Generalizing an idea of Rodrigues (with Christian Hägg and Boris Shapiro)

Rikard Bøgvad

Stockholm University

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The problem

Suppose μ is a probability measure with compact support in the complex plane . Sample the measure n times, producing points $z_i, i = 1, ..., n$ and form the polynomial $q_n(z) = \prod_{i=1}^{n}(z - z_i)$. Now differentiate this $\lfloor An \rfloor$ times, where A is a fix real number:

$$q_n := p_n^{(\lfloor An \rfloor)},$$

and let μ_n^A be the associated probability measure of q_n.

Question: Is there an expected limit ditribution

$$\mu_n^A \to \nu_A,$$

and what is in that case its relation to μ?

If $A = 0$ then $\nu = \mu = \lim \mu_n$(nothing happens), and if $A = 1$, then ν is a measure with support on the empty set. Thus (perhaps) A may be thought of as a deformation parameter, given that the limiting measure exists.

Rodrigues example

- let μ be the discrete uniform probability measure with support on -1 and 1.
- Sample it (deterministically!) $2n$ times producing

$$q_n(z) = (z^2 - 1)^n.$$

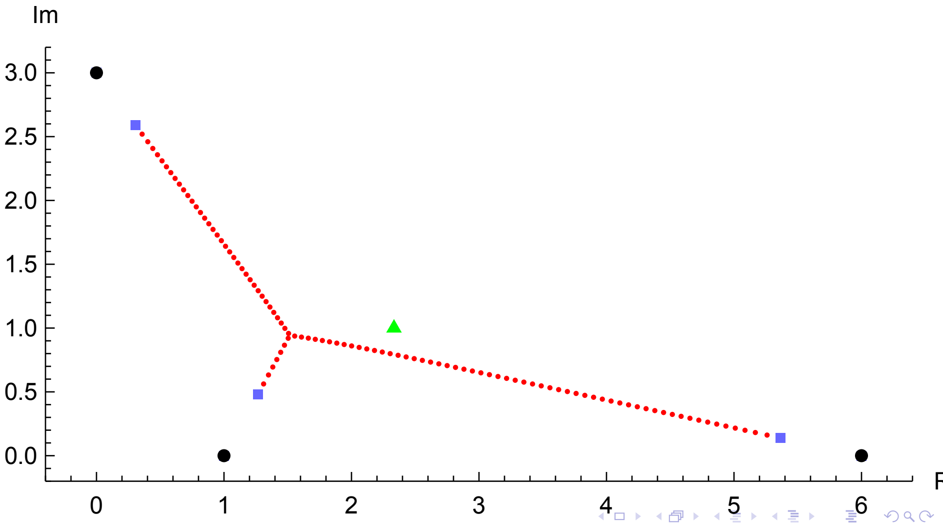
- Differentiate it $(1/2)2n = n$ times :

$$p_n = ((z^2 - 1)^n)^{(n)}.$$

  This is(up to a constant) the Legendre polynomials.
- It is wellknown that $\lim \mu_n$ exists and has a nice description.

Pictures suggest that there is a nice answer...

μ is the uniform discrete probability measure on three points.



How to construct weird measures with support on curves

You need three things:

- a Riemann surface Y (say a smooth algebraic curve).
- a branched cover $\pi : Y \to \mathbb{C}P^1$ (finite number of branch points)
- a real-valued harmonic function H on Y(possibly with logarithmic poles: $H(z) = c\text{Log}|z| + O(1), \quad c \in \mathbb{R}$ in a local coordinate z on Y)

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Definition

Given a real-valued function $H : Y \to \mathbb{R}$, we define its *tropical trace* $\pi_* f$ as

$$\pi_* H(z) = \max_{q_i \in \pi^{-1}(z)} f(q_i).$$

The same definition can be extended to real-valued functions f defined on $Y \setminus S$, where S is a discrete set such that for any $s \in S$, $\lim_{z \to s} H(z)$ always exists as a real number or $\pm\infty$. (In other words, we allow f to attain values $\pm\infty$.)

Properties of the tropical trace

Theorem

Let the poles for H be $P_H = \{z_1, ..., z_s\}$, such that $H(z) = c_i Log|z| + O(1), \quad c \in \mathbb{R}$ in a local coordinate at z_i on Y. Denote by P_H^- the poles with $c_i < 0$,P_H^+ the poles with $c_i > 0$.

- *Assume that $\pi(P^-) = \infty$. Then $\pi_* H(z)$ is a subharmonic function on $\mathbb{C}$.*
- *The Laplacian of $\pi_* H(z)$ has support on level curves*

$$H(p_i(z)) = H(p_j(z)),$$

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Existence of harmonic functions

Theorem

(S. Grushevsky, I. Krichever) For any $(Y, p_1, \ldots, p_n) \in M_{g,n}$*, any set of positive integers* $h_1, \ldots, h_n$*, any choice of* h_i*-jets of local coordinates* z_i *in the neighborhood of marked points* p_i*, and any singular parts (i.e., for* $i = 1 \ldots n$*, the choice of Taylor coefficients* $c_i^1, \ldots, c_i^{h_i}$*, with all ireal residues* $c_i^1 \in \mathbb{R}$ *and the sum of the residues* $\sum c_i^1$ *vanishing), there exists a unique differential* Ψ *on* Y *with purely imaginary periods and prescribed singular parts. In other words, in a neighborhood* U_i *of each* p_i *the differential* Ψ *satisfies the condition*

$$\Psi|_{U_i} = \sum_{j=1}^{h_i} c_i^j \frac{dz}{z^j} + O(1).$$

Moral:

There are many harmonic functions H that induce interesting measures on $\mathbb{C}$. Prescribe that the poles of ω are simple, the residues all real, and define

$$H_\omega(z) = \int_{p_0}^{p} \omega, \quad p \in Y,$$

(where p_0 is a chosen fix point). Then find $\pi : Y \to \mathbb{C}P^1$ such that $\pi(P^-) = \infty$.

Then the trace $\nu_* H$ will be a subharmonic function on $\mathbb{C}$. Note that the $\frac{\partial \nu_* H}{\partial z}$ will satisfy an algebraic equation over the ring of polynomials a.e, since ω is a meromorphic differential.

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Our problem has a solution of this form.

- The curve Y may be identified from the parameterdependent ODE that the polynomials satisfy, or as saddle points in the saddle point method.
- Then the harmonic function may be identified as the integral of a natural differential on Y with imaginary periods.
- finally $L_\nu(z) = \pi_* H(z) + B$.

Main result—abstract version

Theorem

Assume that R is a polynomial of degree d with simple roots. Then there is a plane curve Y of degree d(the number of original points), and a meromorphic differential on Y such that (for a certain constant $B = B(\alpha, d)$)

$$\lim_{n\to\infty} L_{\mu_n}(z) = B + \pi_* H_\omega(z),$$

where the limit is understood in the L^1_{loc}-sense and

$$\lim_{n\to\infty} \mu_n = \nu := \frac{1}{\pi}\frac{\partial^2 \pi_* H(z)}{\partial z \partial \bar{z}},$$

where the limit is understood in the sense of distributions.

Corollary

Theorem

- *The asymptotic measure* ν *has support on a finite number of real-analytic curves and points.*
- *The Cauchy transform* C *satisfies a.e the equation*

$$(d-\alpha)C = \frac{\partial LogR(z + \frac{\alpha}{d-\alpha}C^{-1})}{\partial z} = \frac{R'(z + \frac{\alpha}{d-\alpha}C^{-1}))}{R(z + \frac{\alpha}{d-\alpha}C^{-1}))}.$$

Note that the equation of the Cauchy transform defines a plane curve of degree d(the number of original points). It is related to Y by a birational transformation.

Proof and details of theorem I

We use Cauchy's description of derivation as an integral operator. For any $z_0 \in \mathbb{C} \setminus \{z_1, ..., z_d\}$, where $R(z) = \Pi_{i=1}^{d}(z - z_i)$

$$q_n(z_0) = R^n(z_0)^{([\alpha n])} = \frac{([\alpha n] - 1)!}{2\pi i} \int_C \frac{R^n(z)dz}{(z - z_0)^{[\alpha n]}}, \qquad (1)$$

where C is any curve that encircles z_0.
It is sufficient to understand the L^1_{loc}-limit

$$\lim_{n \to \infty} \mathrm{Log}|q_n|^{1/d_n},$$

(if it exists) where d_n is the degree of q_n. For this it suffices to understand the asymptotic behaviour of the integral. This is given by the saddle point method.

Fact 1:

The curve C may be moved freely, as long as it encircles z_0.

Proof and details of theorem II

By the saddle point method one should rewrite

$$q_n(z_0) = \frac{([\alpha n]-1)!}{2\pi i}\int_C \mathrm{Exp}[n(\mathrm{Log}\ R(z) - \frac{[\alpha n]}{n}\mathrm{Log}\ (z-z_0))], \qquad (2)$$

Fact 2:

The saddle points w belonging to each $z(= z_0$ above) are given by the algebraic plane curve

$$\frac{1}{\alpha}\frac{R'(w)}{R(w)} - \frac{1}{w-z} = 0 \qquad (3)$$

This equation defines (up to a normalisation at infinity) the affine part of the Riemann surface that is used in the theorem Y:

Definition

$Y = \{(w,z) \quad : R'(w)(w-z) = \alpha R(w)\} \subset \mathbb{C}P^1 \times \mathbb{C}P^1.$

Proof and details of theorem III

Fact 3:

The integration contour may be deformed so that it passes through all the saddle points.

Fact 4:

The saddle points $w_1(z), .., w_d(z)$ contribute to the limit of the contour integral in the following way
(up to a certain explicit additive constant $B = B(\alpha, d)$ and $\beta = \frac{d-\alpha}{\alpha}$)

$$\lim_{n\to\infty} \text{Log}|q_n(z)|^{1/d_n} = B + \frac{1}{\alpha\beta}\text{Max}\{Log|R(w_i)| - \alpha Log|(w_i - z)|\}$$

(This is a L^1_{loc}-limit).

Back to abstract version! Reformulation of the solution

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Consider the function on $\mathbb{C}^2$

$$G(z, w) := \frac{1}{1-A}L_\mu(w) - \frac{A}{1-A}\text{Log }|w - z|,$$

and the saddle point locus W of points (z, w) such that

$$\frac{\partial G(z, w)}{\partial w} = 0.$$

W is a plane curve and $G(z, w)$ restricts to a real-valued harmonic function H on it.
Then our result is that the asymptotic root-counting measure ν is given by

$$L_\nu(z) = B + \pi_*(\frac{1}{1-A} L_\mu(w) - \frac{A}{1-A} \mathrm{Log}\, |w - z|)$$

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This is an equation and formulation that makes (eh...some) sense even if ξ is an arbitrary probability measure and p_n is produced by sampling this measure. Hence I conjecture that it holds for much more general measures than those of finite discrete uniform support.

Thank you!