Zeros of orthogonal polynomials related to hyponormal operators

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Overview

1. Gram matrices and their generating functions
2. Hyponormal operators
3. Asymptotics of orthogonal polynomials
4. Numerical experiments on asymptotics of zeros
5. Direct constructions of mother bodies
6. Line bundles, or Riemann-Hilbert

Abstract

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Abstract: In recent joint work with Mihai Putinar and Nikos Stylianopoulos we have investigated zeros of orthogonal polynomials arising in the theory of hyponormal operators. They can equivalently be described as spectra of truncations of such operators. We have several numerical results which show that the zeros in general go deeper into the domain (the interior of the spectrum of the operator in question) compared to what is the case for the Bergman polynomials. However, the theoretical understanding is still poor, only in a few cases we can prove what we see in the pictures.
I will try to explain a little of what we know. See also our recent monograph “Hyponormal quantization of planar domain”, Springer Lecture Notes 2199.

Gram matrices

To make orthogonal: $1, z, z^2, z^3, \ldots$ when

$$\langle z^k, z^\ell\rangle = \text{a given Gram matrix.}$$

Bergman case:

$$\text{Gram matrix} = M_{kj} = \langle z^k, z^j\rangle_{L^2(\Omega)} = \frac{1}{\pi}\int_\Omega z^k \bar{z}^j \, d\text{Area}.$$

"**Exponential**" or "**hyponormal**" case:

$$\text{Gram matrix} = B_{kj} = \langle z^k, z^j\rangle_{\mathcal{H}(\Omega)} = \langle A^k\xi, A^j\xi\rangle_H,$$

defined via

$$\sum_{k,j\geq 0} \frac{B_{kj}}{z^{k+1}\,\bar{w}^{j+1}} = 1 - \exp[-\sum_{k,j\geq 0} \frac{M_{kj}}{z^{k+1}\,\bar{w}^{j+1}}].$$

Degeneracy possible

Since (symbolically)

$$B = 1 - \exp[-M] = M - \frac{M^2}{2!} + \frac{M^3}{3!} - \dots$$

the B-Gram matrix is "weaker" than the M-Gram matrix, and it is only positive **semi**-definite in general.

Example (unit disk)

In this case M_{kj} and B_{kj} are diagonal, with

$$M_{kk} = \frac{1}{k+1} \quad (k = 0, 1, 2, \dots), \quad B_{00} = 1, \ B_{11} = B_{22} = \dots = 0.$$

Thus B_{kj} is degenerate (is only semi-definite), and the corresponding Hilbert space collapses (becomes finite dimensional).

B_{kj} is degenerate if and only if Ω is a *quadrature domain*.
($\Leftrightarrow$ the Cauchy transform of Ω equals a rational function, outside Ω.)

Hyponormal operators, and quantization

- **Recall:** $B_{kj} = \langle A^k\xi, A^j\xi\rangle$, where
- $A : H \to H$, $\xi \in H$ (a complex Hilbert space).
- **Assumptions:** $\mathrm{span}\{\xi, A\xi, A^2\xi, \dots\} = H$, and
- $[A, A^*] = \xi \otimes \xi$.

In particular, A is **cohyponormal**: $[A, A^*] \geq 0$.

Remark

Compare annihilation A and creation A^* operators in quantum mechanics, satisfying (with $\hbar = 1$ and in Dirac's notation),

$$[A, A^*] = \textit{identity} = \sum_n |\psi_n\rangle \otimes \langle\psi_n|,$$

where $|\psi_n\rangle$ is an orthonormal basis. Our case (only one term):

$$[A, A^*] = |\xi\rangle \otimes \langle\xi| = \textit{projection onto a 1D subspace},$$

with the advantage that A is ***bounded****.*

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Relation to subnormal

T is **hyponormal** if $[T^*, T] \geq 0$. In general,

$$\text{hyponomal} \supset \text{subnormal} \supset \text{normal}.$$

Example (relevant when Ω is an ellipse)

$$\mathrm{T} = \begin{pmatrix} 0 & r & 0 & 0 & \dots \\ 1 & 0 & r & 0 & \dots \\ 0 & 1 & 0 & r & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

is subnormal if $r = 0$, hyponormal if $0 < r < 1$, normal if $r = 1$, and cohyponormal if $r > 1$.

With S =unilateral shift $e_n \mapsto e_{n+1}$ we have $T = S + rS^*$. Conformal map from exterior of unit disk to exterior of ellipse: $\phi(\zeta) = \zeta + r\zeta^{-1}$.

Functional model (operator theory $\mapsto$ function theory)

- Hilbert space: $H = \mathcal{H}(\Omega)$, generated by analytic functions in Ω.
- Fixed vector: $\xi = \mathbf{1} \in \mathcal{H}(\Omega)$ (constant function one).
- Operator A: multiplication by independent variable z.
- Inner product: $\langle \cdot, \cdot \rangle$ such that $\langle z^k, z^j \rangle = B_{kj}$, namely:

$$\langle f, g \rangle = +\frac{1}{4\pi^2} \int_\Omega \int_\Omega H(z,w) f(z) \overline{g(w)} dA(z) dA(w)$$
$$= -\frac{1}{4\pi^2} \int_{\partial\Omega} \int_{\partial\Omega} E(z,w) f(z) \overline{g(w)} dz d\bar{w}.$$

$$E(z,w) = \exp\Big[\frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{d\zeta}{\zeta - z} \wedge \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}}\Big],$$ **exponential transform of** Ω,

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Unit disk and ellipse

Example (unit disk)

For $\Omega = \mathbb{D}$ we have

$$E(z,w) = 1 - \frac{1}{z\bar{w}} \quad (z, w \in \mathbb{C} \setminus \Omega),$$

$$H(z,w) = \frac{1}{1 - z\bar{w}} \quad (z, w \in \Omega).$$

Example (ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$)

$$H(z,w) = \frac{\text{const.}}{4a^2b^2 + (a^2 - b^2)(z^2 + \bar{w}^2) - 2(a^2 + b^2)z\bar{w}}$$

$$= \frac{1}{\text{defining function (in polarized form) of } \partial\Omega}.$$

Hypocycloids

Hypocycloid = curve traced out by point on small circle which rolls inside bigger circle. Equation in parametrized form:

$$z(t) = ae^{it} + be^{-\mathrm{i}(d-1)t},$$

$$d = \text{integer} \geq 2, \quad a \geq (d-1)b.$$

Exterior domain is an **unbounded quadrature domain**. Conformal map from exterior disk to exterior domain obtained by $\zeta := e^{\mathrm{i}t}$:

$$\psi(\zeta) = a\zeta + b\zeta^{1-d}.$$

Interior exponential transform becomes again

$$H(z,w) = \frac{1}{\text{defining function of } \partial\Omega} = \frac{1}{\text{polynomial of degree } 2(d-1)}.$$

Ellipse is the case $d = 2$.

Cauchy transforms and quadrature domains

The **Cauchy transform** for bounded domain Ω is

$$C_\Omega(z) := \frac{1}{2\pi i}\int_\Omega \frac{d\zeta \wedge d\bar\zeta}{\zeta - z} = \{\text{when } z \in \mathbb{C}\setminus\overline{\Omega}\} = \frac{-1}{2\pi i}\int_{\partial\Omega}\frac{\bar\zeta d\zeta}{\zeta - z}.$$

If $\partial\Omega$ is analytic, we define analytic functions $S_\pm(z)$ and $S(z)$ by

$$\frac{-1}{2\pi i}\int_{\partial\Omega}\frac{\bar\zeta d\zeta}{\zeta - z} = \begin{cases} - \quad S_+(z), & z \in \Omega, \\ + \quad S_-(z), & z \in \mathbb{C}\setminus\overline{\Omega}, \end{cases}$$

$$S(z) = S_+(z) + S_-(z) = \text{the } \textbf{Schwarz function} \text{ for } \partial\Omega,$$

and it satisfies $S(z) = \bar z$ for $z \in \partial\Omega$.

Definition

- Ω is a **quadrature domain** if $S_-(z)$ is a rational function.
- $\mathbb{C}\setminus\overline{\Omega}$ is (unbounded) **quadrature domain** if $S_+(z)$ is rational.

Complements of unbounded quadrature domains

- If Ω is a (bounded) **quadrature domain** then $\mathcal{H}(\Omega)$ **collapses** to become finite dimensional and there are not enough many orthogonal polynomials. The exponential transform is

$$E(z,w) = \frac{P(z)\overline{P(w)}}{Q(z,\bar{w})}.$$

- **Complements of unbounded quadrature domains** is an **interesting** class in our context. It includes all hypocycloids, for example, and the interior exponential transform is of the form

$$H(z,w) = \frac{p(z)\overline{p(z)}}{q(z,\bar{w})},$$

where $p(z)$ is a polynomial with all zeros in the exterior domain and $q(z,\bar{w})$ is the polynomial defining the the boundary:

$$\partial\Omega: \quad q(z,\bar{z}) = 0.$$

Simply connected case

Simply connected quadrature domain can most conveniently be described in terms of conformal maps from the unit disk:

Theorem (classical: Aharonov, Shapiro, Davis, Sakai, etc.)

A domain is a QD (bounded or unbounded) if and only this conformal map is a rational function.

Comparison with Bergman and Hardy inner products

In terms of the **Double Cauchy transform**

$$C_\Omega(z,w) = \frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{d\zeta}{\zeta - z} \wedge \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}}$$

we have

$$\langle f, g\rangle_{\mathcal{H}(\Omega)} = -\frac{1}{4\pi^2} \int_{\partial\Omega}\int_{\partial\Omega} e^{C(z,w)} f(z)\overline{g(w)}dz d\bar{w},$$

$$(f, g)_{AL^2(\Omega)} = -\frac{1}{4\pi^2} \int_{\partial\Omega}\int_{\partial\Omega} C(z,w) f(z)\overline{g(w)}dz d\bar{w},$$

$$(f, g)_{H^2(\Omega)} \approx -\int_{\partial\Omega}\int_{\partial\Omega} e^{-C(z,w)} f(z)\overline{g(w)}dz d\bar{w}.$$

Feynman integrals (comparison, just for fun)

The inner product in $\mathcal{H}(\Omega)$ has some similarities with **Feynman integrals** in quantum theory. Such an integral may look like

$$\langle \psi_F | \psi_I \rangle = \int D[\Phi] e^{iS[\Psi]},$$

$$S[\Phi] = \int d^4 x \mathcal{L}[\Phi] = \textbf{action},$$

$\mathcal{L}[\Phi]$ being a **Lagrangian density**. The inner product is to be interpreted as a **transition amplitude**, namely the probability (after taking the modulus squared) that an initial state $|\psi_I\rangle$ will after a later measurement show up as a final state $|\psi_F\rangle$. Thus compare our

$$\langle f, g \rangle = \text{const.} \int_{\partial\Omega \times \partial\Omega} e^{C_\Omega(z,w)} f(z) \overline{g(w)} dz d\bar{w},$$

$$C_\Omega(z, w) = \frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{d\zeta}{\zeta - z} \wedge \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} \approx \text{ energy}.$$

Asymptotics of orthogonal polynomials

Gram-Schmidt applied to $\langle z^k, z^j\rangle_{\mathcal{H}(\Omega)}$ gives **exponential orthogonal polynomials**

$$P_n(z) = z^n + \text{lower degree terms},$$

with zeros $z_1^{(n)}, \ldots, z_1^{(n)}$, and corresponding **counting measures**

$$\nu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{z_k^{(n)}}.$$

To identify: all weak star limits ν,

$$\nu_n \rightharpoonup \nu \quad n \to \infty.$$

For many similar polynomials (Bergman, Szegö etc) such limits ν are related to the **equilibrium measure** for the domain, or some kind of potential theoretic skeleton for that. What about the exponential polynomials? Are the limit measures uniquely associated to Ω and do they sit on some kind of “skeleton” for Ω.

Various kinds of skeletons

Somewhat vaguely, a **skeleton** associated to a given mass distribution ρ is a measure μ satisfying at least the following:

- $\operatorname{supp}\mu$ has **measure zero** and **does not separate** the plane.
- the **potentials**, or Cauchy transforms, of μ and ν **agree** near infinity.

Additional requirements may be appropriate, depending on context.

Example

- "Classical" mother body:

$$d\rho = \chi_\Omega \,\mathrm{dArea} \quad (\Omega \subset \mathbb{C} \text{ simply connected})$$

- "Madonna body", or electrostatic skeleton:

$$d\rho = \frac{\partial G}{\partial n}\, ds = \text{ equilibrium measure of given compact set } K$$

($G =$ Green's function for $\overline{\mathbb{C}} \setminus K$ with pole at infinity.)

Spectra of truncated matrices

In abstract setting the Gram matrix is

$$B_{kj} = \langle A^k \xi, A^j \xi \rangle,$$

where $[A, A^*] = \xi \otimes \xi$ with $\xi \in H$ a cyclic vector for A.
Let A_n be truncation of A to subspace $[\xi, A\xi, \dots A^{n-1}\xi] \subset H$.

Theorem (standard)

$P_n(z)$ is the characteristic polynomial of A_n, i.e.,

$$P_n(z) = \det(z - A_n).$$

Example (ellipse again)

Truncation of

$$\mathrm{T} = \begin{pmatrix} 0 & r & 0 & 0 & \dots \\ 1 & 0 & r & 0 & \dots \\ 0 & 1 & 0 & r & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

gives (with $n = 4$)

$$\mathrm{T_n} = \begin{pmatrix} 0 & r & 0 & 0 \\ 1 & 0 & r & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$P_n(z) = \det(z - T_n) = U_n\left(\frac{z}{2\sqrt{r}}\right) = \text{ Chebyshev polynomial.}$$

A few theorems

Theorem (convex hull)

For all $n = 1, 2, 3, \ldots,$

$$\operatorname{supp} \nu_n \subset \operatorname{conv}(\overline{\Omega}).$$

Theorem (disjoint union of domain and quadrature domain)

If Ω is disconnected and one component, D, is a quadrature domain, then

$$D \cap \operatorname{supp} \nu = \emptyset.$$

Theorem (ellipse)

For the ellipse the zeros go to the "Madonna body" (electrostatic skeleton).

The proof (in ellipse case) is by inspection of characteristic polynomials of truncated matrices.

Some general methods for construction of mother bodies

a) By direct construction (using e.g. Schwarz functions/potentials)
b) By solving a suction version of a Hele-Shaw problem (Laplacian growth in the ill-posed direction)
c) By a version of internal DLA
d) Using a “restoring sandpile” algorithm
e) As asymptotic distribution of zeros of orthogonal polynomials
f) As asymptotic distribution of eigenvalues of matrices (for example random matrices, or matrices obtained as truncations of infinite shift operators)

Ellipse

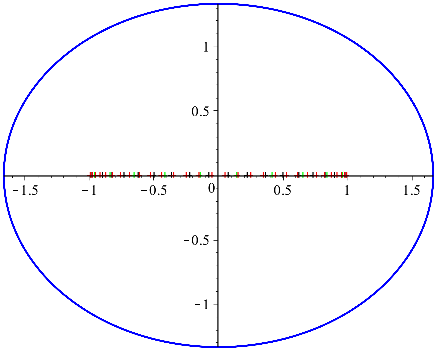


Figure : Zeros of the orthogonal polynomials P_n, $n = 10, 20, 30$, for an ellipse.

Theoretical proof of this location of zeros by inspection of Chebyshev polynomials.

Hypocycloid

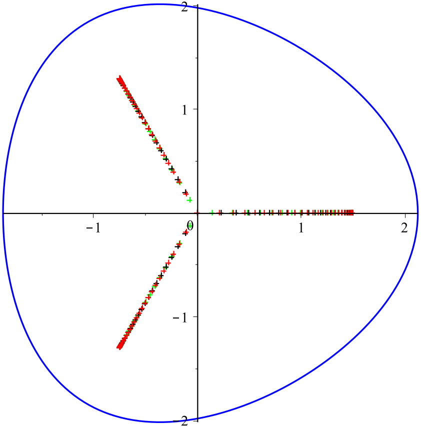


Figure : Zeros of the orthogonal polynomials P_n, $n = 60, 70, 80$, for the hypocycloid defined by $\psi(\zeta) = 2\zeta + 1/(8\zeta^2)$.

Direct construction of mother body (PDE approach)

If Ω has a mother body μ, then the function

$$u = U_\mu - U_\Omega$$

satisfies

$$\begin{cases} \Delta u = 1 & \text{in } \Omega \setminus \operatorname{supp} \mu, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, when seeking a mother body for a domain Ω one may start by trying to find local solutions of the above Cauchy problem (assuming that $\operatorname{supp} \mu$ does not reach $\partial\Omega$) and then try to extend these solutions as far into Ω as possible. There can be no solution in all of Ω, but the places where one has to give up, or has to match different local branches, will be the location for the support of μ. Once u is constructed, μ is obtained as $\mu = 1 - \Delta u$.

Example: Polyhedra

For a polyhedron, $\partial\Omega$ is piecewise flat, and in a neighborhood of each flat piece the local solution of the above Cauchy problem is

$$u(x) = \frac{1}{2}\operatorname{dist}(x, \partial\Omega)^2,$$

in any number of dimensions.

Example (Convex polyhedra)

In case of convex polyhedra this works out perfectly (even in higher dimensions): the branches of u can be glued by simply taking the global minimum of them all. The above formula will then be the global definition of $u(x)$. This gives the mother body (it turns out to be unique), located on the “ridge” of the polyhedron.
Cf. pictures of asymptotic zero distributions for rectangle, pentagon, hexagon below.

Rectangle

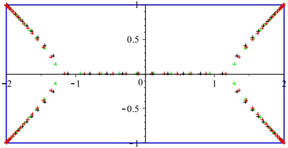


Figure : Zeros of the orthogonal polynomials P_n, $n = 60, 70, 80$, for the 2×1 rectangle with vertices at $2 + i$, $-2 + i$, $-2 - i$ and $2 - i$.

Pentagon

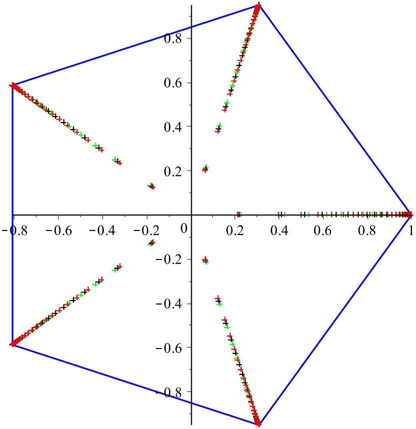


Figure : Zeros of the orthogonal polynomials P_n, $n = 100, 110, 120$, for the canonical pentagon with vertices at the fifth roots of unity

Hexagon

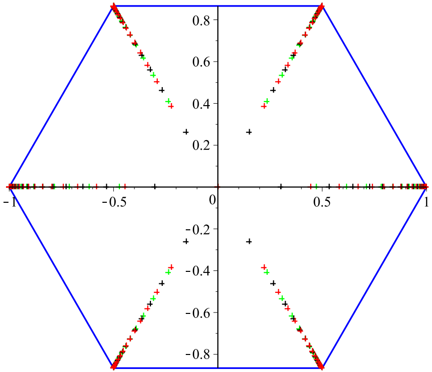


Figure : Zeros of the orthogonal polynomials P_n, $n = 80, 90, 100$, for the canonical hexagon with vertices at the sixth roots of unity

Example: Semidisk (“parade example”)

The above procedure can also be used in more general cases with piecewise analytic boundary. One example is the semidisk

$$\Omega = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0, |z| < 1\}.$$

Here one gets

$$u = \min\{u_1, u_2\},$$

where

$$u_1(z) = \frac{1}{2}x^2, \quad u_2(z) = \frac{1}{4}(|z|^2 - 1 - 2\log|z|).$$

This gives a unique mother body, with support on the curve where $u_1 = u_2$. In the numerical experiment (next slide) the zeros actually go to this curve, with a very high accuracy (but we do not understand why . . .).

Semi-disk

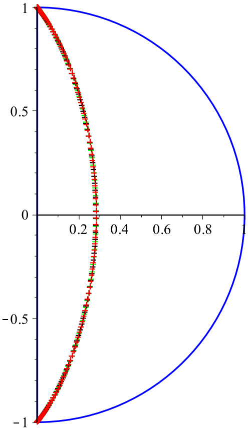


Figure : Zeros of the orthogonal polynomials P_n, $n = 100, 110, 120$, for the half-disk.

Overlapping disks

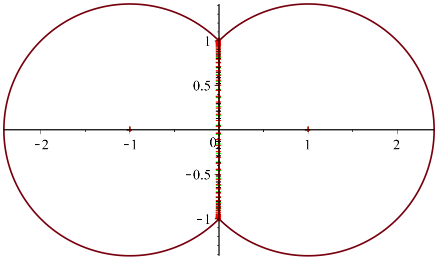


Figure : Zeros of the orthogonal polynomials P_n, $n = 40, 50, 60$, for Ω_{a+b} with b=-a=1.

Pentagon plus disk

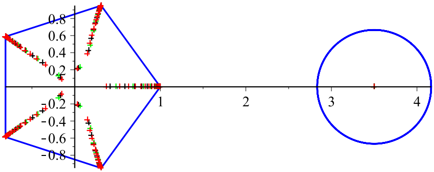


Figure : Zeros of the exponential orthogonal polynomials P_n, $n = 10, 15, 20$, for the disjoint union of an ellipse and a disk

Hexagon plus hypocycloid

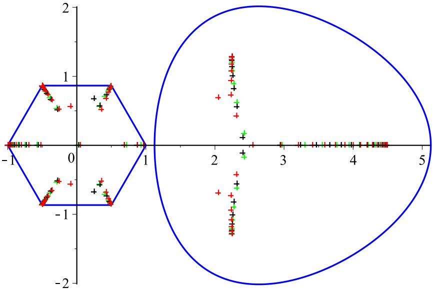
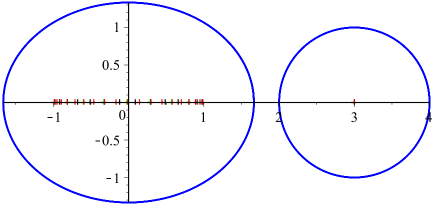


Figure : Zeros of the orthogonal polynomials P_n, $n = 50, 60, 70$, for a disjoint union of a hexagon and a hypocycloid.

Ellipse plus disk



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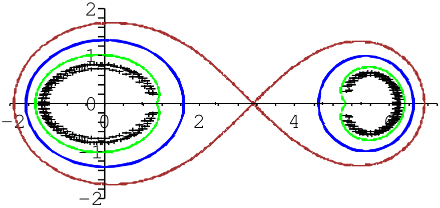
Ellipse plus disk, Bergman



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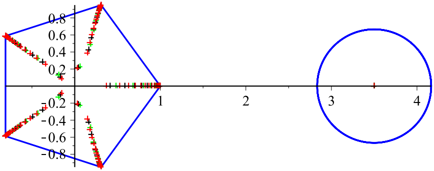
Pentagon plus disk



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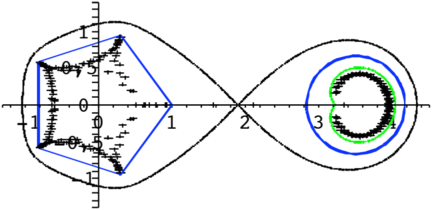
Pentagon plus disk, Bergman



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Riemann-Hilbert (reserve)

The kernels $E(z,w)$, $H(z,w)$ are via a third kernel $G(z,w)$ linked by **Riemann-Hilbert** relations, which characterize the kernels uniquely.

Theorem (link $E \leftrightarrow G$; there is a similar link $G \leftrightarrow H$)

Given $a \in \mathbb{C} \setminus \overline{\Omega}$ there is a unique pair of functions

$$\begin{cases} G(z,a), & \text{analytic in } \Omega, \\ F(z,a), & \text{analytic in } \overline{\mathbb{C}} \setminus \overline{\Omega}, \ \text{with } F(\infty, a) = 1, \end{cases}$$

which match by

$$F(z,a) = (\bar{z} - \bar{a})G(z,a), \quad z \in \partial\Omega.$$

These unique functions are

$$\begin{cases} G(z,a) = \frac{\partial E(z,a)}{\partial \bar{z}}, & z \in \Omega, \\ F(z,a) = E(z,a), & z \in \mathbb{C} \setminus \Omega. \end{cases}$$

Line bundle point of view (reserve)

Using the Schwarz function the matching relation becomes

$$F(z,a) = (S(z) - \bar{a})G(z,a), \quad z \in \partial\Omega.$$

Here $S(z) - \bar{a}$ is a non-vanishing holomorphic function in a neighborhood of $\partial\Omega$, and as such it defines a **line bundle** λ_a on the Riemann sphere, i.e., it defines transition relation relative to the covering $(\mathrm{nbhd}((\overline{\mathbb{C}} \setminus \Omega), \mathrm{nbhd}(\overline{\Omega}))$.
The **Chern class** (= the difference between the number of zeros and number of poles for any meromorphic section) is

$$\mathrm{Chern}(\lambda_a) = -\frac{1}{2\pi i}\int_{\partial\Omega} d\log(S(z) - \bar{a}) = 0,$$

the same as for the trivial line bundle.
It follows that any holomorphic section is determined by its value at one point. So $(F(z,a), G(z,a))$ is determined by $F(\infty, a) = 1$.

Cauchy transform, by $a \to \infty$ (reserve)

The **Cauchy transform** $C_\Omega(z)$ satisfies $C_\Omega(\infty) = 0$ and has the structure

$$C_\Omega(z) = \begin{cases} h(z), & z \in \overline{\mathbb{C}} \setminus \overline{\Omega}, \\ \bar{z} + g(z), & z \in \Omega, \end{cases}$$

with g, h holomorphic. This gives the transition relation

$$e^{h(z)} = e^{S(z)} \cdot e^{g(z)},$$

which exhibits the exponential $e^{h(z)} = e^{C_\Omega(z)}$ as the unique holomorphic section of the line bundle λ_∞ with

$$\text{transition function } = e^{S(z)}$$

and taking the value one at infinity $(\mathrm{Chern}(\lambda_\infty) = 0)$.

THANK YOU!