A multivariate Schmidt–Spitzer theorem

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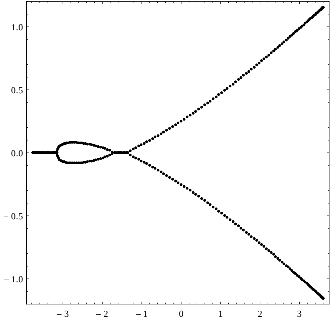
Hausdorff Geometry of Polynomials and Polynomial Sequences

Outline

Goal: Generalize a classical result on asymptotics for eigenvalues of banded Toeplitz matrices.

$$\begin{vmatrix} w & -1 & 3 & 1 & 0 & 0 \\ 0 & w & -1 & 3 & 1 & 0 \\ 1 & 0 & w & -1 & 3 & 1 \\ 0 & 1 & 0 & w & -1 & 3 \\ 0 & 0 & 1 & 0 & w & -1 \\ 0 & 0 & 0 & 1 & 0 & w \end{vmatrix}$$



Key words: Schmidt–Spitzer, Beraha–Kahane–Weiss, linear recurrences, multivariate Chebyshev polynomials, Schur polynomials.

Schmidt–Spitzer Theorem, '60

Let $\mathcal{A}$ be a $k \times k$ Toeplitz matrix and consider $\det(\mathcal{A}_k + w\mathcal{I}_k)$.

Example:

$$\begin{vmatrix} \mathbf{w} & -\mathbf{1} & \mathbf{3} & \mathbf{1} & 0 \\ \mathbf{0} & w & -1 & 3 & 1 \\ \mathbf{1} & 0 & w & -1 & 3 \\ 0 & 1 & 0 & w & -1 \\ 0 & 0 & 1 & 0 & w \end{vmatrix}, \; Q(z,w) = 1 + \underbrace{wz^2}_{c=2} - z^3 + 3z^4 + z^5$$

Let $\alpha_0(w) \leq \alpha_1(w) \leq \cdots \leq \alpha_n(w)$ be the moduli of the roots of $Q(z,w) = 0$. Consider the semi-algebraic set $C_{\mathcal{A}} = \{w : \alpha_c(w) = \alpha_{c+1}(w)\}$.

Then $C_{\mathcal{A}}$ is the limit set of eigenvalues of $\mathcal{A}_k$ as $k \to \infty$.

Some intuition

Why $Q(z,w)$? Why $C_{\mathcal{A}} = \{w : \alpha_c(w) = \alpha_{c+1}(w)\}$?

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Answers: $Q(z,w)$ "behaves like" a characteristic polynomial of a linear recurrence.

Example: $1, 1, 2, 3, 5, 8, 13, \ldots,$ has characteristic polynomial $t^2 - t - 1$. The largest root determine the asymptotic growth.

Each choice of w gives a sequence of determinants satisfying a linear recurrence.
This sequence definitely diverges, unless the two largest roots of the characteristic polynomial are of equal size.
The set $C_{\mathcal{A}}$ describes this set.

Linear algebra

One can show[1] that $\det(\mathcal{A}_k + w\mathcal{I}_k)$ for sufficiently large k satisfy a linear recurrence.
Classic example from Linear algebra 101:

$$\begin{vmatrix} b & c & 0 & 0 & 0 \\ a & b & c & 0 & 0 \\ 0 & a & b & c & 0 \\ 0 & 0 & a & b & c \\ 0 & 0 & 0 & a & b \end{vmatrix} = b \begin{vmatrix} b & c & 0 & 0 \\ a & b & c & 0 \\ 0 & a & b & c \\ 0 & 0 & a & b \end{vmatrix} - ac \begin{vmatrix} b & c & 0 \\ a & b & c \\ 0 & a & b \end{vmatrix}$$

[1] E.g., Schur polynomials, banded Toeplitz matrices and Widom's formula, (2012)

Beraha–Kahane–Weiss, '75

Suppose the polynomials $a_k(\mathbf{x})$ satisfy a linear recurrence:

$$a_k(\mathbf{x}) + c_1(\mathbf{x})a_{k-1}(\mathbf{x}) + \cdots + c_r(\mathbf{x})a_{k-r}(\mathbf{x}) \equiv 0$$

$$a_k(\mathbf{x}) = \sum_{i=1}^{r} g_i(\mathbf{x})\theta_i(\mathbf{x})^k$$

plus some non-degeneracy conditions.

Theorem

Then $\mathbf{x}^*$ is a limit of zeros if and only if the roots θ of the characteristic equation can be numbered so that one of the following is satisfied:

1. $|\theta_1(\mathbf{x}^*)| > |\theta_j(\mathbf{x}^*)|, 2 \leq j \leq b$ *and* $g_1(\mathbf{x}^*) = 0$
2. $|\theta_1(\mathbf{x}^*)| = |\theta_2(\mathbf{x}^*)| = \cdots = |\theta_i(\mathbf{x}^*)| > |\theta_j(\mathbf{x}^*)|, i+1 \leq j \leq r$ *for some* $i \geq 2$.

Proof of Schmidt–Spitzer

The BKW theorem together with linear recurrence implies the Schmidt–Spitzer theorem.

Multivariate eigenvalues

In an $m \times (m+n)$-matrix, there are n "main" diagonals. For $s = 0, \ldots, n$ define the s-th unit matrix

$$\mathcal{I}_s := (\delta_{s+i-j}) \in \mathbb{C}^{m \times (m+n)}.$$

Definition (Shapiro 2009)

Given a matrix $A \in \mathbb{C}^{m \times (m+n)}$ define its ***eigenvalue locus*** $\mathcal{E}_A$ as

$$\mathcal{E}_A := \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} \colon \operatorname{rank}\left(A - \sum_{s=0}^{n} x_s \mathcal{I}_s \right) < m \right\}.$$

For $n = 0$, $\mathcal{E}_A$ coincides with the usual set of eigenvalues of a square matrix A.

Matrix minors

Given an infinite matrix $\mathcal{A} = (a_{ij})$, an integer $n \geq 0$ and a strictly increasing m-tuple of positive integers,

$$I = (i_1, i_2, \ldots, i_m), \text{ with } i_j \leq m + n$$

consider the submatrix A_I of $\mathcal{A} - \sum_{s=0}^{n} x_s \mathcal{I}_s$ formed by the rows and columns indexed by I.
Define

$$P^I_{\mathcal{A}}(x_0, x_1, \ldots, x_n) := \det A_I.$$

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Define

$$P_{\mathcal{A}}^{I}(x_0, x_1, \ldots, x_n) := \det A_I.$$

Let $\mathcal{E}_{\mathcal{A}}^{(m)}$ be the common zeros of all $P_{\mathcal{A}}^{I}(x_0, \ldots, x_n)$ with $|I| = m$.
There are $\binom{m+n}{n+1}$ such zeros (with multiplicity).

Problem

Let $\mathcal{E}_{\mathcal{A}}^{(m)}$ define a root-counting measure, $\mu_{\mathcal{A}}^{(m)}$, as a measure on $\mathbb{C}^{n+1}$.

For what matrices $\mathcal{A}$ does the weak limit $\mu_{\mathcal{A}} := \lim_{m\to\infty} \mu_{\mathcal{A}}^{(m)}$ exists?

Limit set of eigenvalues

Given $\mathcal{A}$, define the ***limit set $B_{\mathcal{A}}$ of eigenvalue loci*** as

$$B_{\mathcal{A}} = \left\{ \mathbf{x} \in \mathbb{C}^{n+1} \colon \mathbf{x} = \lim_{m\to\infty} \mathbf{x}_m, \mathbf{x}_m \in \mathcal{E}_{\mathcal{A}}^{(m)} \right\}.$$

The support of $\mu_{\mathcal{A}}$ (if it exists) is given by $B_{\mathcal{A}}$.

Toeplitz matrices

Let $\mathcal{A} = (c_{i-j})$, with $i, j = 1, 2, \ldots$ be an infinite, banded Toeplitz matrix, where $c_i = 0$ if $i < -k$ or $i > h$.

Let

$$Q(t, \mathbf{x}) = t^k \left(\sum_{j=-k}^{h} c_j t^j - \sum_{j=0}^{n} x_j t^j \right),$$

and let $\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x}), \ldots, \alpha_{k+h}(\mathbf{x})$ be the roots of $Q(t, \mathbf{x}) = 0$.

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Let $C_{\mathcal{A}}$ be the real semi-algebraic set

$$C_{\mathcal{A}} = \{\mathbf{x} \in \mathbb{C}^{n+1} \colon |\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})|\}.$$

Multivariate Schmidt–Spitzer

We conjecture the following multivariate version of the Schmidt–Spitzer theorem:

Conjecture (A., Shapiro 2014)

For any banded Toeplitz matrix $\mathcal{A}$,

$$B_{\mathcal{A}} = C_{\mathcal{A}}.$$

Proof of inclusion

We can prove the inclusion $B_{\mathcal{A}} \subseteq C_{\mathcal{A}}$.

Sketch:

- Consider the $m+1$ submatrices A_I for $I = \{j+1, j+2, \ldots, j+m\}$ for $j = 0, \ldots, m$, and let $D_j^m(\mathbf{x})$ be their determinants.
- Let $\widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)}$ be the set of common zeros of the $D_j^m(\mathbf{x})$. Clearly, $\mathcal{E}_{\mathcal{A}}^{(m)} \subseteq \widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)}$ as there are fewer conditions.
- For fixed j, the sequence $D_j^m(\mathbf{x})$ sequence $m = 1, 2, 3, \ldots$ satisfies a linear recurrence, so by BKW, we have a condition on $\mathbf{x}^*$ to be a limit point of roots.
- Intersection of conditions give $C_{\mathcal{A}}$, so $\widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)} \subseteq C_{\mathcal{A}}$.

Examples

Let $n = 2$ and $\mathcal{A}$ be the infinite Toeplitz matrix with $c_{-k} = c_{n+k} = 1$ and remaining entries 0.

$$\begin{pmatrix} x_0 & x_1 & 1 & 0 & 0 & \cdots \\ 1 & x_0 & x_1 & 1 & 0 & \cdots \\ 0 & 1 & x_0 & x_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \quad \begin{pmatrix} x_0 & x_1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & x_0 & x_1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & x_0 & x_1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

Examples

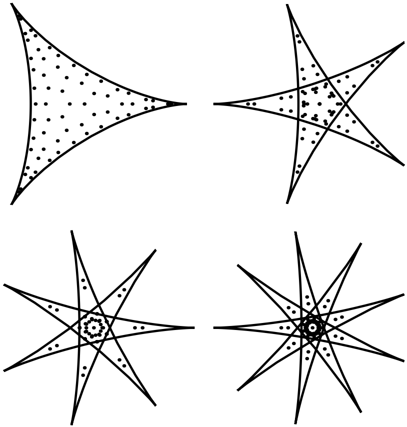

Figure : $\widetilde{\mathcal{E}}^{(m)}_{\mathcal{A}}$ and $C_{\mathcal{A}}$ for $k = 1 \dots 4$. This lies in the complex plane $\overline{x_1} = x_2$.

Thank you for your time

Questions can be emailed to `per.w.alexandersson@gmail.com`.

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