ON ARRANGEMENTS OF ROOTS FOR A REAL HYPERBOLIC POLYNOMIAL AND ITS DERIVATIVES

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Abstract. In this paper we count the number $\sharp_n^{(0,k)}$, $k \leq n-1$ of connected components in the space $\Delta_n^{(0,k)}$ of all real degree n polynomials which a) have all their roots real and simple; and b) have no common root with their k-th derivatives. In this case, we show that the only restriction on the arrangement of the roots of such a polynomial together with the roots of its k-th derivative comes from the standard Rolle's theorem. On the other hand, we pose the general question of counting all possible root arrangements for a polynomial $p(x)$ together with all its nonvanishing derivatives under the assumption that the roots of $p(x)$ are real. Already the first nontrivial case $n = 4$ shows that the obvious restrictions coming from the standard Rolle's theorem are insufficient. We prove a generalized Rolle's theorem which gives an additional restriction on root arrangements for polynomials.

§1. Introduction.

A real polynomial $P(x)$ in one variable is called *hyperbolic* if all its roots (counted with the multiplicities) are real. A hyperbolic $P(x)$ is called *strictly hyperbolic* if all its roots are simple. The standard Rolle's theorem implies that if $P(x)$ is strictly hyperbolic then all its derivatives are strictly hyperbolic as well and for any i the roots of $P^{(i)}(x)$ and $P^{(i+1)}(x)$ are interlacing. If $P(x)$ is of degree n let us denote by $x_1^{(i)} < x_2^{(i)} < \dots < x_{n-i}^{(i)}$, $i = 0, \dots, n-1$ the set of roots of $P^{(i)}(x)$ in the increasing order. Applying the Rolle's theorem several times one gets for any $i < j < n$ *the standard Rolle's restrictions*

$$x_l^{(i)} < x_l^{(j)} < x_{l+j-i}^{(i)} \tag{1}$$

Notation. Denote by Δ_n the space of all strictly hyperbolic polynomials of degree n and denote by $\Delta_n^{(i_1,\dots,i_r)} \subset \Delta_n$ the subset of all strictly hyperbolic polynomials whose derivatives of orders $i_1, i_2, \dots, i_r$ have pairwise no common roots and denote by $\sharp_n^{(i_1,\dots,i_r)}$ the number of connected components in $\Delta_n^{(i_1,\dots,i_r)}$. In this paper we are mostly interested in the spaces $\Delta_n^{(0,1,2,\dots,n-1)}$ and $\Delta_n^{(0,k)}$.

Finally, denote by $\mathcal{EXT}_n^{(i_1,\dots,i_r)}, 0 \leq i_1 < i_2 < \dots < i_r \leq n-1$ the set of all linear extensions of the partial order given by (1) on the family of r groups of elements of the form $\{(x_1^{(i_1)} < x_2^{(i_1)} < ... < x_{n-i_1}^{(i_1)}); (x_1^{(i_2)} < x_2^{(i_2)} < ... < x_{n-i_2}^{(i_2)}); ...; (x_1^{(i_r)} < x_2^{(i_r)} < ... < x_{n-i_r}^{(i_r)})\}$. Set $\flat_n^{(i_1,...,i_r)} = \mathrm{card}(\mathcal{EXT}_n^{(i_1,...,i_r)})$.

Since the roots of $p^{(i_1)}, p^{(i_2)}, ..., p^{(i_r)}$ of any polynomial $p \in \Delta_n^{(i_1,...,i_r)}$ satisfy (1) and are all distinct they form one of the linear extensions in $\mathcal{EXT}_n^{(i_1,...,i_r)}$. It is therefore natural to compare the numbers $\sharp_n^{(i_1,...,i_r)}$ and $\flat_n^{(i_1,...,i_r)}$. But already the first nontrivial case of $\Delta_4^{(0,1,2,3)}$ shows that the situation is far from being obvious, namely $10 = \sharp_4^{(0,1,2,3)} \neq \flat_4^{(0,1,2,3)} = 12$, see Fig.1.

On the figure (i,j) denotes the discriminantal set $\mathrm{Res}(P^{(i)}, P^{(j)}) = 0$ (i.e. the set on which the derivatives $P^{(i)}$ and $P^{(j)}$ have a common real root). The roots of P, P', P'', P''' are denoted respectively by 0, f, s, t. Square brackets indicate coinciding roots.

In this particular degree $n = 4$ the absence of 2 possible linear extensions is explained by the following generalized Rolle type theorem (proved in §3) whose proof is essentially due to S. Tabachnikov.

THEOREM 1. If a strictly hyperbolic polynomial $p(x)$ of degree n satisfies the following $n-2$ inequalities between its roots and the roots of $p''(x)$

$$x_2^{(0)} < x_1^{(2)};\ x_3^{(0)} < x_2^{(2)}; ...;\ x_{n-1}^{(0)} < x_{n-2}^{(2)} \tag{2}$$

then one additionally gets the restriction $x_2^{(n-3)} < x_1^{(n-1)}$.

Nevertheless the main result of this paper is the following one.

THEOREM 2. The connected components in the space $\Delta_n^{(0,k)}$ are in $1-1$-correspondence with all linear extensions of the partial order on two groups of elements $\{(x_1^{(0)} < x_2^{(0)} < ... < x_n^{(0)}); (x_1^{(k)} < x_2^{(k)} < ... < x_{n-k}^{(k)})\}$ given by

$$x_i^{(0)} < x_i^{(k)} < x_{i+k}^{(0)} \tag{3}$$

and, therefore, $\sharp_n^{(0,k)} = \flat_n^{(0,k)}$.

The proof occupies §4.

REMARK. In fact even degenerate configurations of two groups of elements satisfying the above restrictions are realizable (i.e. ones with multiple roots of P and/or equalities between roots of P and of $P^{(k)}$), see Theorems 4.4 and 4.6.

PROPOSITION 3. The generating function $\Psi_k(t)$ for the numbers $\flat_n^{(0,k)}$ is given by

$$\Psi_k(t) = \frac{t^k}{\sum_{i=0}^{\left[\frac{k+1}{2}\right]} (-1)^i \binom{k+1-i}{i} t^i}. \tag{4}$$

The proof is given in §8.

The table below contains the values of $\flat_n^{(0,k)}$ for $1 \le k \le n \le 10$.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		1	2	4	8	16	32	64	128	256
3			1	3	8	21	55	144	277	787
4				1	4	13	40	121	364	1093
5					1	5	19	66	221	728
6						1	6	26	100	364
7							1	7	34	143
8								1	8	43
9									1	9
10										1

The corresponding generating functions are

$$\begin{array}{rr}
\Psi_1(t)=\frac{t}{1-t}; & \Psi_2(t)=\frac{t^2}{1-2t};\\
\Psi_3(t)=\frac{t^3}{1-3t+t^2}; & \Psi_4(t)=\frac{t^4}{1-4t+3t^2};\\
\Psi_5(t)=\frac{t^5}{1-5t+6t^2-t^3}; & \Psi_6(t)=\frac{t^6}{1-6t+10t^2-4t^3};\\
\Psi_7(t)=\frac{t^7}{1-7t+15t^2-10t^3+t^4}; & \Psi_8(t)=\frac{t^8}{1-8t+21t^2-20t^3+5t^4};\\
\Psi_9(t)=\frac{t^9}{1-9t+28t^2-35t^3+15t^4-t^5}; & \Psi_{10}(t)=\frac{t^{10}}{1-10t+36t^2-56t^3+35t^4-6t^5};
\end{array}$$

Main open problem. Enumerate the connected components in the space $\Delta_n^{(0,1,2,\dots,n-1)}$ and find the possible restrictions (i.e. generalized Rolle's inequalities) on root arrangements for a strictly hyperbolic $p(x)$ and all its nonvanishing derivatives. The first unsolved case is $n=5$ for which $\flat_5^{(0,1,2,3,4)}=286$.

Remark. Note that $\flat_n=\flat_n^{(0,1,2,\dots,n-1)}=\binom{n+1}{2}!\frac{1!2!\dots(n-1)!}{1!3!\dots(2n-1)!}$, see e.g. [Ru], [Th].

Some history. Different properties of hyperbolic polynomials and criteria of hyperbolicity were extensively studied at the beginning of the twentieth century, see e.g. [PS], ch. 5-6. In 60's and 70's the interest to hyperbolic polynomials (mostly in the case of several variables) was revived due to the fundamental contributions of I. G. Petrovsky and L. Hörmander to the theory of linear partial differential equations with constant coefficients. But some new results were obtained even in the case of one variable, see e.g. [N]. In the 80's V.Arnold and his students wrote a number of papers on hyperbolic polynomials motivated by their application to potential theory, see [Ar1-2], [Ko1-2].

The authors are grateful to V. Arnold who about 5 years ago mentioned the simplest case of Theorem 1 for $n=4$, see [An]. Sincere thanks are due to S. Tabachnikov for his sketch of proof of Theorem 1. The first author was partially supported by INTAS grant 97-1644. The second author wants to acknowledge the hospitality and financial support of IHES, Paris and Université de Nice in January 2001 when this project was started.

§2. Informal complexification.

We formulate a related question for polynomials with complex coefficients. The usual Rolle's theorem and Theorem 1 generalizing it for polynomials seem to be related to the following fact about the space of polynomials of degree n with complex coefficients. (The authors strongly believe that the question below is worth a separate study.)

Denote by $\mathrm{Pol}_n^{\mathbb{C}}$ (resp. $\mathrm{Pol}_n^{\mathbb{R}}$) the space of all monic degree n polynomials in one variable with complex (resp. real) coefficients. Denote by $\mathcal{PP}_n$ the product space $\mathcal{PP}_n = \mathrm{Pol}_n^{\mathbb{C}} \times \mathrm{Pol}_{n-1}^{\mathbb{C}} \times \ldots \times \mathrm{Pol}_1^{\mathbb{C}}$. A point of $\mathcal{PP}_n$ is an n-tuple of polynomials $(P_n, P_{n-1}, \ldots, P_1)$ of respective degrees and one can decompose the space $\mathcal{PP}_n$ according to the types and multiplicities of multiple and common zeros of the P_i's, see an example on Fig. 2. The combinatorial objects enumerating the strata should be called *colored partitions* since they are partitions of $\binom{n}{2}$ not necessarily distinct points on $\mathbb{C}$ divided into groups of cardinalities $n, n-1, \ldots, 1$ which we can think of as having different colors.

One can easily check that this decomposition is actually a Whitney stratification of $\mathcal{PP}_n$. There is a natural embedding map $\pi : \mathrm{Pol}_n \hookrightarrow \mathcal{PP}_n$ sending each monic polynomial P of degree n to $(P, P'/n, P''/n(n-1), \ldots, P^{(n-1)}/n!)$.

Let λ be a colored partition of $\binom{n}{2}$ colored points, $St_\lambda \subset \mathcal{PP}_n$ be the corresponding stratum and $\pi(St_\lambda) = St_\lambda \cap \pi(\mathrm{Pol}_n^{\mathbb{C}})$ be its (probably empty) intersection with the embedded space of polynomials $\pi(\mathrm{Pol}_n^{\mathbb{C}})$. Note that $\dim St_\lambda$ equals the number of parts in λ.

Definition. The stratum St_λ is called *overdetermined* if the codimension of St_λ in $\mathcal{PP}_n$ is greater than the codimension of $\pi(St_\lambda)$ in $\pi(\mathrm{Pol}_n)$. (Here we assume that $\pi(St_\lambda) \neq \emptyset$.)

Example. A colored partition λ (resp. St_λ) is called a *Rolle's partition* (resp. a *Rolle's stratum*) if λ contains the condition that a multuple root of some P_i should coincide with a root of P_{i+1}. Each such stratum (if $\pi(St_\lambda)$ is nonempty) is overdetermined since a multiple zero of $P^{(i)}$ is automatically a zero of $P^{(i+1)}$.

Example. Another overdetermined stratum is the stratum $(P''|P)$. It is given by the conditions that all (and distinct) roots of P_{n-2} should coincide with roots of P_n and, additionally, the polynomials $P_1, P_3, P_5, \ldots, P_{2[(n-1)/2]+1}$ should have a common zero. This stratum is overdetermined since the condition $P''|P$ implies automatically that $P^{(n-1)}, P^{(n-3)}, P^{(n-5)}, \ldots$ have a common zero. This stratum is closely related to Theorem 1. In the case $n = 4$ this fact is illustrated on Fig.1 where the resultant curve of P' and P''' passes through the intersection point of the two branches of the resultant curve of P and P''.

Problem. Enumerate all overdetermined strata in $\mathcal{PP}_n$.

§3. Proof of Theorem 1.

Let $\Omega_n \subset \Delta_n^{(0,2)}$ denote the open domain satisfying the inequalities (2). (Nonemptiness of Ω_n will be established later in Theorem 2.)

Choose $P(x) = x^n + a_2x^{n-2} + a_3x^{n-3} + \ldots = \sum a_i x^{n-i}$ where $(-1)^i a_i$ is the i-th elementary symmetric function of the roots $x_1^{(0)} < \ldots < x_n^{(0)}$ to $P(x)$. (By shifting $x = \bar{x} + t$ we can always assume that $a_1 = 0$.) Then $\frac{P''(x)}{n(n-1)} = \sum \frac{(n-i-1)(n-i)}{n(n-1)} a_i x^{n-i-2}$. Denote $\bar{a}_i = \frac{(n-i-1)(n-i)}{n(n-1)} a_i$. Obviously, $(-1)^i \bar{a}_i$ is the i-th elementary symmetric function of the roots $x_1^{(2)} < \ldots < x_{n-2}^{(2)}$ to $\frac{P''(x)}{n(n-1)}$. The imposed inequalities (2) imply that if $P(x) \in \Omega_n$ then $P(x_1^{(2)}) > 0, P(x_2^{(2)}) < 0, \ldots$ for any even n and $P(x_1^{(2)}) < 0, P(x_2^{(2)}) > 0, \ldots$ for any odd n. Note that $x_1^{(n-1)} = 0$ due to $a_1 = 0$.

Thus in order to prove the inequality $x_2^{(n-3)} < x_1^{(n-1)}$ it suffices to show that $a_3 < 0$ for any $P(x) \in \Omega_n$ satisfying $a_1 = 0$. This is implied by $P^{(n-3)}(0) = (n-3)!a_3$ together with the fact that $P^{(n-3)}(x)$ is a strictly hyperbolic polynomial. Consider the following symmetric function in the roots $x_1^{(2)}, ..., x_{n-2}^{(2)}$ of $P''(x)$

$$\Psi_3(x_1^{(2)}, ..., x_{n-2}^{(2)}) = \frac{1}{V(x_1^{(2)}, ..., x_{n-2}^{(2)})} \begin{vmatrix} P(x_1^{(2)}) & P(x_2^{(2)}) & ... & P(x_{n-2}^{(2)}) \\ (x_1^{(2)})^{n-4} & (x_2^{(2)})^{n-4} & ... & (x_{n-2}^{(2)})^{n-4} \\ (x_1^{(2)})^{n-5} & (x_2^{(2)})^{n-5} & ... & (x_{n-2}^{(2)})^{n-5} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & ... & 1 \end{vmatrix}$$

where $V(x_1^{(2)}, ..., x_{n-2}^{(2)})$ is the usual Vandermonde determinant. Expanding Ψ_3 along the first row and using the signs of $P(x_i^{(2)})$ we arrive at the inequality $\Psi_3(x_1^{(2)}, ..., x_{n-2}^{(2)}) < 0$ valid in the whole Ω_n. Finally taking into account $P(x) = \sum a_i x^{n-i}$ we get $\Psi_3(x_1^{(2)}, ..., x_{n-2}^{(2)}) = \bar{S}_3 + a_2\bar{S}_1 + a_3$ where $\bar{S}_i$ is the i-th Schur function in $x_1^{(2)}, ..., x_{n-2}^{(2)}$. By definition, the i-th Schur function in the variables $(t_1, \dots, t_{n-2})$ equals

$$\bar{S}_i(t_1, \dots, t_{n-2}) = \frac{1}{V(t_1, ..., t_{n-2})} \begin{vmatrix} t_1^{n-3+i} & t_2^{n-3+i} & ... & t_{n-2}^{n-3+i} \\ t_1^{n-4} & t_2^{n-4} & ... & t_{n-2}^{n-4} \\ t_1^{n-5} & t_2^{n-5} & ... & t_{n-2}^{n-5} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & ... & 1 \end{vmatrix},$$

see [Ma], ch.1, §3. The assumption $\bar{S}_1 = \bar{a}_1 = 0$ implies $\bar{S}_3 = -\bar{a}_3$ and we get $\Psi_3 = \frac{6(n-2)}{n(n-1)}a_3 < 0$. Therefore, $a_3 < 0$ for any $P(x) \in \Omega_n$ with $a_1 = 0$ and the result follows. □

§4. Proof of Theorem 2.

As before consider $P(x) = x^n + a_1x^{n-1} + \ldots + a_n$, $x, a_j \in \mathbb{R}$. Denote by $\mathcal{R}_n^{(0,i)}$ the resultant $\mathrm{Res}(P, P^{(i)})$. By abuse of notation we also write $\mathcal{R}_n^{(0,i)}$ for the discriminantal hypersurface in the space of all polynomials of degree n given by the equation $\{\mathcal{R}_n^{(0,i)} = 0\}$.

Lemma 4.1. Inside the hyperbolicity domain, at a point where a simple root of $P^{(i)}$ concides with a simple root of P, $\mathcal{R}_n^{(0,i)}$ is locally the graph of a smooth function $a_n(a_1, \dots, a_{n-i})$.

Proof. Indeed, a simple root ν of $P^{(i)}$ is a smooth function of $a_1, \dots, a_{n-i}$. The condition $P(\nu, a) = 0$ expresses a_n as a polynomial of $a_1, \dots, a_{n-1}, \nu$, i.e. as a smooth function of $a_1, \dots, a_{n-1}$. □

Remark. Outside the hyperbolicity domain this property is no longer true for $i \geq 2$ (for $i = 1$ it is, see [Ko2]). If, for example, $n = 4$ and for simplicity $a_1 = 0$, then $\mathcal{R}_4^{(0,2)}$ is diffeomorphic to the Whitney umbrella: $P = x^4 + a_2x^2 + a_3x + a_4$,

$P'' = 12x^2 + 2a_2$, the roots of P'' equal $\pm\sqrt{-a_2/6}$, hence, $\mathcal{R}_4^{(0,2)} : \{-5a_2^2/36 \pm a_3\sqrt{-a_2/6} + a_4 = 0\}$, i.e. $\mathcal{R}_4^{(0,2)} : \{(a_4 - 5a_2^2/36)^2 = -a_2a_3^2/6\}$ which after the change of variables $\xi = a_4 - 5a_2^2/36$, $\eta = -a_2/6$, $\zeta = a_3$ becomes $\xi^2 = \eta\zeta^2$. Hence, ξ (resp. a_4) cannot be expressed globally as a function of η, ζ (resp. of a_2, a_3).

Lemma 4.2. A root of multiplicity m of a hyperbolic polynomial P, $0 \leq m < i+1$, can be at most a simple root of $P^{(i)}$.

Proof. For $i = 1$ the lemma is easy to check – each root of P of multiplicity $q \geq 2$ is a root of P' of multiplicity $q-1$ and between any two roots of P there is a root of P' (Rolle's theorem). The latter must be simple, otherwise P must have more than $n-1$ roots (counted with the multiplicities). In the same way one shows (for $j = 2, \ldots, i$) that every root of $P^{(j-1)}$ of multiplicity $q \geq 2$ is a root of $P^{(j)}$ of multiplicity $q-1$ and that between any two roots of $P^{(j-1)}$ there is exactly one root of $P^{(j)}$ which is simple. Hence, a root of $P^{(i)}$ of multiplicity $q > 1$ is necessarily a root of P of multiplicity $q+i$. $\square$

Notation. A *configuration vector (CV)* of *length* n is a vector whose components are either positive integers m_ν (sometimes indexed by the letter a, their sum being n) or the letter a. The integers equal the multiplicities of the roots of P, the letters a indicate the positions of the roots of $P^{(i)}$; m_a means that a root of P of multiplicity $m < i+1$ coincides with a simple root of $P^{(i)}$. A CV is called *a priori admissible* if for the configuration of the roots of P and $P^{(i)}$ defined by it one has $x_l < \xi_l < x_{l+i}$ where $x_l = x_l^{(0)}$, $\xi_l = x_l^{(i)}$, $l = 1, \ldots, n-i$, see (1) (multiple roots of P are allowed).

For a part S of a CV denote by $\Theta(S)$ the sum of its integers m_ν (i.e. of the multiplicities of the roots of P relative to S) and by $\theta(S)$ sum of the number of its letters a (including the indices) and of the positive among the numbers $m_\nu - i$ (i.e. the sum of the multiplicities of the roots of $P^{(i)}$ relative to S).

Example. For $n = 8$, $i = 3$ the CV $(1, a, 1, 2_a, a, a, 4)$ means that the roots x_j and ξ_k are situated as follows: $x_1 < \xi_1 < x_2 < x_3 = x_4 = \xi_2 < \xi_3 < \xi_4 < x_5 = \ldots = x_8 = \xi_5$. The multiplicity 4 is not indexed with a because it automatically implies $x_5 = \ldots = x_8 = \xi_5$.

Lemma 4.3. If a CV is of the form (A, m, C) or (A, m) or (m, C) with $m > i$ and if it is realized by some hyperbolic polynomial, then $\Theta(A) = \theta(A)$ and $\Theta(C) = \theta(C)$.

Proof. By perturbing the polynomial realizing the CV one obtains a polynomial P^* with a CV of the form (A', m, C') or (A', m) or (m, C') with $\Theta(X') = \Theta(X)$ and $\theta(X') = \theta(X)$ for $X = A$ and C where all roots of P^* excluding the one of multiplicity m (denoted by h) are simple. One applies then Rolle's theorem i times to the roots of P^* and its derivatives that are to the left of h, i.e. in A (resp. to the right of h, i.e. in C). $\square$

Theorem 2 follows directly from the following Theorems 4.4. and 4.6. proved respectively in Sections 5 and 7.

Theorem 4.4. Any a priori admissible CV is realized by some hyperbolic polynomial.

Definition. The set of values of the coefficients $a_1, \ldots , a_n$ for which all roots of P are real is called *the hyperbolicity domain.* Define a stratification of the hyperbolicity domain. Call *excess of multiplicity* of a multiplicity vector $(m_1, \ldots , m_d)$ the sum $m = \sum_{j=1}^{d}(m_j - 1)$. A *stratum* of codimension k is defined by a CV which has exactly $k - m$ letters a as indices.

Proposition 4.5. 1) A stratum of codimension k is locally a smooth real algebraic variety of dimension $n - k$ in $\mathbb{R}^n$.

2) A point of a stratum of codimension $k > 1$ defined by a CV $\vec{v}$ belongs to the closure of any stratum of codimension $k - 1$ whose CV $\vec{w}$ is obtained from $\vec{v}$ by means of one of the following three operations:

i) if $\vec{v} = (A, l_a, B)$, $l \leq i$, A and B are non-void, then $\vec{w} = (A, l, a, B)$ or $\vec{w} = (A, a, l, B)$;

ii) if $\vec{v} = (A, r, B)$, $r \geq i + 1$, then $\vec{w} = (A, r_1, a, \ldots , a, r_2, B)$ where $r_1 + r_2 = r$ and there is $\rho = r - i - \max(0, r_1 - i) - \max(0, r_2 - i)$ times a between r_1 and r_2.

iii) if $\vec{v} = (A, r_a, B)$, $r \leq i$, A and B are non-void, then $\vec{w} = (A, r', r''_a, B)$ or $\vec{w} = (A, r'_a, r'', B)$, $r' > 0$, $r'' > 0$, $r' + r'' = r$.

The proposition is proved in §6.

Remark. It follows from the definition of the codimension of a stratum that the three possibilities *i), ii)* and *iii)* from the proposition are the only ones to increase by 1 the dimension of a stratum S when passing to a stratum containing S in its closure.

Theorem 4.6. The set of all hyperbolic polynomials realizing any a priori admissible CV is connected.

§5. PROOF OF THEOREM 4.4.

1^0. If the CV is of the form (n), then there is nothing to prove, so suppose that it has more than one component. Consider the family of polynomials $P(x, h) = (x - h_1)^{m_1} \ldots (x - h_s)^{m_s}$ where $0 = h_1 \leq h_2 \leq \ldots \leq h_s = 1$ $(*)$.

The roots of $P^{(i)}$ depend continuously on $h = (h_2, \ldots , h_{s-1})$. Denote them by $0 < \xi_1 \leq \ldots \leq \xi_g < 1$. Set $\xi_0 = 0$, $\xi_{g+1} = 1$. Define with their help $s-2$ continuous functions $\eta_j(h)$, $j = 2, \ldots , s - 1$, such that if $h_j = \eta_j$, then the roots of P and of $P^{(i)}$ provide the necessary CV.

2^0. If a root h_j of multiplicity $< i + 1$ must coincide with a simple root ξ_k of $P^{(i)}$, then we set $\eta_j = \xi_k$.

3^0. If the roots $h_r < h_{r+1} < \ldots < h_{r+l}$ (among which there might be roots of multiplicity $\geq i + 1$) lie between ξ_k and ξ_{k+w} and all roots $\xi_{k+1}, \ldots, \xi_{k+w-1}$ (if $w > 1$) coincide with roots h_j $(r \leq j \leq r + l)$ of multiplicity $\geq i + 1$, then we set

$\eta_{r+j} = \xi_k + (j + 1)(\xi_{k+w} - \xi_k)/(l + 2)$, $j = 0, 1, \ldots , l$ (A).

More generally, one can set

$\eta_{r+j} = \xi_k + s_{j,r}(\xi_{k+w} - \xi_k)$ where $0 < s_{0,r} < \ldots < s_{l,r} < 1$.

Hence, one has $0 < \eta_2 < \ldots < \eta_{s-1} < 1$ $(**)$.

4^0. We want to achieve the conditions $\eta_j = h_j$, $j = 2, \ldots , s - 1$ $(***)$. The map $h \mapsto (\eta_2, \ldots , \eta_{s-1})$ is a continuous map from the closed unit simplex in $\mathbb{R}^{s-2}$ into itself (see $(*)$ and $(**)$). By the Brouwer fixed point theorem, there exists a point where conditions $(***)$ hold. At this point if all h_j are different, then the roots of P and $P^{(i)}$ provide the necessary CV. However, a priori in $(**)$ only non-strict inequalities can be guaranteed. Therefore one has to check that all h_j at this fixed point are distinct.

In what follows we say "by the CV" when we mean what the *desired* configuration of roots of P and $P^{(i)}$ should be. We say "in reality" when we mean what the true configuration at the fixed point is. For instance, if the CV looks like this: $(\ldots, 1, a, 1, \ldots)$, then by the CV one should have (at this index) $\ldots < h_j < \xi_k < h_{j+1} < \ldots$, h_j and h_{j+1} being simple roots of P. However, at the fixed point one might have in reality $\ldots < h_j = \xi_k = h_{j+1} < \ldots$, i.e. some strict inequalities would be replaced by equalities. We show in $5^0 - 7^0$ that this never happens.

5^0. Suppose that there are equal roots among the roots h_j. Find a maximal sequence of roots h_j such that at the fixed point one has $h_u = h_{u+1} = \ldots = h_{u+v}$ with either $h_{u-1} < h_u$ if $u > 1$ or $h_{u+v} < h_{u+v+1}$ if $u + v < s$ (or both).

Suppose that by the CV h_u or h_{u+v} must be of multiplicity $< i + 1$ and should coincide with a simple root ξ_{j_0} of $P^{(i)}$; suppose that this is the case of h_u. (If it is h_{u+v}, then the reasoning is similar.) This means that η_u is defined like in 2^0. Denote by m the sum of the multiplicities of the roots $h_u, \ldots, h_{u+v}$. Then by the CV there are exactly $m - i$ roots of $P^{(i)}$ (counted with the multiplicities) lying between h_u and h_{u+v} (or equal to h_{u+v} if the latter's multiplicity is $> i$); they are all different from ξ_{j_0}. As $h_u = h_{u+v}$, in reality all these roots coincide with h_u and with ξ_{j_0} which means that h_u is a root of P of multiplicity m and a root of $P^{(i)}$ of multiplicity $\geq m - i + 1$ – a contradiction.

6^0. Suppose that the CV is such that η_u is defined like in 3^0, i.e. h_u plays the role of one of the roots $h_r < h_{r+1} < \ldots < h_{r+l}$, say, of h_{r+l_1} (if it is h_{u+v} which plays this role, then the reasoning is similar). Suppose also that by the CV one should have $h_{r+l} < h_{u+v}$ (if this is not the case, then see 7^0). Then $h_u = \xi_k + (l_1 + 1)(\xi_{k+w} - \xi_k)/(l + 2)$ (B), see 3^0, by the CV the root ξ_{k+w} must lie between h_u and h_{u+v}. As in reality one has $\xi_{k+w} = h_u$, one must have (see (B)) $\xi_k = h_u$.

Like in 5^0, by the CV there must be $m - i$ roots of $P^{(i)}$ (counted with their multiplicities) lying between h_u and h_{u+v} or equal to one of them when its multiplicity is $> i$, and ξ_k is not among them. Hence, again h_u is a root of P of multiplicity m and a root of $P^{(i)}$ of multiplicity $\geq m - i + 1$ – a contradiction.

7^0. If in 6^0 one has $h_{r+l} \geq h_{u+v}$, then η_u and η_{u+v} are obtained from formula (A) (see 3^0) for two different indices j and $\eta_u = h_u$, $\eta_{u+v} = h_{u+v}$. This means that $\xi_k = \xi_{k+w}$, see (A) from 3^0, and that $\xi_k = \xi_{k+w} = h_u = h_{u+v}$. However, by the CV ξ_k and ξ_{k+w} are not among the m roots of $P^{(i)}$ (when counted with the multiplicities) which lie between h_u and h_{u+v} or coincide with one of them when its multiplicity is $> i$. This means that h_u is a root of P of multiplicity m and of $P^{(i)}$ of multiplicity $\geq m - i + 2$ – a contradiction again.

Theorem 4.4. is proved. $\square$

§6. PROOF OF PROPOSITION 4.5.

1^0. Part 1) of the proposition in the case when the CV contains no indices a follows from the known results about the strata of the hyperbolicity domain. For a CV with β indices a (which has the form (A, l_a, B), see 2) *i)*) one can construct a deformation leading to a CV of the form (A, l, a, B) (hence, with $\beta - 1$ such indices). The construction is explained in 2^0. This means that the variety defined by the first CV belongs to the closure of the variety defined by the second one, hence, is of lower dimension.

On the other hand, every a priori admissible CV can be incorporated into a chain of CVs each of which is obtained from the previous one by imposing one more equality (either between two roots of P or between a root of P and a root of $P^{(i)}$). The last CV of the chain is by definition of dimension 0. On the other hand, it is realized by some polynomial, hence, the stratum it defines is of dimension ≥ 0. This means that every time we impose an equality between roots we decrease the dimension of the stratum exactly by 1. This proves 1) of the proposition.

2^0. Use the notation from 2^0 and 3^0 of the proof of Theorem 4.4. To realize possibility *i)* one replaces the function η_j from 2^0 of that proof (which equals the root of P of multiplicity l) by $(1-u)\eta_j+u\eta_{j+1}$ or by $(1-u)\eta_j+u\eta_{j-1}$, $0<u\in[0,\epsilon]$, $\epsilon>0$, u being a deformation parameter.

To realize possibility *ii)* or *iii)* one first increases the number of functions η_j by 1 (i.e. one changes the indices) attributing two of them (say, η_j and η_{j+1}) to the root of P of multiplicity r which is to be replaced by two roots of multiplicities r_1 and r_2 or r' and r''. After this one sets $\eta_{j+1}=(1-u)\eta_j+u\eta_{j+2}$ or $\eta_j=(1-u)\eta_{j+1}+u\eta_{j-1}$ with u as above. □

§7. Proof of Theorem 4.6.

1^0. For convenience in the proof below we call by ‘strata’ the connected components of the above strata defined by given CVs. We prove the following statement from which the theorem follows:

Statement 7.1. Suppose that $Q(x,a)$ is a germ of a family of polynomials of degree $l\geq n$ which is a versal deformation of x^n ($a\in(\mathbb{R}^m,0)$, $m\geq n$). Denote by H the subset of $(\mathbb{R}^m,0)$ such that for $a\in H$ the polynomial Q has n real roots close to 0. Define a stratification of H by the configuration of the roots of Q and $Q^{(i)}$ which are close to 0 (the distant real roots if any are not taken into account). Then for each a priori admissible CV there exists a single stratum in H.

If $n\leq i$, then there are no roots of $Q^{(i)}$ at 0 and the statement follows from the fact that $Q(x,a)$ is a versal deformation of x^n (this is the induction base).

2^0. Suppose that $n\geq i+1$. In a neighbourhood of a point in H of the stratum defined by the CV $(n-1,1)$ (or $(1,n-1)$) the family of polynomials Q is a versal deformation of $(x-\alpha)^{n-1}$ for some $\alpha\in\mathbb{R}$ and one can apply the inductive assumption.

Call *preCV* the CV of the roots of Q and $Q^{(i)}$ which are close to α. Knowing the preCV A allows to restitute the CV as well because in a small neighbourhood of $(n-1,1)$ the CV of a stratum of H equals $(A,a,1)$ where A is the preCV. To every preCV (hence, to every CV) defining a stratum (or strata) containing $(n-1,1)$ in its closure there corresponds actually a single stratum of H. Call these strata *of first generation.* By definition, the strata of *second generation* are the ones that are not of first generation and belong to the closures of the strata of first generation.

3^0. A stratum S_0 of second generation, of dimension 0 and belonging to the closure of a stratum S of first generation of dimension 1 must be the only one defined by its CV. Indeed, S is a rational real algebraic curve, i.e. a connected one-dimensional variety whose border consists of S_0 and of $(n-1,1)$. If S_1 is another stratum defined by the same CV as S_0, then it must also belong to the closure of S which is impossible.

4^0. A stratum S' of second generation (dim$S' = 1$) belonging to the closure of a stratum S'' of first generation (dim$S'' = 2$) and containing in its closure a stratum S_0 like in 3^0 must be the only one defined by its CV. Here and later in the proof we use the fact that locally the stratification resembles the standard stratification of $\mathbb{R}^n$ by the coordinates.

Indeed, the closure of S'' at S_0 is homeomorphic to the closed upper half-plane with the real axis, the open upper half-plane being the image under this homeomorphism of S'', the origin being the image of S_0, the left and right half-axes being the images of the strata S' and S (see 3^0). Any other stratum defined by the same CV as S' must contain S_0 in its closure and must belong to the border of S in a neighbourhood of S_0 together with S' which is impossible.

5^0. As in 3^0 and 4^0 one proves that all CVs defining strata of dimension 0 or 1 which are of second generation define each a single stratum. E.g. the CV defining a stratum S^* of dimension 0 from the closure of S' must define a single stratum; this is proved by analogy with the uniqueness of S_0 from 3^0 because S' like S is unique. Then one proves the uniqueness of all strata of second generation and of dimension 1 which belong to the closure of S'' and contain S^* in their closure like this was done for S' in 4^0 etc.

Having finished with the strata of dimension 0 and 1 from the closure of S'', the same is to be done for all two-dimensional strata of first generation in the place of S''.

6^0. Suppose we have proved that for each a priori admissible CV defining a second generation stratum of dimension g there exists a single such stratum. Prove that the same is true for the second generation strata of dimension $g+1$ (one has $g < m-1$ because there are no second generation strata for $g = m$).

This is done like in dimension 1, see 4^0. Namely, suppose that a stratum S' of second generation (dim$S' = g+1$) belongs to the closure of a stratum S'' of first generation (dim$S'' = g+2$) and contains in its closure a stratum S_0 of second generation (dim$S_0 = g$). Suppose that the stratum of first or second generation S contains in its closure S_0 and that S is the only stratum defined by its CV.

The closure of S'' at S_0 is homeomorphic to the half-space $\{(c_1, \ldots, c_{g+2}) | c_i \in \mathbb{R}, c_1 \geq 0\}$. The open half space ($c_1 > 0$) is the image of S'', the subspace $c_1 = c_2 = 0$ is the image of S_0 and the ones of S' and of S are respectively $c_1 = 0 < c_2$ and $c_1 = 0 > c_2$. This leaves no place for another stratum defined by the same CV as S' which must also contain S_0 in its closure and belong to the closure of S''.

7^0. Call strata of *third generation* the strata of dimension m which are not of first generation and which contain in their closures strata of second generation of dimension $m-1$.

Each CV defining a third generation stratum defines a single such stratum.

Indeed, a stratum of dimension $m-1$ is locally a smooth hypersurface in $\mathbb{R}^m$. On the one side of this hypersurface is a first generation stratum, on the other side is the given third generation one. The stratum of second generation being the only one defined by its CV so is the third generation one adjacent to it.

8^0. A stratum of *fourth generation* is a not second generation one from the closure of a stratum of third generation.

Each CV defining a fourth generation stratum defines a single such stratum.

The proof is performed by analogy with 6^0. However, we start with the fourth generation strata of dimension $m-1$ and finish with the 0-dimensional ones. In more detail – we prove first the uniqueness of all fourth generation strata T^1 of dimension

d containing in their closure a second generation stratum T' of dimension $d-1$ which is also contained in the closure of a second generation stratum T^0, $\dim T^0 = d$, for $d = m-1, m-2, \ldots, 1$; for $d = m-1$, T^1, T' and T^0 belong to the closure of one and the same third generation stratum, for $d < m-1$ they belong to the closure of one and the same fourth generation stratum of dimension $d+1$.

After this we prove the uniqueness of all strata contained in the closures of the strata T^1 like above. Then we prove in the same way the uniqueness of all fourth generation strata T^2 of dimension d containing in their closure a fourth generation stratum T'' of dimension $d-1$ which belongs to the closure of a stratum T^1 like above ($\dim T^1 = d$); then we prove the uniqueness of all strata from the closures of the strata T^2 etc.

9^0. One continues like this till all a priori admissible CVs are exhausted. The strata of odd generation (excluding the first one) are of dimension m, the ones of the next even generation belong to their closures and are not from previous generations. As there are finitely many strata (and H is connected because Q is a versal deformation of x^n), after finitely many steps we prove for all a priori admissible CVs that they define each a single stratum. Theorem 4.6 (and thus Theorem 2) is finally proved. $\square$

§8. PROOF OF PROPOSITION 3.

We will first prove that $\flat_{k,n} = \flat_n^{(0,k)}$, $k < n$ satisfy the recurrence relation

$$\flat_{k,n} + \sum_{i=1}^{\left[\frac{k+1}{2}\right]} (-1)^i \binom{k+1-i}{i} \flat_{k,n-i} = 0. \tag{5}$$

For the sake of convenience we set $\flat_{k,j} = 0$, $j = 1, \ldots, k-1$ and $\flat_{k,k} = 1$. For all bigger n the numbers $\flat_{k,n}$ are determined by (5). Consider the following nonstandard representation of Pascal's triangle:

$signs \backslash k$	-1	0	1	2	3	4	5	6	7	8	9	10
$+$	1	1	1	1	1	1	1	1	1	1	1	1
$-$			1	2	3	4	5	6	7	8	9	10
$+$					1	3	6	10	15	21	28	36
$-$							1	4	10	20	35	56
$+$									1	5	15	35
$-$											1	6

The usual rows of the Pascal's triangle are located along the diagonals on the above picture. The j-th column of the triangle above is obtained by adding the $(j-1)$-st column with the $(j-2)$-nd column shifted by 1 down. Note that the k-th column contains the coefficients in relation (5) with the signs given in the first column. Let $\flat_{k,n}^j$, $1 \le j \le k$ denote the number of linear extensions of the partial order determined in (3) with the additional restriction $x_j^{(0)} < x_1^{(k)} < x_{j+1}^{(0)}$. Obviously, $\flat_{k,n} = \sum_{j=1}^k \flat_{k,n}^j$. We show now that $\flat_{k,n}^j$ satisfy the similar recurrence relation

$$\flat_{k,n}^j + \sum_{i=1}^{\left[\frac{j+1}{2}\right]} (-1)^i \binom{j-i}{i-1} \flat_{k,n-i} = 0. \tag{6}$$

In particular, $\flat^1_{k,n} = \flat^2_{k,n} = \flat_{k,n-1}$, $\flat^3_{k,n} = \flat_{k,n-1} - \flat_{k,n-2}$, $\flat^4_{k,n} = \flat_{k,n-1} - 2\flat_{k,n-2}$, $\flat^5_{k,n} = \flat_{k,n-1} - 3\flat_{k,n-2} + \flat_{k,n-3}$ etc. The recurrence (6) is the immediate corollary of the relation

$$\flat^j_{k,n} = \flat^{j-1}_{k,n} - \flat^{j-2}_{k,n-1} \tag{7}$$

together with the boundary cases $\flat^1_{k,n} = \flat_{k,n-1}$ and $\flat^{\leq 0}_{k,n} = 0$. To prove (7) notice that all linear extensions with $x^{(0)}_{j-1} < x^{(k)}_1 < x^{(0)}_j$ (i.e. whose number equals $\flat^{j-1}_{k,n}$) can be changed into the linear extensions with $x^{(0)}_j < x^{(k)}_1 < x^{(0)}_{j+1}$ (i.e. whose number equals $\flat^j_{k,n}$) except for those with $x^{(0)}_{j-1} < x^{(k)}_1 < x^{(0)}_j$, $x^{(0)}_{j-1} < x^{(k)}_2 < x^{(0)}_j$. The later are counted by $\flat^{j-2}_{k,n-1}$. We show now that (7) implies (6) by induction on the pair of indices (n, j). Indeed, for any fixed n the relation (6) is valid for $j = 1$ since $\flat^1_{k,n} = \flat_{k,n-1}$. Assume now that (6) is proved for all pairs (n', j') which are lexicographically smaller than (n, j), i.e. $n' \leq n$ and $j' \leq j$ with at least one of the inequalities being strict. In particular,

$$\begin{cases} \flat^{j-1}_{k,n} = \sum_{i=1}^{\left[\frac{j}{2}\right]} (-1)^{i-1} \binom{j-1-i}{i-1} \flat_{k,n-i} \\ \flat^{j-2}_{k,n-1} = \sum_{i=1}^{\left[\frac{j-1}{2}\right]} (-1)^{i-1} \binom{j-2-i}{i-1} \flat_{k,n-1-i} \end{cases} \tag{8}$$

Thus, for even j (i.e. when $\frac{j}{2}$ is an integer) one gets

$$\begin{aligned} \flat^j_{k,n} = \flat^{j-1}_{k,n} - \flat^{j-2}_{k,n-1} &= \sum_{i=1}^{\left[\frac{j}{2}\right]} \flat_{k,n-i} \left[(-1)^{i-1} \binom{j-1-i}{i-1} - (-1)^{i-2} \binom{j-1-i}{i-2} \right] \\ &= \sum_{i=1}^{\left[\frac{j}{2}\right]} (-1)^{i-1} \binom{j-i}{i-1} \flat_{k,n-i}. \end{aligned}$$

The proof of (6) is completed by noticing that for odd j (i.e. when $\frac{j-1}{2}$ is an integer) the right-hand side of the last identity contains an additional term coming from $i = \left[\frac{j-1}{2}\right]$ in the second equality of (8) (which also changes the range of summation for i from $\left[\frac{j}{2}\right]$ to $\left[\frac{j+1}{2}\right]$).

Finally, in order to get (5) we should add the terms in (6) for all $j = 1, \dots, k$ since $\flat_{k,n} = \sum_{j=1}^{k} \flat^j_{k,n}$. The result of this summation gives (5). Namely, equalizing the coefficients at each $\flat_{k,n-i}$ in (5) and in $\sum_{j=1}^{k} \flat^j_{k,n}$ we need to show that for any pair of positive integers (k, i) satisfying $k + 1 \geq 2i$; $i \geq 1$ one has

$$\binom{k+1-i}{i} = \sum_{j=\epsilon}^{k} \binom{j-i}{i-1},$$

where ϵ is the minimal solution of the inequality $\left[\frac{j+1}{2}\right] \geq i$ (with $i \geq 1$ fixed). Up to an index shift the last identity coincides with the classical

$$\binom{m+1}{i} = \sum_{j=i-1}^{m} \binom{j}{i-1}.$$

In order to prove formula (4) for the generating function $\Psi_k(t)$ note that any sequence satisfying the recurrence (5) has a generating function of the form

$$\frac{t^l Q(t)}{\sum_{i=0}^{\left[\frac{k+1}{2}\right]}(-1)^i\binom{k+1-i}{i}t^i}$$

where $l \geq 0$ and $Q(t)$ is a polynomial with a nonvanishing constant term and of degree at most $\left[\frac{k+1}{2}\right]$, see e.g. [St], ch. 4. A more detailed consideration in our situation shows that the recurrence relation (5) is actually valid for all $n > k \geq 1$ (and not just for sufficiently big n) if we additionally set $\flat_{k,n} = 0$ for $k > n$ and $\flat_{k,k} = 1$. Thus $Q(t)$ has to be equal to 1 since otherwise the recurrence relation (5) should be wrong for some value of n satisfying $k < n < 2k$. This gives exactly formula (4). $\square$

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