolicity. Then* $\forall f \in \mathcal{L} - \mathcal{P}$ *all the zeros of* $T(f)$ *are simple.*

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Example

For $n \in \mathbb{N}$ we consider $L_n(x) = x^n \in \mathcal{HP}$.

$$Q_n(x,\theta) := T_{\theta,1}(x^n) = \frac{e^{i\theta}(x+i)^n - e^{-i\theta}(x-i)^n}{i}$$

$$= 2\sin\theta \prod_{k=1}^{n}\left(x - cot\frac{-\theta+\pi k}{n}\right), \text{if } sin\,\theta \neq 0,$$

and

$$Q_n(x,2\pi k) = -Q_n(x,\pi+2\pi k) = Q_n(x,0) =$$

$$\frac{(x+i)^n - (x-i)^n}{i} = 2n\prod_{k=1}^{n-1}\left(x - cot\frac{\pi k}{n}\right), \text{ if } sin\,\theta = 0.$$

Example

Denote by

$$x_k = x_k(\theta) = cot\frac{-\theta + \pi k}{n}, \quad k = 1, 2, \cdots, N,$$

the zeros of the polynomial $Q_n(x, \theta)$,

where $N = n$ if $sin\,\theta \neq 0$ and $N = n - 1$ if $sin\,\theta = 0$.

So, the minimal distance between different zeros of Q_n approaches zero as $n \to \infty$.

General case

$$P_n(x) = x^n + ax^{n-1} + bx^{n-2} + \sum_{k=0}^{n-3} c_k x^k \in \mathbb{C}[z].$$

For $\theta \in \mathbb{R}$, $h > 0$ consider the polynomial

$$T_{\theta,h}(P_n)(x) = \frac{e^{i\theta} P_n(x+ih) - e^{-i\theta} P_n(x-ih)}{i}.$$

Denote by

$$X_1(h,\theta), X_2(h,\theta), \dots, X_N(h,\theta)$$

the roots of this polynomial numerated under the condition:

$$\text{Re } X_1(h,\theta) \le \text{Re } X_2(h,\theta) \le \dots \le \text{Re } X_N(h,\theta),$$

where $N = n$ if $sin\,\theta \neq 0$ and $N = n-1$ if $sin\,\theta = 0$.

Asymptotic behavior of $X_j(h, \theta)$ as $h \to \infty$

Theorem

Theorem 5 *(KTV). For every $\theta \in \mathbb{R}$, $h > 0$, and $j = 1, 2, \cdots, N$*

$$X_j(h, \theta) =$$

$$x_j \cdot h - \frac{a}{n} + \left(\frac{a^2(n-1)}{2n^2} - \frac{b}{n} \right) \frac{Q_{n-2}(x_j, \theta)}{Q_{n-1}(x_j, \theta)} \cdot \frac{1}{h} + O\left(\frac{1}{h^2} \right), \quad h \to \infty,$$

where polynomials Q_{n-1}, Q_{n-2} and numbers x_j are taken from the above Example.

Central finite difference operator with non-constant coefficients

Definition

Definition. Let $M_1, M_2 : \mathbb{C} \to \mathbb{C}$, $M_1 \not\equiv 0$, $M_2 \not\equiv 0$, and h be a complex number, $h \neq 0$.

$$T_{M_1,M_2,h}(f)(z) = M_1(z)f(z+h) + M_2(z)f(z-h). \tag{4}$$

Problem. To describe all such operators preserving the Laguerre-Pólya class.

Necessary conditions on M_1 , M_2 and h for the operator $T_{M_1,M_2,h}$ to preserve $\mathcal{L}-\mathcal{P}$

Theorem

Theorem 6 *(KTV). Let* $M_1, M_2 : \mathbb{C} \to \mathbb{C}$, $M_1 \not\equiv 0, M_2 \not\equiv 0$, *and h be a complex number,* $h \neq 0$.
Assume $T_{M_1,M_2,h}(\mathcal{L}-\mathcal{P}) \subseteq \mathcal{L}-\mathcal{P}$.
Then
1. M_1 *and* M_2 *are entire functions.*
2. $ih \in \mathbb{R}$

Further we will consider the operator

$$T_{M_1,M_2}(f)(z) = M_1(z)f(z+i) + M_2(z)f(z-i). \tag{5}$$

Necessary and sufficient conditions on M_1 and M_2 for the operator T_{M_1,M_2} to preserve the property of real-rootedness

Theorem

Theorem 7 *(KTV). Let M_1 and M_2 be polynomials not identically zero. Then for every polynomial $p \in \mathbb{R}[x]$ with only real zeros, the polynomial*

$$M_1(z)p(z+i) + M_2(z)p(z-i)$$

has only real zeros if, and only if, either $\left|\frac{M_1(z)}{M_2(z)}\right| \equiv 1$, or $M_1(z) = e^{i\theta} \cdot \overline{M_2(\bar{z})}$, $\theta \in [0, 2\pi)$, and all the zeros of the polynomial M_2 lie in the half-plane $Im\, z \geq 0$.

Necessary and sufficient conditions on M_1 and M_2 for the operator T_{M_1,M_2} to preserve $\mathcal{L}-\mathcal{P}$

Theorem

Theorem 8 *(KTV). Let M_1 and M_2 be entire functions not identically zero. Then for every function $f \in \mathcal{L}-\mathcal{P}$,*

$$M_1(z)f(z+i) + M_2(z)f(z-i) \in \mathcal{L}-\mathcal{P}$$

if, and only if, M_1 and M_2 satisfy the conditions:

Necessary and sufficient conditions on M_1 and M_2 for the operator T_{M_1,M_2} to preserve $\mathcal{L}-\mathcal{P}$

1)$M_1(z) = \overline{M_2(\bar{z})}$;

2) $\left|\frac{M_2(z)}{M_1(z)}\right| < 1$ for every z with Im $z > 0$, or $\frac{M_2(z)}{M_1(z)}$ is a constant function with $\left|\frac{M_2(z)}{M_1(z)}\right| \equiv 1$

3) The function M_2 is of the form

$$M_2(z) = Cz^n e^{-az^2+bz} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) e^{\frac{z}{\alpha_k}}, \qquad (6)$$

where $C, b \in \mathbb{C}$, Im $b \geqslant 0$, $n \in \mathbb{N} \cup \{0\}$, $a \geqslant 0$, $\alpha_k \neq 0$, Im $\alpha_k \geqslant 0$, and $\sum_{k=1}^{\infty} \frac{1}{|\alpha_k|^2} < \infty$.

Relation to the class $\overline{\mathcal{HB}}$

It follows from the above theorem that the entire function M_2 belongs to the class $\overline{\mathcal{HB}}$.

Definition

Definition. An entire function $M \in \overline{\mathcal{HB}}$ if

1. M doesn't have roots in the open half-plane $\{x : \ \text{Im } z < 0\}$;
2. $\left|\dfrac{M(z)}{\overline{M(\bar{z})}}\right| \leq 1$ for every z with $\text{Im } z > 0$.

The class $\overline{\mathcal{HB}}$ plays an important role in transcendental analogue of Hermite-Biehler Theory.
It was introduced by M. Krein in 1930's in his work devoted to transcendental analogue of Hurwitz criterion. M. Krein obtained a full description of this class.

Description of operators T_{M_1,M_2} preserving the Laguerre-Pólya class

Theorem

Theorem 8 *(KTV). Let M_1 and M_2 be non-zero entire functions. Then operator $T_{M_1,M_2}(f)(z) = M_1(z)f(z+i) + M_2(z)f(z-i)$ preserves the class $\mathcal{L}-\mathcal{P}$ if, and only if, the functions M_1 and M_2 satisfy the conditions:*

1. $M_1(z) = \overline{M_2(\bar{z})},\ z \in \mathbb{C}$.

2. The function M_2 is of the form

$$M_2(z) = Cz^n e^{-az^2+bz} \prod_{k=1}^{\infty}\left(1 - \frac{z}{\alpha_k}\right) e^{\frac{z}{\alpha_k}} \prod_{k=1}^{\infty}\left(1 - \frac{z}{x_k}\right) e^{\frac{z}{x_k}},$$

where Im $\alpha_k > 0$, $x_k \in \mathbb{R}\setminus\{0\}$, $\sum_{k=1}^{\infty}\left(|\alpha_k|^{-2} + x_k^{-2}\right) < \infty$, $a \geqslant 0$, $C,\ b \in \mathbb{C}$, $n \in \mathbb{N}\cup\{0\}$, and

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$$\operatorname{Im} b \geqslant 0 \quad \text{whenever} \quad \sum_{k=1}^{\infty} \frac{1}{|\alpha_k|} < \infty,$$

or

$$\left(\operatorname{Im} b - \sum_{k=1}^{\infty} \frac{\operatorname{Im} \alpha_k}{|\alpha_k|^2}\right) \geqslant 0 \quad \text{whenever} \quad \sum_{k=1}^{\infty} \frac{1}{|\alpha_k|} = \infty.$$

Thanks for attention!

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