Generating relations of hyperbolic polynomials

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The general problem

In general we want to understand the connection

$$\bigcup_{m=0}^{\infty} \mathcal{Z}\left(H_m(z)\right) \sim G(t, z)$$

for a sequence of polynomials given by the generating relation

$$\sum_{m=0}^{\infty} H_m(z) t^m = G(t, z).$$

More specifically, we are interested in the cases when the zeros of $\{H_m(z)\}_{m=0}^{\infty}$ lie on a line (or in particular are real).

Some examples

- the Hermite polynomials generated by:

$$\sum_{m=0}^{\infty} \frac{H_m(z)}{m!} t^m = e^{2zt-t^2}$$

- The (simple) Laguerre polynomials generated by:

$$\sum_{m=0}^{\infty} L_m(z) t^m = \frac{1}{(1-t)} \exp\left(-\frac{zt}{1-t}\right)$$

- The Legendre polynomials generated by:

$$\sum_{m=0}^{\infty} P_m(z) t^m = \frac{1}{\sqrt{1-2zt+t^2}}$$

Families of generating functions

Theorem (Tran, 2015).

Let $P_m(z)$ be a sequence of polynomials generated by is

$$\sum_{m=0}^{\infty} P_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^r},$$

where $A(z)$ and $B(z)$ are polynomials in z with complex coefficients.There is a constant $C = C(r)$ such that for all $m > C$, the roots of $P_m(z)$ which satisfy $A(z) \neq 0$ lie on a fixed curve given by

$$\Im\left(\frac{B^r(z)}{A(z)}\right) = 0 \qquad \text{and} \qquad 0 \leq (-1)^r \frac{B^r(z)}{A(z)} \leq \frac{r^r}{(r-1)^{r-1}}$$

and are dense there as $m \to \infty$.

Observations

- In this much generality, there is no reason to expect that the zero locus described above is a line (or a subset of $\mathbb{R}$), and indeed it generally is not.
- The result is about zero-curves, not zero attractors
- the result gives the location of zeros only for $m \gg 1$.

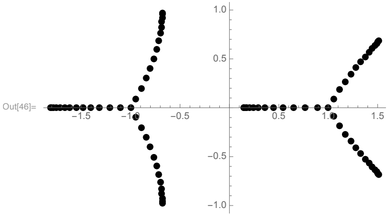

Figure: The root location of the polynomials generated by the function

$$\frac{1}{1+(z^2-1)t+zt^3}$$

Main results

Theorem 2 (–, Tran, 2016).

Let $n, r \in \mathbb{N}$ such that $\max\{r, n\} > 1$. For all large m, the zeros of the polynomial $P_m(z)$ generated by the relation

$$\sum_{m=0}^{\infty} P_m(z) t^m = \frac{1}{(1-t)^n + zt^r} \tag{1}$$

lie on the interval

$$I = \begin{cases} (0, \infty) & \text{if } n, r \geq 2 \\ (0, n^n/(n-1)^{n-1}) & \text{if } r = 1 \\ ((r-1)^{r-1}/r^r, \infty) & \text{if } n = 1 \end{cases}$$

Furthermore, if $\mathcal{Z}(P_m)$ denotes the set of zeros of the polynomial $P_m(z)$, then $\bigcup_{m \gg 1} \mathcal{Z}(P_m)$ is dense in I.

Main ideas of the proof

(i) Assume $z \in \mathbb{R}$, and find an expression for $z(\theta)$ where $t = |t|e^{-i\theta}$ is a zero of $(1-t)^n + zt^r$.

(ii) Show that $z(\theta)$ is a monotone function on the interval $(0, \pi/r)$.

(iii) Find a function $R_m(\theta)$ with the property that $R_m(\theta) = 0$ if and only if $P_m(z(\theta)) = 0$.

(iv) Understand the behavior or $R_m(\theta)$ (find "dominant part") and count the number of sign changes in $R_m(\theta)$ on the interval $(0, \pi/r)$

(v) Make sure we have enough zeros.

Some details for the proof of Theorem 1

For this theorem, we have
(i)

$$z(\theta) = \frac{\sin^n \theta}{\sin^{n-r}(\phi - \theta)\sin^r(\phi)}, \qquad \left(\phi = \frac{(n-1)\pi + r\theta}{n}\right)$$

(ii)

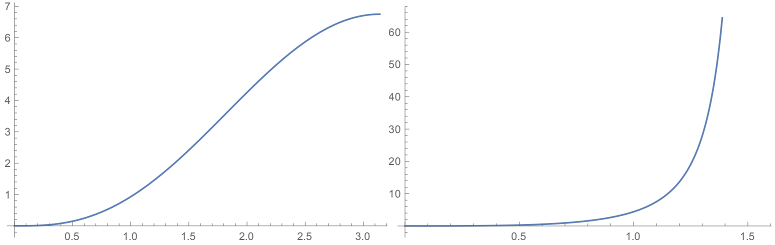

Figure: The curve $z(\theta)$ when $n = 3, r = 1$ (left), and $n = 3, r = 2$ (right)

The function $R_m(\theta)$ when $\max\{n, r\} = 3$

(iii) Suppose $\theta \in (0, \pi/r)$ and let $z \in I$ be the point so that $t_0 = |t_0|e^{-i\theta},\ t_1 = t_0 e^{2i\theta}$ and $t_2 \in \mathbb{R}$ are the zeros of $(1-t)^n + zt^r$ which are distinct in this case. By partial fraction decomposition we obtain

$$\begin{aligned}
\frac{1}{D_{n,r}(t,z)} &= -\frac{1}{(t-t_0)(t_0-t_1)(t_0-t_2)} - \frac{1}{(t-t_1)(t_1-t_0)(t_1-t_2)} \\
&- \frac{1}{(t-t_2)(t_2-t_0)(t_2-t_1)} \\
&= \sum_{m=0}^{\infty}\left(\frac{1}{(t_0-t_1)(t_0-t_2)}\frac{1}{t_0^{m+1}} + \frac{1}{(t_1-t_0)(t_1-t_2)}\frac{1}{t_1^{m+1}}\right. \\
&+ \left.\frac{1}{(t_2-t_0)(t_2-t_1)}\frac{1}{t_2^{m+1}}\right) t^m.
\end{aligned}$$

The function $R_m(\theta)$ when $\max\{n, r\} = 3$

Consequently, for all $m \geq 0$,

$$P_m(z) = \frac{1}{t_0^{m+1}(t_0 - t_1)(t_0 - t_2)} + \frac{1}{t_1^{m+1}(t_1 - t_0)(t_1 - t_2)} + \frac{1}{t_2^{m+1}(t_2 - t_0)(t_2 - t_1)}. \qquad (2)$$

We set $q_1 = t_1/t_0$, $q_2 = t_2/t_0$,and divide the right hand side of (2) by t_0^{m+3} to conclude that z **is a zero of** $P_m(z)$ **if and only if**

$$\frac{1}{(1-q_1)(1-q_2)} + \frac{1}{q_1^{m+1}(q_1 - 1)(q_1 - q_2)} + \frac{1}{q_2^{m+1}(q_2 - 1)(q_2 - q_1)} = 0. \qquad (3)$$

After some algebra and trig manipulation, we see that z **is a zero of** $P_m(z)$ **if and only if** θ **is a zero of**

$$R_m(\theta) = \frac{(\cos\theta - \zeta_2)\sin((m+1)\theta)}{\sin\theta} + \cos((m+1)\theta) + \frac{1}{\zeta_2^{m+1}}, \qquad (|\zeta_2| > 1)$$

The function $R_m(\theta)$ - general case

(iii) In the general case,

$$R_m(\theta) = \sum_{k=0}^{\max\{n,r\}-1} \frac{1}{\zeta_k^{m+1} Q'(\zeta_k)}$$

where $\zeta_k = e^{-i\theta} t_k / t_0$, $0 \le k < \max\{n, r\} - 1$, are the roots of

$$Q(\zeta) = \left(\frac{\sin(\phi - \theta)}{\sin \theta} - \zeta \frac{\sin \phi}{\sin \theta} \right)^n + \zeta^r$$

with $\phi = ((n-1)\pi + r\theta)/n$. We separate the first two terms in the sum (two smallest zeros of Q in magnitude) and write

(iv)

$$\begin{aligned} R_m(\theta) &= \frac{2}{|Q'(e^{i\theta})|^2} \left(A(\theta) \cos((m+r)\theta) - B(\theta) \sin((m+r)\theta) \right) \\ &+ \sum_{k=2}^{\max\{n,r\}-1} \frac{1}{\zeta_k^{m+1} Q'(\zeta_k)} \end{aligned}$$

Counting the zeros

(v) Finally, if $\theta_h = \dfrac{h\pi}{m+r}, (h = 1, \ldots, \lfloor m/r \rfloor)$, denote the values of θ in $(0, \pi/r)$ which give $\cos((m+r)\theta) = \pm 1$, then

(i) $\operatorname{sgn}(R_m(\theta_h)) = (-1)^h$, and

(ii) $\operatorname{sgn}(R_m(\pi/r^-)) = (-1)^{\lfloor m/r \rfloor + 1}$.

for all m sufficiently large.
Comparing the number of zeros with the degree of $P_m(z)$ shows that $P_m(z)$ is hyperbolic for all $m \gg 1$.

Main results ctd.

Theorem 3 (–,Tran, 2017).

Suppose $P(t)$ is a real polynomial, whose zeros are positive numbers, and suppose that $P(0) > 0$. Let r be a positive integer satisfying $\max\{\deg P, r\} > 1$. *If t_a and t_b are the smallest positive, and the largest nonpositive real zeros of the polynomial $t^{2r}\dfrac{d}{dt}(-P(t)/t^r)$ respectively, then for all large integers m, the zeros of the polynomial $H_m(z)$ generated by*

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{P(t) + zt^r} \qquad (4)$$

lie in the interval (a, b), where $a = -P(t_a)/t_a^r$ and

$$b = \begin{cases} -P(t_b)/t_b & \text{if } t_b \neq 0 \\ \infty & \text{if } t_b = 0 \end{cases}.$$

Moreover, the set $\mathcal{Z} = \bigcup_{m \gg 1}\{z \mid H_m(z) = 0\}$ is dense on (a, b).

Current work

Note that the ‘linear’ coefficient cases can combined by studying generating functions whose denominator is $P(t) + zQ(t)t^r$.
Let

$$P(t) = \prod_{-p_- < k \leq p_+} (t - \tau_k), \qquad \text{and} \qquad Q(t) = \prod_{-q_- < k \leq q_+} (t - \gamma_k)$$

be two hyperbolic polynomials with p_+, q_+ (and p_-, q_-) positive (and negative) zeros respectively.

When $P(0), Q(0) \neq 0$, we consider the sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$ generated by the relation

$$\sum_{m=0}^{\infty} H_m(z) t^m = \frac{1}{P(t) + zt^r Q(t)} = \frac{1}{D(t,z)}, \qquad r \geq 2.$$

Current work

Set

$$R(t) = r - \frac{tP'(t)}{P(t)} + \frac{tQ'(t)}{Q(t)}.$$

For each $x > 0$, we let $n_+^P(x)$ and $n_+^Q(x)$ be the number of positive zeros of $P(t)$ and $Q(t)$ on $(0, x]$ counting multiplicity.
Similarly, for each $x < 0$, we define $n_-^P(x)$ and $n_-^Q(x)$ as the number of negative zeros of $P(t)$ and $Q(t)$ on $[x, 0)$.
Under the conditions (1) and (3) of Theorem 4, we can show that $P(t)R(t)$ has a smallest positive real zero, t_a, which satisfies $\tau_1 \leq t_a \leq \tau_2$.

Current work

Theorem 4 (–, Tran).

Suppose

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{P(t) + zt^r Q(t)} \quad (r \geq 2)$$

If

1. $n_+^P(x) - n_+^Q(x) \geq 2, \forall x \geq \tau_2$, *and* $n_+^Q(x) = 0, \forall x \in (0, \tau_2]$
2. $n_-^Q(x) - n_-^P(x) \geq 0, \forall x < 0,$
3. $\operatorname{Im} R(t) > 0$ *on the sector* $\{t | 0 < |t| < \tau_2, 0 < \operatorname{Arg} t < \pi/r\}$,
4. $\operatorname{Im} R(t) > 0$ *on the semi-disk* $\{t | 0 < |t| < t_a, 0 < \operatorname{Arg} t < \pi\}$,

then the zeros of $H_m(z)$ *are real and of the same sign* $(-1)^{p_+ - q_+}$.

Moreover, $\bigcup_{m=1}^{\infty} \mathcal{Z}(H_m)$ *is dense between* $a = -\dfrac{P(t_a)}{t_a^r Q(t_a)}$ *and* $(-1)^{p_+ - q_+}\infty$.

Current work

Theorem 4 (–, Piotrowski, Tran).

For any $r \in \mathbb{N}$, the zeros of all the polynomials $H_m(z)$ generated by

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{e^{-t} + zt^r}$$

lie on the real interval I where

$$I = \begin{cases} (0, e) & \text{if } r = 1 \\ (0, \infty) & \text{if } r > 1 \end{cases}.$$

Furthermore, if $\mathcal{Z}(H_m)$ denotes the set of zeros of the polynomial $H_m(z)$, then $\bigcup_{m=0}^{\infty} \mathcal{Z}(H_m)$ is dense on I.

Connections to combinatorics

- A *Riordan matrix* (g, f) is an infinite lower triangular matrix $R := [r_{n,k}]_{n,k\geq 0}$ whose kth column has the generating function $g(z)f^k(z)$ for $g(z), f(z) \in \mathbb{C}[[z]]$ such that $f(0) = 0$, i.e.

$$r_{n,k} = [z^n]g(z)f^k(z)$$

  where $[z^n]$ is the coefficient extraction operator.
- The Riordan group is the group of invertible Riordan matrices under multiplication. In terms of generating functions,

$$(g, f)^{-1} = \left(1/g \circ f^{-1}, f^{-1}\right)$$

Sheffer sequences

Definition.

A *Sheffer sequence* is a sequence of polynomials $s_n(x)$ defined by

$$\sum_{n\geq 0} s_n(z)\frac{t^n}{n!} = \frac{1}{g(f^{-1}(t))}e^{zf^{-1}(t)}$$

for $g(t), f(t) \in \mathbb{C}[[t]]$ with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. The coefficient matrix of $(s_n(z))_{n\geq 0}$ is the inverse of the exponential Riordan matrix $(g(t), f(t))_{\mathcal{E}}$.

Future work

Conjecture 1 (–, Kim, Tran)

Let $Q(z)$ be a quadratic polynomial whose zeros are real and $Q(0) \neq 0$ and $Q'(0) \neq 0$. The zeros $z = \sigma + it$ of the polynomials $H_m(z)$ generated by

$$\sum_{m=0}^{\infty} H_m(z) \frac{t^m}{m!} = Q(t)^z Q(-t)^{1-z} \qquad (z \neq 0, 1)$$

lie on the critical line $\sigma = 1/2$.

Conjecture 2 (–, Kim, Tran)

Suppose $\alpha, \beta \in \mathbb{R}$ and $\beta < 0$. The zeros of the sequence of polynomials $H_m(x)$ generated by

$$\sum_{m=0}^{\infty} H_m(x) t^m = e^{xt + \alpha t^2 + \beta t^4}$$

are either real or purely imaginary.

Two open questions

1. Strengthen the previous theorems to **show that all the generated polynomials are hyperbolic, not just the ones with large enough degree**.
A potentially viable approach could be to identify a tuple of operators $(J_1(x,D), J_2(x,D), \ldots, J_k(x,D))$ to which the k-subsequences (of simple sets) belong.
A note of interest: while the generated sequence may all be hyperbolic even if the operator(s) it belongs to aren't all hyperbolicity preservers, the Borcea-Brändén theorems are certainly a tool one can use to understand whether

$$J_\ell(x,D) = \sum_{i=0}^{\infty} T_{i,\ell}(x) D^i, \qquad (\ell = 1, 2, \ldots, k)$$

are hyperbolicity preservers.

Two open questions

2. Suppose $\{s_n(x)\}$ is a Sheffer sequence, whose coefficients are given by $(g, f)_{\mathcal{E}}^{-1}$. **How does the combinatorics inform the properties of the polynomials $s_n(x)$ (in particular root location)?** Equally interesting - how does the root location of $s_n(x)$ inform the combinatorics? What is the connection to the Riordan matrix (g, f)?

Thank you!