ON THE HOMOTOPY TYPE OF INTERSECTION OF TWO REAL BRUHAT CELLS. I

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To Andrei Zelevinsky who left us far too early

ABSTRACT. In this paper, we continue the line of research initiated in [12, 13, 9, 10]. We introduce a new stratification of the intersection of two arbitrary top-dimensional Bruhat cells in SL_m/B over $\mathbb{R}$ and present new topological results about such intersection.

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1. Introduction

Bruhat/Schubert cell decompositions of Grassmannians and various spaces of flags have been used in mathematics for more than a century and are standard objects/tools in e.g. topology, enumerative geometry and representation theory. Intersections of pairs and more general collections of Bruhat cells appear naturally in several areas such as singularity theory, Kazhdan-Lusztig theory, matroid theory, to mention a few. In spite of their importance, to the best of our knowledge, there is hardly any topological information available about such intersections, see e.g. [14] and references therein.

One exception from this general situation is the problem of counting connected components in pairwise intersections of big (i.e. top-dimensional) Bruhat cells over the reals where substantial progress was obtained in the late 90's, see [12, 13, 15, 9, 10, 4, 16]. In short, this problem can be reduced to counting the orbits of a certain finite group of symplectic transvections acting on a finite-dimensional vector space over the two element field $\mathbb{F}_2$; both the group and the vector space are uniquely determined by the pair of Bruhat cells under consideration, see [15]. Further information about counting such orbits can be found in [11].

For example, in case of two opposite big Bruhat cells over the reals in the standard space of complete flags $\mathrm{Fl}_m = \mathrm{SL}_m/B$ where B is the Borel subgroup of $m \times m$ upper-triangular matrices, the number $\sharp_m$ of connected components in their intersection equals $2, 6, 20, 52$ for $m = 2, 3, 4, 5$ respectively. Starting from $m = 6$, the number of connected components stabilizes and is given by $\sharp_m = 3 \cdot 2^{m-1}$ which is explained by the possibility to embed, for $m \geq 6$, the lattice E_6 in a certain lattice arising in this problem, see [13].

Observe that the relative positions of two big Bruhat cells in Fl_m are in $1-1$-correspondence with permutations of length m, i.e. with the elements of the symmetric group S_m. In particular, opposite big Bruhat cells correspond to the longest permutation $\eta = (m, m-1, \dots, 1)$. The study of the number $\sharp_m(\sigma)$, $\sigma \in S_m$ of connected components in the intersection of two big cells in a given relative position σ was initiated in § 7 of [13]. For each concrete σ, the number $\sharp_m(\sigma)$ can, in principle, be deduced from the results of [11] obtained about two decades ago. However, to the best of our knowledge, there is no closed formula for $\sharp_m(\sigma)$ in terms of σ and this problem seems hard.

The main goal of the present paper and its sequel [1] is to introduce the appropriate tools which allow us to study the higher homotopy and homology groups of the latter intersections. In particular, we introduce a novel stratification of (every connected component of) an arbitrary pairwise intersection of big Bruhat cells over the reals and show that the dual CW-complex of this stratification is homotopy equivalent to the pairwise intersection under consideration. This stratification depends on a reduced decomposition of the permutation encoding the relative position of the cells. Although it is not a Whitney stratification meaning that the closure of a stratum is not necessarily a union of low dimensional strata, we can still extract from it important topological information using a certain partial order of the strata. We illustrate our stratification in several examples, consider in details the cases $m = 4$ and $m = 5$ and show that in these cases each connected component of any such pairwise intersection is contractible.

Starting with $m = 6$ the situation becomes more complicated and we postpone its study as well as further discussions of combinatorial and topological aspects of our stratification till [1]. Our technique heavily relies on the use of the spin group which is a double cover of the special orthogonal group which, in its turn, is a multiple cover of the space of complete flags over the reals. Besides that, in the present paper, we apply some of our technique to obtain new results about the number of connected components.

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2. Group-theoretical and other preliminaries

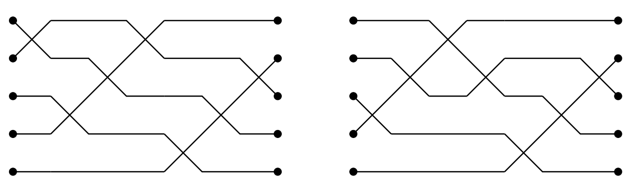
2.1. **Main notions.** In what follows, it will be convenient to shift the index by 1, i.e. to use $n = m - 1$. For instance, we have $\mathrm{Fl}_{n+1} := \mathrm{SL}_{n+1}/B = \mathrm{SL}_m/B$. For a permutation σ in the symmetric group S_{n+1}, define the *Bruhat cell* of σ in GL_{n+1} as

$$\mathrm{Bru}_\sigma^{\mathrm{GL}} = \{U_0 M_\sigma U_1;\ U_0, U_1 \in \mathrm{Up}_{n+1}\} \subset \mathrm{GL}_{n+1}\,. \tag{1}$$

Here GL_{n+1} is the group of all real invertible $(n+1)\times(n+1)$ matrices, $\mathrm{Up}_{n+1} \subset \mathrm{GL}_{n+1}$ is the subgroup of upper triangular matrices and M_σ is the permutation matrix defined by $e_i^\top M_\sigma = e_{i^\sigma}^\top$. This stratification of GL_{n+1} has been extensively studied, is well understood and has multiple applications.

The quotient space Fl_{n+1} is standardly interpreted as the space of complete flags in $\mathbb{R}^{n+1}$ (resp. $\mathbb{C}^{n+1}$) i.e., the space whose elements are sequences of enclosed subspaces of all dimensions from 1 to $n+1$. Fixing a complete flag $f \in \mathrm{Fl}_{n+1}$, we define the top-dimensional Bruhat cell $\mathrm{Bru}_f^{\mathrm{Fl}} \subset \mathrm{Fl}_{n+1}$ as the set of all complete flags whose subspaces are in general position with all subspaces of f. One can show that using the action of SL_{n+1} any two flags can be transformed into two coordinate flags. Therefore topology of intersection of two top-dimensional Bruhat cells is the same as for the coordinate Bruhat cells where the first one corresponds to the standard coordinate flag while the second one is an arbitrary coordinate flag f_σ. In fact, we can identify $\mathrm{Bru}_\eta^{\mathrm{Fl}}$ with Lo_{n+1}^1 and $\mathrm{Bru}_\sigma^{\mathrm{Fl}}$ with $Lo_{n+1}^1 \cap \mathrm{Bru}_\sigma^{\mathrm{GL}}$. REWRITE

Below we extensively use the standard Coxeter-Weyl generators $a_1, \dots, a_n$ of the symmetric group S_{n+1}; a_i being the simple transposition $(i, i+1)$. A *reduced word* (also called *reduced decomposition*) for $\sigma \in S_{n+1}$ is a product $\sigma = a_{i_1} \cdots a_{i_\ell}$ where $\ell = \mathrm{inv}(\sigma)$ is the length of σ in the generators a_i, $i \in [\![n]\!]$. A reduced word can be drawn as wiring diagram like the one shown in Figure 1. (Notice that there are different conventions in the literature; in our system, each crossing is a generator, read left-to-right.)



{fig:reduced} FIGURE 1. The words $a_1a_3a_2a_1a_4a_3a_2$ and $a_3a_2a_1a_2a_4a_3a_2$ are both reduced and represent the same permutation $\sigma = (4, 3, 5, 1, 2)$.

Consider the natural projection $\nu : \mathrm{SO}_{n+1} \to \mathrm{Fl}_{n+1}$ sending an orthogonal matrix o to the complete flag $\nu(o)$ whose i-dimensional space is spanned by the fist i columns of o, $i = 1, 2, \dots, n+1$. Recall that the spin group Spin_{n+1} is a double covering space of the special orthogonal group SO_{n+1}; we denote this covering map by $\Pi : \mathrm{Spin}_{n+1} \to \mathrm{SO}_{n+1}$. The composition map $\nu * \Pi : \mathrm{Spin}_{n+1} \to \mathrm{Fl}_{n+1}$ will be denoted by Θ.

Obviously, one can identify the Lie algebras $\mathfrak{spin}_{n+1} \simeq \mathfrak{so}_{n+1}$. A convenient family of generators of $\mathfrak{so}_{n+1}$ consists of $\mathfrak{a}_j$, $j \in [\![n]\!]$; the only nonzero entries of the skew-symmetric matrix $\mathfrak{a}_j$ being $(\mathfrak{a}_j)_{(j+1,j)} = 1$ and $(\mathfrak{a}_j)_{(j,j+1)} = -1$. Recall that $\mathfrak{a}_j \in \mathfrak{spin}_{n+1} = \mathfrak{so}_{n+1}$, $j \in [\![n]\!]$, is the skew-symmetric matrix whose only nonzero entries are $(\mathfrak{a}_j)_{(j+1,j)} = 1$ and $(\mathfrak{a}_j)_{(j,j+1)} = -1$.

Denote by $B_{n+1}^+ \subset \mathrm{SO}_{n+1}$ the finite group of signed permutation matrices with positive determinant (which is the corresponding Coxeter group) and set

$\tilde{B}^+_{n+1} := \Pi^{-1}[B^+_{n+1}] \subset \mathrm{Spin}_{n+1}$. We also have the "forgetful" group homomorphism $\mu : B^+_{n+1} \to S_{n+1}$ sending a signed permutation matrix to the corresponding usual permutation matrix, i.e., removing all the negative signs from the signed permutation matrix. We denote by $\Theta : \tilde{B}^+_{n+1} \to S_{n+1}$ the composition $\mu * \Pi : \tilde{B}^+_{n+1} \to B^+_{n+1} \to S_{n+1}$.

Let us introduce the following important elements of $\tilde{B}^+_{n+1}$:

$$\acute{a}_j = \exp\left(\frac{\pi}{2}\,\mathfrak{a}_j\right), \quad \grave{a}_j = (\acute{a}_j)^{-1}, \quad \hat{a}_j = (\acute{a}_j)^2. \tag{2}$$

Observe that $\Theta(\acute{a}_i) = \tilde{\Pi}(\grave{a}_i) = a_i$.

The next statement is straightforward.

Claim 2.1. *The n-tuple $(\acute{a}_j)$, $j \in [\![n]\!]$ is a system of generators of the group $\tilde{B}^+_{n+1}$. The kernel $\mathrm{Quat}_{n+1} \subset \tilde{B}^+_{n+1}$ of the group homomorphism $\Theta : \tilde{B}^+_{n+1} \to S_{n+1}$ is the subgroup generated by $\hat{a}_j$, $j \in [\![n]\!]$. Additionally, the following relations hold:*

$$(\hat{a}_i)^2 = -1, \quad \hat{a}_i\hat{a}_j = (-1)^{[|i-j|=1]}\hat{a}_j\hat{a}_i. \tag{3}$$

(Here we use the Iverson bracket notation so that $[|i-j|=1] = 1$ if $|i-j| = 1$ and $[|i-j|=1] = 0$ otherwise.)

One has the natural short exact sequence

$$1 \to \mathrm{Quat}_{n+1} \to \tilde{B}^+_{n+1} \to S_{n+1} \to 1.$$

For $n \geq 2$, the center $\mathbb{Z}(\mathrm{Quat}_{n+1})$ of Quat_{n+1} contains $\{\pm 1\}$ and the quotient $\mathrm{Quat}_{n+1}/\{\pm 1\})$ is isomorphic to $\{\pm 1\}^n$.

Recall the standard inclusion $\mathrm{Spin}_{n+1} \subset \mathrm{Cl}^0_{n+1} \subset \mathrm{Cl}_{n+1}$ of the spin group in (the even subalgebra of) the standard Clifford algebra, see [2, 5, 8]. (In what follows we will only use Cl^0_{n+1} and not the whole Cl_{n+1}. By a slight abuse of notation we call it the *Clifford algebra*). Namely, the spin group Spin_{n+1} is a subset of Cl^0_{n+1}, generated by the 1-parameter subgroups

$$\alpha_i : \mathbb{R} \to \mathrm{Spin}_{n+1}, \quad \alpha_i(\theta) = \exp(\theta\mathfrak{a}_i) = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{a}_i, \quad i \in [\![n]\!]. \tag{4}$$

The associative algebra Cl^0_{n+1} is a real vector space of dimension 2^n with a linear basis given by

$$\mathrm{HQuat}_{n+1} = \{1, \hat{a}_1, \hat{a}_2, \hat{a}_1\hat{a}_2, \hat{a}_3, \hat{a}_1\hat{a}_3, \hat{a}_2\hat{a}_3, \hat{a}_1\hat{a}_2\hat{a}_3, \ldots, \hat{a}_1\hat{a}_2\cdots\hat{a}_n\}.$$

The set $\mathrm{Quat}_{n+1} = \mathrm{HQuat}_{n+1} \sqcup (-\mathrm{HQuat}_{n+1}) \subset \mathrm{Spin}_{n+1}$ is a subgroup of Spin_{n+1} of cardinality 2^{n+1} generated by $\hat{a}_i$, $i \in [\![n]\!]$.

As an algebra, Cl^0_{n+1} is generated by the elements $\hat{a}_i$, $i \in [\![n]\!]$ and multiplication in Cl^0_{n+1} satisfies the relations (3). In Cl^0_{n+1} we have the relations

$$\mathfrak{a}_j = \frac{1}{2}\hat{a}_j,\ j \in [\![n]\!].$$

Additionally, in Cl^0_{n+1} relations (2) can be rewritten as

$$\acute{a}_i = \alpha_i\left(\frac{\pi}{2}\right) = \frac{1+\hat{a}_i}{\sqrt{2}}, \qquad \grave{a}_i = (\acute{a}_i)^{-1} = \alpha_i\left(-\frac{\pi}{2}\right) = \frac{1-\hat{a}_i}{\sqrt{2}}. \tag{5}$$

As the vector space, Cl^0_{n+1} is standardly equipped with the inner product $\langle .,. \rangle$ with respect to which its basis HQuat_{n+1} is orthonormal. Finally, for $z \in \mathrm{Cl}^0_{n+1}$, we define its *real part* as $\Re(z) := \langle z, 1\rangle$. Thus, for $z = \sum_{q \in \mathrm{HQuat}_{n+1}} c_q\, q$, one has $\Re(z) = c_1$.

Definition 2.2. Given $\sigma \in S_{n+1}$ and a reduced word $\sigma = a_{i_1}\cdots a_{i_\ell}$, define the mapping $P : \{\pm 1\}^{[\![\ell]\!]} \to \tilde{B}^+_{n+1}$ given by

$$P(\varepsilon) = (\acute{a}_{i_1})^{\varepsilon(1)} \cdots (\acute{a}_{i_\ell})^{\varepsilon(\ell)}, \tag{6}$$

where $\varepsilon : \{1,\dots,\ell\} \to \{+1,-1\}$ is any sign sequence.

It is easy to verify that, for any sign sequence $\varepsilon \in \{\pm 1\}^{[\![\ell]\!]}$, one has $\Theta(P(\varepsilon)) = \sigma$.

{section:Bz}

2.2. **The sets** Bru_z**.** Here we define a preliminary crude stratification of the sets $\mathrm{Bru}_\sigma \subset \mathrm{Lo}^1_{n+1}$ by similar Bruhat cells Bru_z, $z \in \tilde{B}^+_{n+1}$ where $\Theta(z) = \sigma$. (Our main stratification will be introduced later.)

In (1) we defined, for $\sigma \in S_{n+1}$, the Bruhat cells $\mathrm{Bru}^{\mathrm{GL}}_\sigma \subset \mathrm{GL}_{n+1}$. For the standard projection $\Pi : \mathrm{Spin}_{n+1} \to \mathrm{SO}_{n+1} \subset \mathrm{GL}_{n+1}$, define

$$\mathrm{Bru}^{[\mathrm{Spin}]}_\sigma := \Pi^{-1}[\mathrm{Bru}^{\mathrm{GL}}_\sigma \cap \mathrm{SO}_{N+1}] \subset \mathrm{Spin}_{n+1}\,.$$

(When the group is clear from the context, we will omit the superscript. For example, we can just write Bru_σ for $\mathrm{Bru}^{[\mathrm{Spin}]}_\sigma \subset \mathrm{Spin}_{n+1}$). In [5] it is proven that every $\mathrm{Bru}_\sigma \subset \mathrm{Spin}_{n+1}$ has precisely 2^{n+1} connected components, each one containing a single element of $\Theta^{-1}[\{\sigma\}] \subset \tilde{B}^+_{n+1}$. Using this fact, for each $z \in B^+_{n+1}$, $\sigma = \Theta(z)$, define $\mathrm{Bru}_z \subset \mathrm{Spin}_{n+1}$ as the connected component of Bru_σ containing z. Moreover, each Bru_z is a smooth submanifold diffeomorphic to $\mathbb{R}^{\mathrm{inv}(\Theta(z))}$, i.e., a cell of dimension $\mathrm{inv}(\Theta(z))$. These cells define a stratification

$$\mathrm{Spin}_{n+1} = \bigsqcup_{z \in \tilde{B}^+_{n+1}} \mathrm{Bru}_z\,. \tag{7}$$

Furthermore, for $L \in \mathrm{Lo}^1_{n+1}$, define $\mathbf{Q}(L) := Q \in \mathrm{SO}_{n+1}$ where $L = Q \cdot R$ with $R \in \mathrm{Up}^+_{n+1}$; here Up^+_{n+1} is the group of upper triangular matrices with positive diagonal entries. The smooth map $\mathbf{Q} : \mathrm{Lo}^1_{n+1} \to \mathrm{SO}_{n+1}$ is essentially given by the QR-decomposition, i.e., the Gram-Schmidt orthogonalization of matrices in

Lo_{n+1}^1. Lifting the above map to the spin group we define $\mathbf{Q} : \mathrm{Lo}_{n+1}^1 \to \mathrm{Spin}_{n+1}$ with $\mathbf{Q}(I) = 1$. Let $\mathcal{U}_1 \subset \mathrm{Spin}_{n+1}$ be the image of the map $\mathbf{Q}$ which is an open contractible neighborhood of the unit element $1 \in \mathrm{Spin}_{n+1}$.

Definition 2.3. The stratification of the spin group given by (7) allows us, for each $z \in \tilde{B}_{n+1}^+$, to define the (possible empty) set

:countBsigma}

$$\mathrm{Bru}_z = \mathbf{Q}^{-1}[\mathrm{Bru}_z] \subset \mathrm{Lo}_{n+1}^1 .$$

Lemma 2.4. *For any permutation $\sigma \in S_{n+1}$, the manifold Bru_σ decomposes as a disjoint union:*

$$\mathrm{Bru}_\sigma = \bigsqcup_{z \in \Theta^{-1}[\{\sigma\}]} \mathrm{Bru}_z \tag{8}$$

of submanifolds in the group Lo_{n+1}^1. In particular, the number of connected components of Bru_σ is the sum of the number of connected components of Bru_z where z runs over $\Theta^{-1}[\{\sigma\}]$.

Proof. This follows directly from the definitions and the fact that each set Bru_z is either empty or a smooth submanifold of dimension $\mathrm{inv}(\Theta(z))$. □

Let $\mathfrak{lo}_{n+1}^1$ be the Lie algebra of Lo_{n+1}^1, i.e., the set of strictly lower triangular matrices. For $j \in [\![n]\!]$, let $\mathfrak{l}_j \in \mathfrak{lo}_{n+1}^1$ be the matrix whose only nonzero entry is $(\mathfrak{l}_j)_{j+1,j} = 1$. Denote $\lambda_j(t) = \exp(t\mathfrak{l}_j)$.

Given a reduced word $\sigma = a_{i_1} \cdots a_{i_\ell} \in S_{n+1}$ where $\ell = \mathrm{inv}(\sigma)$, consider the product

$$L = \lambda_{i_1}(t_1) \cdots \lambda_{i_\ell}(t_\ell). \tag{9}$$

It is well known that $L \in \mathrm{Bru}_\sigma$ and that if $L \in \mathrm{Bru}_\sigma$ can be written as in (9) then the vector $(t_1, \ldots, t_\ell)$ is unique, see [3]. Also, for almost all $L \in \mathrm{Bru}_\sigma$, there exists a vector $(t_1, \ldots, t_\ell) \in (\mathbb{R} \smallsetminus \{0\})^\ell$ for which (9) holds. Now, for a sign sequence $\varepsilon \in \{\pm 1\}^\ell$, define

$$B_\varepsilon = \{\lambda_{i_1}(t_1) \cdots \lambda_{i_\ell}(t_\ell);\ t_j \in \mathbb{R} \smallsetminus \{0\}, \mathrm{sign}(t_j) = \varepsilon(j)\} \subseteq \mathrm{Bru}_\sigma \subset \mathrm{Lo}_{n+1}^1 . \tag{10}$$

Clearly, B_ε is open in Bru_σ. Corollary 6.5 from [5] implies that $B_\varepsilon \subseteq \mathrm{Bru}_z$, $z = P(\varepsilon)$.

3. Introducing stratification of Bru_σ: first examples

stratexample}

In the next few sections we define the most essential construction of the article which is a certain stratification of the sets Bru_σ and $\mathrm{Bru}_z \subset \mathrm{Bru}_\sigma$, for $z \in \acute{\sigma}\,\mathrm{Quat}_{n+1}$. WHY NOT IN $\Theta^{-1}(\sigma)$? (This construction is in many ways similar to the one presented in [6].)

Remark 3.1. It is important to mention right away that

(i) our stratification depends on the choice of a reduced word for σ;

(ii) our stratification is not a Whitney stratification, i.e., the closure of a stratum is not necessarily the union of some strata of lower dimension, see example below??? $\diamond$

In this section we start with a few preliminary notations and examples.

ion:directlabel} Fix a reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$ where $\ell = \mathrm{inv}(\sigma)$.

Definition 3.2. A sequence $\varepsilon \in \{\pm1, \pm2\}^{[\![\ell]\!]}$ is called a valid *ε-label* (or just a *label*) if it allows us to define an associated sequence of permutations $(\rho_k)_{0 \leq k \leq \ell}$, $\rho_k \in S_{n+1}$, satisfying the conditions:

(1) $\rho_0 = \rho_\ell = \eta = (n+1, n, \ldots, 1)$;
(2) If $|\varepsilon(k)| = 1$ then $\rho_k = \rho_{k-1}$;
(3) If $\varepsilon(k) = -2$ then $\rho_k < \rho_{k-1} = \rho_k a_{i_k}$;
(4) If $\varepsilon(k) = +2$ then $\rho_{k-1} < \rho_k = \rho_{k-1} a_{i_k}$;
(5) If $\rho_{k-1} < \rho_{k-1} a_{i_k}$ then $\varepsilon(k) = +2$.

(The partial order under consideration is the standard Bruhat order.)

For a valid label ε, we define its codimension as

$$d = \mathrm{codim}(\varepsilon) = |\{k \mid \varepsilon(k) = -2\}| = |\{k \mid \varepsilon(k) = +2\}|.$$

Next we extend the definition of the mapping P to ε-labels as:

$$P(\varepsilon) = (\acute{a}_{i_1})^{\mathrm{sign}(\varepsilon(1))} \cdots (\acute{a}_{i_\ell})^{\mathrm{sign}(\varepsilon(\ell))} \in \acute{\sigma}\, \mathrm{Quat}_{n+1}\,. \tag{11}$$

For each label ε, we will later define the stratum $B_\varepsilon \subset B_{P(\varepsilon)} \subseteq \mathrm{Bru}_\sigma$ and prove that B_ε is a non-empty smooth contractible submanifold of codimension $d = \mathrm{codim}(\varepsilon)$. Moreover, we will show that for distinct valid ϵ-labels, the respective strata are disjoint and that their union over all valid ϵ-labels is the whole Bru_σ.

To start with, if ε is a valid label of codimension 0, then, by definition,

$$B_\varepsilon := \{\lambda_{i_1}(t_1) \cdots \lambda_{i_\ell}(t_\ell),\ \mathrm{sign}(t_k) = \varepsilon(k)\}, \quad \lambda_j(t) = \exp(t\mathfrak{l}_j),$$

where $\mathfrak{l}_j$ is the $(n+1)\times(n+1)$-matrix whose only non-zero entry is $(\mathfrak{l}_j)_{j+1,j} = 1$. After a couple of simple examples we describe how, given $L \in \mathrm{Bru}_\sigma$, we determine
{example:aba} the label ε such that $L \in B_\varepsilon$.

Example 3.3. Take $n = 2$ and $\eta = a_1 a_2 a_1 = (3, 2, 1) \in S_3$. We have

$$\mathrm{Lo}_3^1 = \left\{ L = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}; x, y, z \in \mathbb{R} \right\},$$

$\mathrm{Bru}_\eta = \{L \mid z \neq 0, z \neq xy\} \subset \mathrm{Lo}_3^1\,.$ $\qquad B_{(1-\hat{a}_1\hat{a}_2)/\sqrt{2}} = \{L \mid z > \max\{0, xy\}\} \subset \mathrm{Bru}_\eta\,.$

The ten valid labels are:

$$(\pm 1, \pm 1, \pm 1), \quad d = 0; \qquad (-2, +1, +2), (-2, -1, +2), \quad d = 1.$$

For two labels with $d = 1$ we have $\rho_1 = \rho_2 = a_1 a_2$. Notice that the three sequences $(1, -1, -1)$, $(-1, 1, 1)$ and $(-2, 1, 2)$ are the only ones with $P(\varepsilon) = \acute{a}_1 \grave{a}_2 \grave{a}_1 = (1 - \hat{a}_1 \hat{a}_2)/\sqrt{2}$.

A simple computation gives

$$\lambda_1(t_1)\lambda_2(t_2)\lambda_1(t_3) = \begin{pmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2 t_3 & t_2 & 1 \end{pmatrix}$$

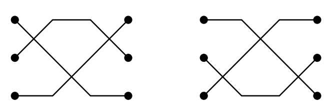
and therefore

$$\begin{aligned} B_{(-1,+1,+1)} &= \{L \mid z > \max\{0, xy\}, y > 0\}, \\ B_{(+1,-1,-1)} &= \{L \mid z > \max\{0, xy\}, y < 0\} \end{aligned}$$

are both contained in $B_{(1-\hat{a}_1\hat{a}_2)/\sqrt{2}}$. As we shall see later

$$\begin{aligned} B_{(-2,+1,+2)} &= \{L \mid y = 0, z > 0\}, \\ B_{(1-\hat{a}_1\hat{a}_2)/\sqrt{2}} &= B_{(-1,+1,+1)} \sqcup B_{(-2,+1,+2)} \sqcup B_{(+1,-1,-1)}. \end{aligned}$$

The subset $B_{(-2,+1,+2)} \subset B_{(1-\hat{a}_1\hat{a}_2)/\sqrt{2}}$ is a contractible submanifold of codimension $d = 1$. A similar decomposition holds for $B_{(1+\hat{a}_1\hat{a}_2)/\sqrt{2}}$.



{fig:aba-bab} FIGURE 2. Two reduced words for $\eta = a_1 a_2 a_1 = a_2 a_1 a_2 \in S_3$.

As shown in Figure 2, the permutation η also admits the reduced word $\eta = a_2 a_1 a_2$. A similar decomposition exists for the other reduced word, but the strata are different. For this other word, the set $B_{(1-\hat{a}_1\hat{a}_2)/\sqrt{2}}$ contains two open strata:

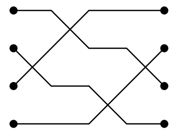
$$\begin{aligned} B_{(+1,+1,-1)} &= \{L \mid z > \max\{0, xy\}, x > 0\}, \\ B_{(-1,-1,+1)} &= \{L \mid z > \max\{0, xy\}, x < 0\} \end{aligned}$$

and a third stratum of codimension 1:

$$B_{(-2,-1,+2)} = \{L \mid z > \max\{0, xy\}, x = 0\}.$$

Notice that the meaning of the label depends on the choice of the used reduced word. ◇

example:bacb} **Example 3.4.** Take $n = 3$ and $\sigma = a_2 a_1 a_3 a_2 = (3, 4, 1, 2) \in S_4$. Up to transposing adjacent commuting generators, as in $a_2 a_1 a_3 a_2 = a_2 a_3 a_1 a_2$, the permutation σ admits only one reduced word, shown in Figure 3.



{fig:bacb} FIGURE 3. The permutation $a_2a_1a_3a_2 \in S_4$.

The 20 valid sequences are

$$(\pm1, \pm1, \pm1, \pm1), \quad d = 0; \qquad (-2, \pm1, \pm1, +2), \quad d = 1.$$

For the four valid sequences with $d = 1$, we have $\rho_1 = \rho_2 = \rho_3 = a_1a_2a_3a_2a_1$. The three valid sequences with $P(\varepsilon) = -\acute{\sigma}\hat{a}_3 = (1 - \hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_3 - \hat{a}_2\hat{a}_3)/2$ are $(+1, +1, -1, -1)$, $(-1, -1, +1, +1)$ and $(-2, -1, +1, +2)$.

Write $L \in \mathrm{Lo}_4^1$ as

$$\mathrm{Lo}_4^1 = \left\{ L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ u & y & 1 & 0 \\ w & v & z & 1 \end{pmatrix}, u, v, w, x, y, z \in \mathbb{R} \right\}.$$

By applying the definition, the set Bru_σ is

$$\mathrm{Bru}_\sigma = \{L \mid w = 0, u \neq 0, v \neq 0, xyz = xv + zu\}.$$

If $L = \lambda_2(t_1)\lambda_1(t_2)\lambda_3(t_3)\lambda_2(t_4)$ then

$$u = t_1t_2, \quad v = t_3t_4, \quad w = 0, \quad x = t_2, \quad y = t_1 + t_4, \quad z = t_3.$$

Let $U \subset (0, +\infty)^2 \times \mathbb{R}^2$ (with coordinates (u, v, x, y)) be the contractible open set defined by $xy < u$. Consider the map $\Phi : U \to \mathrm{Bru}_\sigma$ taking (u, v, x, y) to the matrix $L \in \mathrm{Bru}_\sigma$ with the prescribed values of u, v, x, y and $z = xv/(xy - u)$.

If $x > 0$ we have $L = \Phi(u, v, x, y) \in B_{(+1,+1,-1,-1)}$; if $x < 0$ we have $L = \Phi(u, v, x, y) \in B_{(-1,-1,+1,+1)}$. We will see that

$$B_{(-2,-1,+1,+2)} = \Phi[(0, +\infty)^2 \times \{0\} \times \mathbb{R}],$$
$$B_{-\acute{\sigma}\hat{a}_3} = B_{(-1,-1,+1,+1)} \sqcup B_{(-2,-1,+1,+2)} \sqcup B_{(-1,-1,+1,+1)}.$$

It follows that $B_{-\acute{\sigma}\hat{a}_3}$ is contractible. Similarly, all connected components of Bru_σ are contractible. $\diamond$

4. ϵ-LABELS VERSUS ξ-LABELS

{section:label}

Again consider a fixed permutation $\sigma \in S_{n+1}$ and a fixed reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$, $\ell = \mathrm{inv}(\sigma)$. In Definition 3.2 we explained how, for the given reduced word for σ, to define valid ε-labels which are certain sequences $\varepsilon \in \{\pm1, \pm2\}^{[\![\ell]\!]}$. In particular, given a valid label ε, we have a sequence $(\rho_k)_{0 \leq k \leq \ell}$ of permutations with the following properties:

(1) $\rho_0 = \rho_\ell = \eta$;
(2) for any k, either $\rho_k = \rho_{k-1}$ or $\rho_k = \rho_{k-1} a_{i_k}$;
(3) if $\rho_{k-1} < \rho_{k-1} a_{i_k}$ then $\rho_k = \rho_{k-1} a_{i_k}$.

Conversely, given a sequence (ρ_k) with the above properties, consider a sequence $\varepsilon \in \{\pm 1, \pm 2\}^{[\![\ell]\!]}$ such that, for all k,

$$\rho_k < \rho_{k-1} \to \varepsilon(k) = -2, \quad \rho_k > \rho_{k-1} \to \varepsilon(k) = +2, \quad \rho_k = \rho_{k-1} \to |\varepsilon(k)| = 1.$$

One can easily check that ε is a valid label. Furthermore, there are $2^{\ell - 2d}$ such labels, all of dimension $d = |\{k \mid \rho_k < \rho_{k-1}\}|$.

Let us now modify the sequence (ρ_k) to define a ξ-*label*. It will turn out that
inition:dual} a ξ-label contains the same information as a ε-label.

Definition 4.1. A valid ξ-*label* is a sequence $\xi \in \{0, 1, 2\}^{[\![\ell]\!]}$ such that the sequence $\varrho = (\varrho_k)_{0 \le k \le \ell}$ of elements of $\tilde{B}^+_{n+1}$ recursively defined by

$$\varrho_0 = \acute{\eta}, \qquad \varrho_k = \varrho_{k-1} (\acute{a}_{i_k})^{\xi(k)}, \tag{12}$$

has the following properties:

(1) $\varrho_0 = \acute{\eta}$ and $\Theta(\varrho_\ell) = \eta$;
(2) if $\Theta(\varrho_{k-1}) < \Theta(\varrho_{k-1}) a_{i_k}$ then $\xi(k) = 1$.

As above, the partial order is the (strong) Bruhat order. Notice that if ξ is a valid label and (ϱ_k) is defined by (12) then the sequence (ρ_k) defined by $\rho_k = \Pi(\varrho_k)$ satisfies the conditions in the beginning of this section.

Let us now describe a bijection between ε-labels and ξ-labels. The definition of this bijection is recursive in k. The sequence $\varrho = (\varrho_k)$ of elements of $\tilde{B}^+_{n+1}$ is defined by (12). The sequence $(\rho_k)_{0 \le k \le \ell}$ of permutations and the sequence $(q_k)_{0 \le k \le \ell}$ of elements of Quat_{n+1} can be obtained from (ϱ_k) via the relations $\rho_k = \Pi(\varrho_k)$ and $q_k = (\acute{\rho}_k)^{-1} \varrho_k$. The recursive definition is:

$$\xi(k) = \begin{cases} 0, & \varepsilon(k) = [\hat{a}_{i_k}, q_{k-1}], \\ 2, & \varepsilon(k) = -[\hat{a}_{i_k}, q_{k-1}], \\ 1, & |\varepsilon(k)| = 2. \end{cases} \tag{13}$$

Here $[\hat{a}_{i_k}, q_{k-1}] = (\hat{a}_{i_k})^{-1} q_{k-1}^{-1} \hat{a}_{i_k} q_{k-1} \in \{\pm 1\}$ is the commutator in the group-theoretical sense.

Conversely, given a valid label ξ, consider ϱ_k, $\rho_k = \Pi(\varrho_k)$ and $q_k = (\acute{\rho}_k)^{-1} \varrho_k$ as above. We then define

$$\varepsilon(k) = \begin{cases} -2, & \xi(k) = 1, \ \rho_k < \rho_{k-1}, \\ +2, & \xi(k) = 1, \ \rho_k > \rho_{k-1}, \\ (1 - \xi(k))[\hat{a}_{i_k}, q_{k-1}], & \xi(k) \ne 1. \end{cases} \tag{14}$$

{lemma:xlabel}

Lemma 4.2. *Given a permutation $\sigma \in S_{n+1}$ and a fixed reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$, $\ell = \operatorname{inv}(\sigma)$, one has a bijection and its inverse from the set of ε-labels to the set of ξ-labels defined by formulas* (13) *and* (14)*. Furthermore, if ξ and ε correspond to each other, then $P(\varepsilon) = \acute{\sigma} q_\ell^{-1}$.*

Proof. Compute, compute, and compute. □

We write $P(\xi) = P(\varepsilon)$. The codimension of a label ξ equals $d = \operatorname{codim}(\xi) = \frac{1}{2}|\{k \mid \xi(k) = 1\}|$, and therefore equal to $\operatorname{codim}(\varepsilon)$ where ε is the label corresponding to ξ.

The Bruhat order in S_{n+1} can be defined by

$$\sigma_0 \leq \sigma_1 \quad \Longleftrightarrow \quad \mathrm{Bru}_{\sigma_0} \subseteq \overline{\mathrm{Bru}_{\sigma_1}}.$$

We extend it to a partial order in the group $\tilde{B}^+_{n+1}$, which we also call the *Bruhat order*, by

$$z_0 \leq z_1 \quad \Longleftrightarrow \quad \mathrm{Bru}_{z_0} \subseteq \overline{\mathrm{Bru}_{z_1}}.$$

As in S_{n+1}, the above condition is equivalent to $\mathrm{Bru}_{z_0} \cap \overline{\mathrm{Bru}_{z_1}} \neq \{\emptyset\}$. Clearly, $z_0 \leq z_1$ implies that $\tilde{\Pi}(z_0) \leq \Theta(z_1)$ (where $\Theta : \tilde{B}^+_{n+1} \to S_{n+1}$ is the usual quotient map), but the converse does not hold. For instance, if $\Theta(z_0) = \tilde{\Pi}(z_1)$ then $z_0 \leq z_1$ if and only if $z_0 = z_1$.

We now define a partial order on the sets of ε- and ξ-labels. Given two dual labels ξ and $\tilde{\xi}$, consider the corresponding valid labels ε and $\tilde{\varepsilon}$ as in (13) and (14). Let (ϱ_k) and $(\tilde{\varrho}_k)$ be defined by (12). We define

$$\xi \succeq \tilde{\xi} \quad \Longleftrightarrow \quad \varepsilon \succeq \tilde{\varepsilon} \quad \Longleftrightarrow \quad (\forall k, \varrho_k \geq \tilde{\varrho}_k). \tag{15}$$

Notice that $\varepsilon \succeq \tilde{\varepsilon}$ implies $P(\varepsilon) = P(\tilde{\varepsilon})$. The fact that this is a partial order is straightforward.

:stratification}

5. A stratification of Bru_σ: formal definition

For a fixed permutation and reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$, define recursively $\sigma_0 = 1$, $\sigma_1 = a_{i_1}$, $\sigma_k = \sigma_{k-1} a_{i_k} = a_{i_1} \cdots a_{i_k}$ so that $\sigma = \sigma_\ell$. Recall that $\alpha_j : \mathbb{R} \to \mathrm{Spin}_{n+1}$ are homomorphisms defined in (4). Theorem 1 from [5] claims that, for $z_k \in \mathrm{Bru}_{\acute{\sigma}_k} \subset \mathrm{Spin}_{n+1}$, there exist unique $z_{k-1} \in \mathrm{Bru}_{\acute{\sigma}_{k-1}}$ and $\theta_k \in (0, \pi)$ such that $z_k = z_{k-1} \alpha_{i_k}(\theta_k)$.

Thus, given $z_\ell \in \mathrm{Bru}_{\acute{\sigma}_\ell}$, we have well-defined sequences $(\theta_k)_{0<k\leq\ell}$ and $(z_k)_{0\leq k\leq\ell}$, $z_k \in \mathrm{Bru}_{\acute{\sigma}_k}$. The mapping

$$(0, \pi)^\ell \to \mathrm{Bru}_{\acute{\sigma}_\ell}, \quad (\theta_1, \dots, \theta_\ell) \mapsto \alpha_{i_1}(\theta_1) \cdots \alpha_{i_\ell}(\theta_\ell), \tag{16}$$

is a diffeomorphism. Similarly, the functions $\mathrm{Bru}_{\acute{\sigma}_\ell} \to \mathrm{Bru}_{\acute{\sigma}_k}$, $z_\ell \mapsto z_k$, are smooth submersions.

Recall that the maps $\mathbf{Q}: \mathrm{Lo}^1_{n+1} \to \mathcal{U}_1 \subset \mathrm{Spin}_{n+1}$ and $\mathbf{L}: \mathcal{U}_1 \to \mathrm{Lo}^1_{n+1}$ are diffeomorphisms. The set $\mathcal{U}_1 := \grave{\eta}\,\mathrm{Bru}_{\acute{\eta}} \subset \mathrm{Spin}_{n+1}$ is an open contractible neighborhood of $1 \in \mathrm{Spin}_{n+1}$. We identify $L \in \mathrm{Bru}_\sigma$ with

$$\tilde{z}_\ell = \mathbf{Q}(L) \in \mathrm{Bru}_\sigma \cap \mathcal{U}_1 \subset \mathrm{Spin}_{n+1}\,.$$

There exist unique $z_\ell \in \mathrm{Bru}_{\acute{\sigma}}$ and $q_\ell \in \mathrm{Quat}_{n+1}$ with $z_\ell = \tilde{z}_\ell q_\ell$. Moreover, if $L \in \mathrm{Bru}_z$ (with $z \in \tilde{B}^+_{n+1}$) then, by definition, $\tilde{z}_\ell \in \mathrm{Bru}_z$ and therefore $z_\ell = \tilde{z}_\ell q_\ell \in \mathrm{Bru}_{zq_\ell}$ and therefore $zq_\ell = \acute{\sigma}$. Summing up, q_ℓ is characterized by $L \in B_{\acute{\sigma}q_\ell^{-1}}$, so that the notation is consistent with Lemma 4.2.

Use the diffeomorphism in (16) to define $(\theta_k)_{k\le\ell}$, $\theta_k \in (0,\pi)$, such that $z_\ell = \alpha_{i_1}(\theta_1)\cdots\alpha_{i_\ell}(\theta_\ell)$. Recursively define

$$z_0 = 1, \quad z_k = z_{k-1}\alpha_{i_k}(\theta_k) \in \mathrm{Bru}_{\acute{\sigma}_k}\,. \tag{17}$$

Notice that the functions $\mathrm{Bru}_\sigma \to (0,\pi)$, $L \mapsto \theta_k$, are smooth. The functions $\mathrm{Bru}_\sigma \to \mathrm{Bru}_{\acute{\sigma}_k}$, $L \mapsto z_k$, are also smooth.

The sequence (ϱ_k) corresponding to L is defined by $z_k \in \grave{\eta}\,\mathrm{Bru}_{\varrho_k}$. The Bruhat stratification of Spin_{n+1} shows that $\varrho_k \in \tilde{B}^+_{n+1}$ is well-defined; we proceed to show consistency with the notation of the previous section.

lemma:varrho}

Lemma 5.1. *Given $L \in \mathrm{Bru}_\sigma$, define sequences (z_k) and (ϱ_k) as above. There exists a unique label ξ such that (12) obtains (ϱ_k) from ξ.*

Proof. Using Theorem 1 from [5], we proceed by induction on k. We have $z_0 = 1 \in \grave{\eta}\,\mathrm{Bru}_{\acute{\eta}}$ and therefore $\varrho_0 = \acute{\eta}$. Assume that $z_{k-1} \in \grave{\eta}\,\mathrm{Bru}_{\varrho_{k-1}}$ and let $\rho_{k-1} = \Pi(\varrho_{k-1})$.

If $\rho_{k-1} < \rho_{k-1}a_{i_k}$ then $z_{k-1}\alpha_{i_k}(\theta) \in \mathrm{Bru}_{\varrho_{k-1}\acute{a}_{i_k}}$ for any $\theta \in (0,\pi)$. It follows that $z_k = z_{k-1}\alpha_{i_k}(\theta_k) \in \mathrm{Bru}_{\varrho_k}$ for $\varrho_k = \varrho_{k-1}\acute{a}_{i_k}$, $\xi(k) = +1$, as desired.

If $\rho_{k-1} > \rho_\star = \rho_{k-1}a_{i_k}$, let $\varrho_\star = \varrho_{k-1}\grave{a}_{i_k}$. Given z_{k-1}, there exists a unique $\theta_\star \in (0,\pi)$ such that $z_{k-1}\alpha_{i_k}(-\theta_\star) \in \mathrm{Bru}_{\varrho_\star}$. If $\theta_k \in (0,\pi-\theta_\star)$ then $z_k = z_{k-1}\alpha_{i_k}(\theta_k) \in \mathrm{Bru}_{\varrho_k}$ for $\varrho_k = \varrho_{k-1}$, $\xi(k) = 0$. If $\theta_k \in (\pi-\theta_\star,\pi)$ then $z_k = z_{k-1}\alpha_{i_k}(\theta_k) \in \mathrm{Bru}_{\varrho_k}$ for $\varrho_k = \varrho_{k-1}\hat{a}_{i_k}$, $\xi(k) = 2$. Finally, if $\theta_k = \pi - \theta_\star$ then $z_k = z_{k-1}\alpha_{i_k}(\theta_k) \in \mathrm{Bru}_{\varrho_k}$ for $\varrho_k = \varrho_{k-1}\acute{a}_{i_k}$, $\xi(k) = 1$. ☐

Given a label ξ, let B_ξ be the set of matrices $L \in \mathrm{Bru}_\sigma$ with corresponding label ξ. Thus, by Lemmas 4.2 and 5.1,

$$\mathrm{Bru}_\sigma = \bigsqcup_\xi B_\xi, \qquad \mathrm{Bru}_z = \bigsqcup_{P(\xi)=z} B_\xi.$$

In the next section we shall prove that the sets B_ξ are diffeomorphic to open balls. First, however, we provide an alternative description, using ε-labels. Let

$$\mathcal{U}_1^\diamond = \bigsqcup_{\sigma\in S_{n+1}} \grave{\eta}\,\mathrm{Bru}_{\acute{\sigma}}, \qquad \mathcal{U}_1 \subset \mathcal{U}_1^\diamond \subset \overline{\mathcal{U}_1} \subset \mathrm{Spin}_{n+1}\,.$$

The set $\mathcal{U}_1^\diamond$ is a fundamental domain for the action of Quat_{n+1} on Spin_{n+1}: given $z \in \mathrm{Spin}_{n+1}$ there exists a unique $q \in \mathrm{Quat}_{n+1}$ such that $zq \in \mathcal{U}_1^\diamond$ [5]. As above, we have $z_k \in \grave{\eta}\,\mathrm{Bru}_{\varrho_k}$. Set $\rho_k = \Pi(\varrho_k)$ and $q_k = (\acute{\rho}_k)^{-1}\varrho_k$ so that

$$z_k q_k^{-1} \in \grave{\eta}\,\mathrm{Bru}_{\varrho_k} q_k^{-1} = \grave{\eta}\,\mathrm{Bru}_{\acute{\rho}_k} \subset \mathcal{U}_1^\diamond.$$

emma:tildetheta} We therefore define $\tilde{z}_k \in \mathcal{U}_1^\diamond$ by $z_k = \tilde{z}_k q_k$.

Lemma 5.2. *There exist unique $\tilde{\theta}_k \in (-\pi, 0) \cup (0, \pi)$ such that $\tilde{z}_k = \tilde{z}_{k-1}\alpha_{i_k}(\tilde{\theta}_k)$. Furthermore, $\tilde{\theta}_k = \pm\theta_k$ or $\tilde{\theta}_k = \pm(\pi - \theta_k)$.*

Proof. Assume $z_{k-1} = \tilde{z}_{k-1}q_{k-1}$ and $z_k = \tilde{z}_k q_k$ with $z_{k-1} \in \mathrm{Bru}_{\acute{\sigma}_{k-1}}$, $z_k \in \mathrm{Bru}_{\acute{\sigma}_k}$, $q_{k-1}, q_k \in \mathrm{Quat}_{n+1}$ and $\tilde{z}_{k-1}, \tilde{z}_k \in \mathcal{U}_1^\diamond$. Assume furthermore that $z_k = z_{k-1}\alpha_{i_k}(\theta_k)$, $\theta_k \in (0, \pi)$. We thus have $\tilde{z}_k = \tilde{z}_{k-1}q_{k-1}\alpha_{i_k}(\theta_k)q_k^{-1}$. Since $\mathfrak{a}_{i_k}, q_{k-1} \in \mathrm{Quat}_{n+1}$ we either have $q_{k-1}\mathfrak{a}_{i_k} = \mathfrak{a}_{i_k}q_{k-1}$ or $q_{k-1}\mathfrak{a}_{i_k} = (-\mathfrak{a}_{i_k})q_{k-1}$. In the either case there exists $\varepsilon_\star \in \{\pm 1\}$ such that $q_{k-1}\alpha_{i_k}(\theta) = \alpha_{i_k}(\varepsilon_\star\theta)q_{k-1}$ for all $\theta \in \mathbb{R}$. We thus have

$$\tilde{z}_k = \tilde{z}_{k-1}\alpha_{i_k}(\varepsilon_\star\theta_k)(q_{k-1}q_k^{-1}) \in \mathcal{U}_1^\diamond. \tag{18}$$

Since $\tilde{z}_{k-1} \in \mathcal{U}_1^\diamond$ there exists a permutation ρ_{k-1} such that $\tilde{z}_{k-1} \in \grave{\eta}\,\mathrm{Bru}_{\acute{\rho}_{k-1}}$. Let $\rho_\star = \rho_{k-1}a_{i_k}$. We either have $\rho_{k-1} < \rho_\star$ or $\rho_{k-1} > \rho_\star$; we consider the two cases separately.

If $\rho_{k-1} < \rho_\star$ then $\tilde{z}_{k-1}\alpha_{i_k}(\theta) \in \grave{\eta}\,\mathrm{Bru}_{\acute{\rho}_\star}$ for all $\theta \in (0, \pi)$. Thus, for all $\theta \in (0, \pi)$ and $q \in \mathrm{Quat}_{n+1}$ we have $\tilde{z}_{k-1}\alpha_{i_k}(\theta)q \in \grave{\eta}\,\mathrm{Bru}_{\acute{\rho}_\star q}$. We therefore have $\tilde{z}_{k-1}\alpha_{i_k}(\theta)q \in \mathcal{U}_1^\diamond$ if and only if $q = 1$ (assuming $\theta \in (0, \pi)$). If $\varepsilon_\star = +1$, it follows from (18) that $\tilde{\theta}_k = \theta_k$ and $q_{k-1}q_k^{-1} = 1$. If $\varepsilon_\star = -1$, we have $\tilde{\theta}_k = \pi - \theta_k$ and $q_{k-1}q_k^{-1} = -\hat{a}_{i_k}$.

On the other hand, if $\rho_{k-1} > \rho_\star$ there exists a unique $\vartheta \in (0, \pi)$ such that $\tilde{z}_{k-1}\alpha_{i_k}(-\vartheta) \in \grave{\eta}\,\mathrm{Bru}_{\acute{\rho}_\star}$. For $\theta \in [-\vartheta, \pi - \vartheta)$ and $q \in \mathrm{Quat}_{n+1}$, we have $\tilde{z}_{k-1}\alpha_{i_k}(\theta)q \in \mathcal{U}_1^\diamond q$. If $\varepsilon_\star = +1$, it follows from (18) that we either have $\tilde{\theta}_k = \theta_k \in (0, \pi - \vartheta)$ and $q_{k-1}q_k^{-1} = 1$ or $\theta_k \in [\pi - \vartheta, \pi)$, $\tilde{\theta}_k = \theta_k - \pi \in [-\vartheta, 0)$ and $q_{k-1}q_k^{-1} = -\hat{a}_{i_k}$. If
mark:tildetheta} $\varepsilon_\star = -1$, ... This completes the proof of the claim and of the lemma. □

Remark 5.3. The case $\rho_{k-1} < \rho_\star$ corresponds to the label $+2$. For $\rho_{k-1} > \rho_\star$, the case $\tilde{\theta}_k = \theta_k \in (0, \pi - \vartheta)$ corresponds to the label $+1$, $\theta_k = \pi - \vartheta$ and $\tilde{\theta}_k = -\vartheta$ to -2 and $\theta_k \in (\pi - \vartheta, \pi)$ and $\tilde{\theta}_k = \theta_k - \pi \in (-\vartheta, 0)$ to -1. We give a different but related definition of labels below. ◇

The sequence of permutations $(\rho_k)_{0 \le k \le \ell}$ in the definition of ε-labels is defined by $\acute{\eta}\tilde{z}_k \in \mathrm{Bru}_{\acute{\rho}_k}$. Notice that $\rho_0 = \rho_\ell = \eta$. Finally, set

$$\varepsilon(k) = \mathrm{sign}(\tilde{\theta}_k)(1 + [\rho_k \ne \rho_{k-1}]).$$

It is easy to verify that this is indeed a label. The set $B_\varepsilon \subset \mathrm{Bru}_\sigma$ is defined to be the set of matrices $L \in \mathrm{Lo}_{n+1}^1$ with label equal to ε.

example:abaL}

Example 5.4. As in Example 3.3, set $n = 2$ and $\sigma = \eta = a_1a_2a_1$. Consider

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \tilde{z}_3 = \mathbf{Q}(L_0) = \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}.$$

(More correctly, the matrix shown is $\Pi(\tilde{z}_3) \in \mathrm{SO}_3$. Here and in other occasions it is easier to do computations in SO_{n+1} instead of Spin_{n+1}.)

We have $\tilde{z}_3 = \alpha_1(-\frac{\pi}{2})\alpha_2(\frac{\pi}{4})\alpha_1(\frac{\pi}{2})$. We have $\rho_1 = \rho_2 = a_1a_2$ and therefore (in ε-label notation) $L_0 \in B_{(-2,1,2)}$. More generally, it is not hard to verify that, for $L \in \mathrm{Lo}_3^1$, we have $L \in B_{(-2,1,2)}$ if and only if $y = 0$ and $z > 0$. $\diamond$

xample:bacbL}

Example 5.5. As in Example 3.4, set $n = 3$ and $\sigma = a_2a_1a_3a_2$. Consider

$$L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad \tilde{z}_4 = \mathbf{Q}(L_0) = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

We have $\tilde{z}_4 = \alpha_2(-\frac{\pi}{2})\alpha_1(-\frac{\pi}{4})\alpha_3(\frac{\pi}{4})\alpha_2(\frac{\pi}{2})$, $\rho_1 = \rho_2 = \rho_3 = a_1a_2a_3a_2a_1$ and therefore (in ε-label notation) $L_0 \in B_{(-2,-1,1,2)}$. $\diamond$

6. The strata B_ξ

ratification}

:submanifold}

We now prove that the strata B_ξ (or B_ε) are reasonably well-behaved.

Lemma 6.1. *Consider a permutation and reduced word $\sigma = a_{i_1} \cdots a_{i_\ell} \in S_{n+1}$ and a valid label ξ. The subset $B_\xi \subset \mathrm{Bru}_\sigma$ is a smooth submanifold of codimension $d = \mathrm{codim}(\xi)$.*

Proof. Consider $L \in B_\xi$. As above, construct $\tilde{z}_\ell = \mathbf{Q}(L) \in \mathcal{U}_1$, $z_\ell = \tilde{z}_\ell q_\ell \in \mathrm{Bru}_{\acute{\sigma}}$, $q_\ell \in \mathrm{Quat}_{n+1}$. Write

$$z_\ell = \alpha_{i_1}(\theta_1) \cdots \alpha_{i_\ell}(\theta_\ell)$$

and recursively define $z_0 = 1$, $z_k = z_{k-1}\alpha_{i_k}(\theta_k) \in \mathrm{Bru}_{\acute{\sigma}_k} \cap (\grave{\eta}\,\mathrm{Bru}_{\varrho_k})$. Let $K_0 = \{k \in \mathbb{Z}, 1 \leq k \leq \ell, \varrho_k < \varrho_{k-1}\}$ so that $|K_0| = d$. Let $V = \{x \in \mathbb{R}^\ell \mid k \in K_0 \to x_k = 0\} \subset \mathbb{R}^\ell$, a linear subspace of codimension d. We construct a compact set $U \subset \mathbb{R}^\ell$ and a smooth local diffeomorphism $\Phi : U \to \mathrm{Bru}_{\acute{\sigma}}$ such that $\Phi(0) = z_l$ and $\Phi(x) \in \mathbf{Q}[B_\xi]q_\ell$ if and only if $x \in V$.

The set U has the form

$$U = [-\epsilon_1, \epsilon_1] \times \cdots \times [-\epsilon_\ell, \epsilon_\ell];$$

let $U_k = U \cap \mathbb{R}^k$ (by $\mathbb{R}^k \subset \mathbb{R}^\ell$ we mean of course the subspace spanned by the first k unit vectors). We recursively domains $\epsilon_k > 0$ and maps $\Phi_k : U_k \to \mathrm{Bru}_{\acute{\sigma}_k}$ with $\Phi_k(0) = z_k$. In every case we shall have $\Phi_k(x_{k-1}, \theta) = \Phi_{k-1}(x_{k-1})\alpha_{i_k}(*)$, where $*$

stands for a smooth function of $x_{k-1} \in U_{k-1}$ and $\theta \in [-\epsilon_k, \epsilon_k]$. The case $k = 0$ is of course trivial.

If $k \notin K_0$, take $\epsilon_k > 0$ sufficiently small such that the following two conditions hold. For all $x_{k-1} \in U_{k-1}$ and $\theta \in [-\epsilon_k, \epsilon_k]$ we have $\Phi_{k-1}(x_{k-1})\alpha_{i_k}(\theta_k + \theta) \in \mathrm{Bru}_{\acute{\sigma}_k}$. For all $x_{k-1} \in U_{k-1} \cap V$ and $\theta \in [-\epsilon_k, \epsilon_k]$ we have $\Phi_{k-1}(x_{k-1})\alpha_{i_k}(\theta_k + \theta) \in \grave{\eta}\,\mathrm{Bru}_{\varrho_k}$. The existence of such $\epsilon_k > 0$ follows from Theorem 1 of [5]. We then define $\Phi_k(x_{k-1}, \theta) = \Phi_{k-1}(x_{k-1})\alpha_{i_k}(\theta_k + \theta)$.

If $k \in K_0$, there exists a smooth function $\vartheta : U_{k-1} \cap V \to (0, \pi)$ such that, for all $x_{k-1} \in U_{k-1} \cap V$ we have $\Phi_{k-1}(x_{k-1})\alpha_{i_k}(\theta_k + \vartheta(x_{k-1})) \in \mathrm{Bru}_{\varrho_k}$ (here we again use Theorem 1 of [5]). Notice that $\vartheta(0) = 0$. Let $\Pi : U_{k-1} \to U_{k-1} \cap V$ be the orthogonal projection. Extend ϑ to U_{k-1} by defining $\vartheta(x_{k-1}) = \vartheta(\Pi(x_{k-1}))$; notice that this is a smooth function. Define

$$\Phi_k(x_{k-1}, \theta) = \Phi_{k-1}(x_{k-1})\alpha_{i_k}(\theta_k + \vartheta(x_{k-1}) + \theta).$$

Notice that, for $x_{k-1} \in U_{k-1} \cap V$ we have $\Phi_k(x_{k-1}, \theta) \in \mathrm{Bru}_{\varrho_k(\acute{a}_{i_k})^{\mathrm{sign}(\theta)}}$. Choose sufficiently small $\epsilon_k > 0$ and we are done. ☐

ma:contractible}

Lemma 6.2. *Consider a permutation σ, a reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$ and a label ξ with $d = \mathrm{codim}(\xi)$. The smooth submanifold $B_\varepsilon \subset \mathrm{Bru}_\sigma$ is diffeomorphic to $\mathbb{R}^{\ell-d}$.*

Proof. Let (ϱ_k) be the usual sequence of elements of $\tilde{B}^+_{n+1}$. Let $\Psi_k : (0, \pi)^k \to \mathrm{Bru}_{\acute{\sigma}_k}$ be the diffeomorphism

$$\Psi_k(\theta_1, \ldots, \theta_k) = \alpha_{i_1}(\theta_1) \cdots \alpha_{i_k}(\theta_k).$$

We recursively define subsets $X_k \subseteq X_{k-1} \times (0, \pi) \subseteq (0, \pi)^k$. The set X_0 has a single element, the empty sequence. For $(\theta_1, \ldots, \theta_k) \in X_{k-1} \times (0, \pi)$ we have $(\theta_1, \ldots, \theta_k) \in X_k$ if and only if $\Psi_k(\theta_1, \ldots, \theta_k) \in \grave{\eta}\,\mathrm{Bru}_{\varrho_k}$. In particular, $\theta_1 \in X_1$ if and only if $\alpha_{i_1}(\theta_1) \in \grave{\eta}\,\mathrm{Bru}_{\varrho_1}$. By definition, the restriction $\Psi_\ell : X_\ell \to B_\xi$ is a bijection; it follows from Lemma 6.1 that it is a diffeomorphism. We recursively prove that X_k is diffeomorphic to an open ball of the appropriate dimension.

We again divide our discussion into cases. If $\xi(k) = 1$ and $\varrho_{k-1} < \varrho_k = \varrho_{k-1}\acute{a}_{i_k}$ then $X_k = X_{k-1} \times (0, \pi)$ and we are done.

Otherwise, we have $\rho_{k-1}a_{i_k} < \rho_{k-1}$. Let $\varrho_\star = \varrho_{k-1}\grave{a}_{i_k}$: there exists a smooth function $\vartheta : X_{k-1} \to (0, \pi)$ such that, for all $\Theta \in X_{k-1}$ we have

$$\Psi_{k-1}(\Theta)\alpha_{i_k}(-\vartheta(\Theta)) \in \mathrm{Bru}_{\varrho_\star}.$$

If $\xi(0) = 0$ we have

$$X_k = \{(\Theta, \theta_k) \in X_{k-1} \times (0, \pi) \mid \theta_k < \pi - \vartheta(\Theta)\},$$

which is diffeomorphic to $X_{k-1} \times (0, 1)$ and therefore to an open ball. If $\xi(0) = 2$ we have

$$X_k = \{(\Theta, \theta_k) \in X_{k-1} \times (0, \pi) \mid \theta_k > \pi - \vartheta(\Theta)\},$$

and we are done. Finally, if $\xi(0) = 1$ we have

$$X_k = \{(\Theta, \theta_k) \in X_{k-1} \times (0, \pi) \mid \theta_k = \pi - \vartheta(\Theta)\},$$

which is diffeomorphic to X_{k-1}. This completes the proof. $\square$

{lemma:poset} The following lemma is the reason why we defined a partial order among labels.

Lemma 6.3. *Let $\xi, \tilde{\xi}$ be valid ξ-labels. If $B_{\tilde{\xi}} \cap \overline{B_\xi} \neq \emptyset$ then $\xi \succeq \tilde{\xi}$.*

Notice that we do not claim equivalence, or that either of the above conditions imply $B_{\tilde{\xi}} \subseteq \overline{B_\xi}$.

Proof of Lemma 6.3. Assume that $B_{\tilde{\xi}} \cap \overline{B_\xi} \neq \emptyset$. Thus, there exists a sequence (L_j) of elements of B_ξ converging to an element L_∞ of $B_{\tilde{\xi}}$. Since each subset $B_{\acute{\sigma} q} \subseteq \mathrm{Bru}_\sigma$, $q \in \mathrm{Quat}_{n+1}$, is both closed and open, we may assume that all L_j and L_∞ belong to the same such set $B_{\acute{\sigma} q_\ell}$. We may therefore write $z_{j,\ell} = \tilde{z}_{j,\ell} q_\ell$, $z_{\infty,\ell} = \tilde{z}_{\infty,\ell} q_\ell$, $\tilde{z}_{j,\ell} = \mathbf{Q}(L_j)$, $\tilde{z}_{\infty,\ell} = \mathbf{Q}(L_\infty)$, $\lim_{j\to\infty} z_{j,\ell} = z_{\infty,\ell}$. Let $z_{j,k}$ and $z_{\infty,k}$ be as usual; we have $\lim_{j\to\infty} z_{j,k} = z_{\infty,k}$. We have $z_{j,k} \in \grave{\eta}\,\mathrm{Bru}_{\varrho_k}$ and $z_{\infty,k} \in \grave{\eta}\,\mathrm{Bru}_{\tilde{\varrho}_k}$. We therefore have $\mathrm{Bru}_{\tilde{\varrho}_k} \cap \overline{\mathrm{Bru}_{\varrho_k}} \neq \emptyset$, which implies $\varrho_k \geq \tilde{\varrho}_k$ (for all k), completing the proof. $\square$

subsection:N}

6.1. **Counting preimages.** The next three subsections are useful for counting connected components and doing examples of our stratification...

Define

$$N(z) = |P^{-1}[\{z\}]| = |\{\varepsilon \in \{\pm 1\}^{[\![\ell]\!]} \mid P(\varepsilon) = z\}|. \tag{19}$$

Here P is the mapping introduced ... As we shall see, the choice of the reduced word affects the map P, but not the value of $N(z)$.

Definition 6.4. We say that a permutation $\sigma \in S_{n+1}$ *blocks at the entry* k, $k \in [\![n]\!] = \{1, 2, \dots, n\}$, if and only if for all j, $j \leq k$ implies $j^\sigma \leq k$. Given σ, let $\mathrm{Block}(\sigma) \subset [\![n]\!]$ be the set of values of k such that σ blocks at k. A pair $(i_0, i_1) \in [\![n+1]\!]^2$ is an *inversion* of $\sigma \in S_{n+1}$ if $i_0 < i_1$ and $i_0^\sigma > i_1^\sigma$; also, $\mathrm{inv}(\sigma) \in \mathbb{N}$ denotes the number of inversions of σ.

Observe that, given a subset $B \subseteq [\![n]\!]$, the set H_B of all permutations σ such that $\mathrm{Block}(\sigma) \supseteq B$ is the subgroup of S_{n+1} generated by a_i, $i \notin B$. Denote by $\tilde{H}_B \subseteq \tilde{B}^+_{n+1}$ the subgroup generated by $\acute{a}_i$, $i \in [\![n]\!] \smallsetminus B$. In the next theorem we
{theo:N} will calculate the cardinalities of preimages under the mapping P.

Theorem 1. *Given $z \in \tilde{B}^+_{n+1}$ and $\sigma = \Theta(z) \in S_{n+1}$, set $\ell = \mathrm{inv}(\sigma)$, $B = \mathrm{Block}(\sigma) \subseteq [\![n]\!]$, and $b = |B|$. In the above notation, if $z \notin \tilde{H}_B$ then $N(z) = N(-z) = 0$; otherwise*

$$N(z) = 2^{\ell-n+b-1} + 2^{\frac{\ell}{2}-1}\,\Re(z),$$

where $\Re : \mathrm{Cl}^0_{n+1} \to \mathbb{R}$ is the real part defined above.

Remark 6.5. Notice that since $z \in \mathrm{Spin}_{n+1}$, the real part $\Re(z)$ takes values in the interval $[-1,1]$. Thus Theorem 1 implies that $|N(z) - 2^{\ell-n+b-1}| \leq 2^{\frac{\ell}{2}-1}$. For n large and most $\sigma \in S_{n+1}$, we have that $\mathrm{Block}(\sigma) = \{\emptyset\}$ and $\ell = \mathrm{inv}(\sigma) \gg n$. Therefore $N(z) \approx 2^{\ell-n-1}$, with an "error term" bounded by $2^{\frac{\ell}{2}-1} = 2^{\frac{n-1}{2}}\sqrt{2^{\ell-n-1}}$. $\diamond$

For $\sigma = a_{i_1} \cdots a_{i_\ell} \in S_{n+1}$, $\ell = \mathrm{inv}(\sigma)$, define $S(\sigma) \in \mathrm{Cl}^0_{n+1}$ given by

$$S(\sigma) := \sum_{z \in \Theta^{-1}[\{\sigma\}]} N(z)\, z = \sum_{\varepsilon \in \{\pm 1\}^{[\![\ell]\!]}} P(\varepsilon) = \sum_{\varepsilon \in \{\pm 1\}^{[\![\ell]\!]}} (\acute{a}_{i_1})^{\varepsilon(1)} \cdots (\acute{a}_{i_\ell})^{\varepsilon(\ell)}. \tag{20}$$

{lemma:S} The following result gives a simple formula for $S(\sigma)$.

Lemma 6.6. *For any $\sigma \in S_{n+1}$, we have $S(\sigma) = 2^{\frac{\ell}{2}} \cdot 1 \in \mathrm{Cl}^0_{n+1}$ where $\ell = \mathrm{inv}(\sigma)$.*

Proof. The last formula in (20) can be rewritten as

$$S(\sigma) = (\acute{a}_{i_1} + \grave{a}_{i_1})(\acute{a}_{i_2} + \grave{a}_{i_2}) \cdots (\acute{a}_{i_\ell} + \grave{a}_{i_\ell}).$$

{lemma:minus} However, for any $i \in [\![n]\!]$, we have $\acute{a}_i + \grave{a}_i = \sqrt{2}$, implying the desired result. $\square$

Lemma 6.7. *For any $z \in \tilde{B}^+_{n+1}$, we have*

$$N(z) - N(-z) = 2^{\frac{\ell}{2}} \Re(z)$$

where $\ell = \mathrm{inv}(\sigma)$ and $\sigma = \Theta(z)$).

Proof. Take $\sigma = \tilde{\Pi}(z)$. Let us compute $c = \langle S(\sigma), z\rangle$ in two different ways. Using (20) we get

$$c = \sum_{\tilde{z} \in \tilde{\Pi}^{-1}[\{\sigma\}]} N(\tilde{z})\langle \tilde{z}, z\rangle = \sum_{q \in \mathrm{Quat}_{n+1}} N(zq)\langle zq, z\rangle = \sum_{q \in \mathrm{Quat}_{n+1}} N(zq)\langle q, 1\rangle,$$

which implies that $c = N(z) - N(-z)$. On the other hand, by Lemma 6.6 we have $c = 2^{\frac{\ell}{2}}\langle 1, z\rangle = 2^{\frac{\ell}{2}}\Re(z)$. The result follows. $\square$

{lemma:plus} Recall that, for $B \subseteq [\![n]\!]$, $\tilde{H}_B \leq \tilde{B}^+_{n+1}$ is generated by $\acute{a}_i$, $i \in [\![n]\!] \smallsetminus B$.

Lemma 6.8. *For $z \in \tilde{B}^+_{n+1}$, take $\sigma = \Theta(z)$, $\ell = \mathrm{inv}(\sigma)$, $B = \mathrm{Block}(\sigma)$ and $b = |B|$. If $z \in \tilde{H}_B$, we have $N(z) + N(-z) = 2^{\ell-n+b}$; otherwise, we have $N(z) = N(-z) = 0$.*

Proof. For any $\varepsilon \in \{+1,-1\}^{[\![\ell]\!]}$, we have $P(\varepsilon) = (\acute{a}_{i_1})^{\varepsilon(1)} \cdots (\acute{a}_{i_\ell})^{\varepsilon(\ell)} \in \tilde{H}_B$. Also $-1 \in \tilde{H}_B$ and therefore $z \in \tilde{H}_B$ if and only if $(-z) \in \tilde{H}_B$, completing the proof of the second claim.

We want to compute $N(z)+N(-z)$ which is the number of solutions of $P(\varepsilon)=\pm z$. We may therefore compute $P(\varepsilon)$ in the quotient group B^+_{n+1}. In other words, we want to compute and count in this quotient the products

$$(\grave{\sigma})^{-1}P(\varepsilon)=\acute{a}_{i_\ell}\cdots\acute{a}_{i_1}(\acute{a}_{i_1})^{\varepsilon(1)}\cdots(\acute{a}_{i_\ell})^{\varepsilon(\ell)}\in\operatorname{Quat}_{n+1}/\{\pm1\}.$$

It is convenient to work in the group algebra $\mathbb{Z}[B^+_{n+1}]$ and calculate the product:

$$C=\acute{a}_{i_\ell}\cdots\acute{a}_{i_1}(\acute{a}_{i_1}+\grave{a}_{i_1})\cdots(\acute{a}_{i_\ell}+\grave{a}_{i_\ell})=\acute{a}_{i_\ell}\cdots\acute{a}_{i_2}(1+\hat{a}_{i_1})(\acute{a}_{i_2}+\grave{a}_{i_2})\cdots(\acute{a}_{i_\ell}+\grave{a}_{i_\ell}).$$

In the algebra $\mathbb{Z}[B^+_{n+1}]$, for all $i,j\in[\![n]\!]$, we have the relation

$$(1+\hat{a}_i)(\acute{a}_j+\grave{a}_j)=(\acute{a}_j+\grave{a}_j)(1+\hat{a}_i).$$

We therefore obtain

$$\begin{aligned}C&=\acute{a}_{i_\ell}\cdots\acute{a}_{i_3}\acute{a}_{i_2}(\acute{a}_{i_2}+\grave{a}_{i_2})(1+\hat{a}_{i_1})(\acute{a}_{i_3}+\grave{a}_{i_3})\cdots(\acute{a}_{i_\ell}+\grave{a}_{i_\ell})\\&=\acute{a}_{i_\ell}\cdots\acute{a}_{i_3}(1+\hat{a}_{i_2})(1+\hat{a}_{i_1})(\acute{a}_{i_3}+\grave{a}_{i_3})\cdots(\acute{a}_{i_\ell}+\grave{a}_{i_\ell})\\&=\acute{a}_{i_\ell}\cdots\acute{a}_{i_4}(1+\hat{a}_{i_3})(1+\hat{a}_{i_2})(1+\hat{a}_{i_1})(\acute{a}_{i_4}+\grave{a}_{i_4})\cdots(\acute{a}_{i_\ell}+\grave{a}_{i_\ell})\\&=(1+\hat{a}_{i_\ell})\cdots(1+\hat{a}_{i_2})(1+\hat{a}_{i_1})\in\mathbb{Z}[\operatorname{Quat}_{n+1}/\{\pm1\}].\end{aligned}$$

The group algebra $\mathbb{Z}[\operatorname{Quat}_{n+1}/\{\pm1\}]$ is commutative and in it we have

$$(1+\hat{a}_i)^2=2(1+\hat{a}_i).$$

We thus get

$$C=2^{\ell-n+b}\prod_{i\in[\![n]\!]\smallsetminus B}(1+\hat{a}_i)=2^{\ell-n+b}\sum_{q\in(\operatorname{Quat}_{n+1}\cap\tilde{H}_B)/\{\pm1\}}q.$$

The coefficient of $q=(\grave{\sigma})^{-1}z$ in C is $N(z)+N(-z)$, completing the proof. □

Proof of Theorem 1. The result follows directly from Lemmas 6.7 and 6.8. Notice that these lemmas also imply that if $z\notin\tilde{H}_B$ then $\Re(z)=0$. □

ion : Nthin}

6.2. **Computing** $N_{\text{thin}}(z)$**.**

Definition 6.9. Given a permutation $\sigma\in S_{n+1}$, a reduced word $\sigma=a_{i_1}\cdots a_{i_\ell}$ where $\ell=\operatorname{inv}(\sigma)$, $B=\operatorname{Block}(\sigma)$, and a sign sequence $\tilde{\varepsilon}:[\![n]\!]\smallsetminus B\to\{\pm1\}$, define the element

$$\tilde{P}(\tilde{\varepsilon})=P(\tilde{\varepsilon}\circ i)=(\acute{a}_{i_1})^{\tilde{\varepsilon}(i_1)}\cdots(\acute{a}_{i_\ell})^{\tilde{\varepsilon}(i_\ell)}\in\Theta^{-1}[\{\sigma\}]\subset\operatorname{Spin}_{n+1}.$$

Important special cases are the sign sequences $\varepsilon_{+1},\varepsilon_{-1}\in\{\pm1\}^{([\![n]\!]\smallsetminus B)}$ defined by $\varepsilon_s(i)=s$, for all $i\in[\![n]\!]\smallsetminus B$. We then have (in the notation of [5])

$$\tilde{P}(\varepsilon_{+1})=\acute{a}_{i_1}\cdots\acute{a}_{i_\ell}=\acute{\sigma};\qquad\tilde{P}(\varepsilon_{-1})=\grave{a}_{i_1}\cdots\grave{a}_{i_\ell}=\grave{\sigma}.\tag{21}$$

Definition 6.10. If $\tilde{P}(\tilde{\varepsilon})=z$ we call $\varepsilon=\tilde{\varepsilon}\circ i$ a *thin solution* of equation $P(\varepsilon)=z$. A solution of $P(\varepsilon)=z$ which is not thin is called *thick*. VERY UNCLEAR! NEEDS COMMENTS!

Let $N_{\text{thin}}(z)$ be the number of thin solutions:

$$N_{\text{thin}}(z) = |\tilde{P}^{-1}[\{z\}]| = |\{\varepsilon \in \{\pm 1\}^{(\llbracket n \rrbracket \smallsetminus B)} \mid \tilde{P}(\varepsilon) = z\}|. \tag{22}$$

The multiplicative abelian group $\mathcal{E} = \{\pm 1\}^n$ acts as an automorphism group of Cl^0_{n+1}, defined by $(\hat{a}_i)^{[\varepsilon]} = (\hat{a}_i)^{\varepsilon(i)} = \varepsilon(i)\hat{a}_i$. This group also acts on Spin_{n+1} and $\tilde{B}^+_{n+1}$ by automorphisms. For $z \in \mathrm{Spin}_{n+1}$, define $\mathcal{E}_z \subseteq \mathcal{E}$ as the isotropy group of z, i.e.

$$\mathcal{E}_z = \{\varepsilon \in \mathcal{E} \mid z^{[\varepsilon]} = z\}, \qquad \mathcal{E} = \{\pm 1\}^n. \tag{23}$$

Given a permutation $\sigma \in S_{n+1}$, let $\mathrm{nc}(\sigma)$ be the *number of cycles of* σ (counting cycles of length 1, i.e., fixed positions). In particular, for the Coxeter element $\eta \in S_{n+1}$, i.e. $\eta = (n+1, n, \ldots, 1)$, we get $\ell(\eta) = n(n+1)/2$ and $\mathrm{nc}(\eta) = 1 + \lfloor \frac{n}{2} \rfloor$.

{theo:thin} **Theorem 2.** *Consider $\sigma \in S_{n+1}$, $b = |\operatorname{Block}(\sigma)|$, and $\acute{\sigma} \in \tilde{B}^+_{n+1}$ as defined in (21). Take the isotropy group $\mathcal{E}_{\acute{\sigma}}$ as in (23) and set $e := 2^{-b}|\mathcal{E}_{\acute{\sigma}}|$. Then, for $z \in \Pi^{-1}[\{\sigma\}]$, $N_{\text{thin}}(z) \in \{0, e\}$. WHAT IS e HERE? We have $N_{\text{thin}}(z) = e$ if there exists $\varepsilon \in \mathcal{E}$ with $z = (\acute{\sigma})^{[\varepsilon]}$; otherwise, we have $N_{\text{thin}}(z) = 0$.*

Furthermore, $|\mathcal{E}_{\acute{\sigma}}| = 2^{(c-1-c_{\text{anti}})}$ where $c = \mathrm{nc}(\sigma)$ and $c_{\text{anti}} = 1$ if there exists $\varepsilon \in \mathcal{E}$ with $(\acute{\sigma})^{[\varepsilon]} = -\acute{\sigma}$, $c_{\text{anti}} = 0$ otherwise.

Recall that the multiplicative abelian group $\mathcal{E} = \{\pm 1\}^n$ acts by automorphisms on the Clifford algebra Cl^0_{n+1}. Further, for $\varepsilon \in \mathcal{E} = \{\pm 1\}^n$, set

$$(\hat{a}_j)^{[\varepsilon]} = (\hat{a}_j)^{\varepsilon(j)} := \varepsilon(j)\hat{a}_j.$$

Restrictions of this action to the groups Spin_{n+1} and $\tilde{B}^+_{n+1}$ act by automorphisms as well.

Let $\mathrm{Diag}_{n+1} \subset \mathrm{O}_{n+1}$ be the subgroup of diagonal matrices, so that if $E \in \mathrm{Diag}_{n+1}$ then $E_{i,j} = 0$ for $i \neq j$ and $E_{j,j} \in \{\pm 1\}$ for all $j = 1, \ldots, n+1$. Consider homomorphisms $\Pi : \mathrm{Diag}_{n+1} \to \mathcal{E}$ and $\Psi : \mathcal{E} \to \mathrm{Diag}_{n+1}$ defined by

$$(\Pi(E))(j) = E_{j,j}\, E_{j+1,j+1}, \qquad (\Psi(\varepsilon))_{j,j} = \prod_{i<j} \varepsilon(i). \tag{24}$$

The composition $\Pi \circ \Psi$ equals identity and $\ker(\Pi) = \{\pm I\}$ where I is the identity matrix. The maps Π and Ψ thus provide us with an identification between $\mathcal{E}$ and $\mathrm{Diag}_{n+1}/\{\pm I\}$.

The group Diag_{n+1} acts on GL_{n+1}, SO_{n+1} or Lo^1_{n+1} by conjugation. In each case, this induces an action of $\mathcal{E}$ on these groups. This is closely related to our action of $\mathcal{E}$ on Spin_{n+1}.

{lemma:actionso} **Lemma 6.11.** *For $\varepsilon \in \mathcal{E}$, let $E = \Psi(\varepsilon)$. For $z \in \mathrm{Spin}_{n+1}$ and $Q = \Pi(z) \in \mathrm{SO}_{n+1}$ we have $\Pi(z^{[\varepsilon]}) = EQE$. Also, $Q = EQE$ if and only if $z^{[\varepsilon]} = \pm z$.*

Proof. Consider the one-parameter subgroups $\alpha_j : \mathbb{R} \to \mathrm{Spin}_{n+1}$ defined in (4). The projection $\Pi : \mathrm{Spin}_{n+1} \to \mathrm{SO}_{n+1}$ gives us $\tilde{Q} = \Pi(\alpha_j(\theta)) \in \mathrm{SO}_{n+1}$ with $\tilde{Q}_{i,i} = 1$ for $i \notin \{j, j+1\}$, $\tilde{Q}_{j,j} = \tilde{Q}_{j+1,j+1} = \cos(\theta)$ and $\tilde{Q}_{j+1,j} = -\tilde{Q}_{j,j+1} = \sin(\theta)$; the other entries vanish. The action is given by $(\Pi(\alpha_j(\theta)))^{[\varepsilon]} = \Pi(\alpha_j(\varepsilon(j)\theta))$. A straightforward computation verifies that this matches the statement. $\square$

mma:Aepsilon} From now on we write $Q^{[\varepsilon]} = \Psi(\varepsilon)Q\Psi(\varepsilon)$ (and similarly for the other groups).

Lemma 6.12. *Let $\varepsilon \in \mathcal{E}$, $E = \Psi(\varepsilon)$, $z_0 \in \tilde{B}^+_{n+1}$. The map $A_{[\varepsilon]} : B_{z_0} \to B_{z_1}$, $z_1 = z_0^{[\varepsilon]}$, $A_{[\varepsilon]}(L) = ELE$, is a diffeomorphism. Furthermore, thin connected components are mapped by $A_{[\varepsilon]}$ to thin connected components.*

Proof. We first prove that the map is well-defined (and smooth). Indeed, by definition, if $L \in B_{z_0}$ then $\mathbf{Q}(L) \in \mathrm{Bru}_{z_0} \subset \mathrm{Spin}_{n+1}$. We also have $\mathbf{Q}(ELE) = (\mathbf{Q}(L))^{[\varepsilon]} \in \mathrm{Bru}_{z_1}$ and therefore $A_\varepsilon(L) \in B_{z_1}$, as desired.

We similarly have that $A_{[\varepsilon]} : B_{z_1} \to B_{z_0}$ is well-defined. Since $\varepsilon^2 = 1$, one map is the inverse of the other. The final claim follows from the definition of a thin connected component. $\square$

Given $\sigma \in S_{n+1}$, the group $\mathcal{E}$ acts on the finite sets

$$\acute{\sigma}\,\mathrm{Quat}_{n+1} = \Theta_{\tilde{B}}^{-1}[\{\sigma\}] \subset \tilde{B}^+_{n+1} \quad \text{and} \quad \Pi_B^{-1}[\{\sigma\}] \subset B^+_{n+1}$$

of cardinalities 2^{n+1} and 2^n by permutations. CORRECT NOTATION? Let us describe these actions.

For $Q \in \mathrm{SO}_{n+1}$, we define the isotropy groups

$$\mathcal{E}_Q = \{\varepsilon \in \mathcal{E} \mid Q^{[\varepsilon]} = Q\}, \qquad (\mathrm{Diag}_{n+1})_Q = \{E \in \mathrm{Diag}_{n+1} \mid EQE = Q\}. \tag{25}$$

Thus, if $z \in \mathrm{Spin}_{n+1}$ we have

$$\mathcal{E}_z = \{\varepsilon \in \mathcal{E} \mid z^{[\varepsilon]} = z\} \leq \mathcal{E}_{\Pi(z)} = \{\varepsilon \in \mathcal{E} \mid z^{[\varepsilon]} = \pm z\}.$$

Recall that a set $X \subseteq [\![n+1]\!]$ is σ-invariant if and only if $X^\sigma = X$, where $X^\sigma = \{x^\sigma, x \in X\}$. This happens if and only if X is a disjoint union of cycles of σ. Given $\sigma \in S_{n+1}$, there exist 2^c σ-invariant invariant sets $X \subseteq [\![n+1]\!]$ where $c := \mathrm{nc}(\sigma)$ is the number of cycles of the permutation σ.

a:isotropyso} **Lemma 6.13.** *Consider a permutation $\sigma \in S_{n+1}$ with $c = \mathrm{nc}(\sigma)$ cycles and the action of $\mathcal{E}$ on the set $\Theta_B^{-1}[\{\sigma\}]$. For any $Q \in \tilde{\Pi}_B^{-1}[\{\sigma\}]$, the isotropy group $\mathcal{E}_Q$ is the same. Indeed, consider $\varepsilon \in \mathcal{E}$ and $E = \Psi(\varepsilon) \in \mathrm{Diag}_{n+1}$. We have $\varepsilon \in \mathcal{E}_Q$ if and only if $X = \{j \in [\![n+1]\!], E_{j,j} = -1\}$ is σ-invariant. Also, given $Q_0, Q_1 \in \tilde{\Pi}_B^{-1}[\{\sigma\}]$, Q_0 and Q_1 are in the same $\mathcal{E}$-orbit if and only if, for each cycle $C \subseteq [\![n+1]\!]$ of σ,*

$$\prod_{i \in C} (Q_0)_{i,i^\sigma} = \prod_{i \in C} (Q_1)_{i,i^\sigma}. \tag{26}$$

We have $|\mathcal{E}_Q| = 2^{c-1}$*; there are* 2^{c-1} *orbits, each of cardinality* 2^{n-c+1}.

STRANGE FORMULATION?

Proof. For $i \le n+1$, we have $Q_{i,i^\sigma} \neq 0$ and $(EQE)_{i,i^\sigma} = E_{i,i}E_{i^\sigma,i^\sigma}Q_{i,i^\sigma}$. Thus, $(EQE)_{i,i^\sigma} = Q_{i,i^\sigma}$ if and only if $E_{i,i} = E_{i^\sigma,i^\sigma}$. Thus, $E \in (\mathrm{Diag}_{n+1})_Q$ if and only if $E_{i,i}$ is constant in each cycle of σ, proving that $\mathcal{E}_Q$ is as in the statement. Also, if Q_0 and Q_1 are in the same orbit then the condition in (26) holds; the converse follows by counting. $\square$

Consider $\sigma \in S_{n+1}$, $z_0 \in \tilde{\Pi}_{\tilde{B}}^{-1}[\{\sigma\}]$ and $Q_0 = \Pi(z_0) \in \Pi_B^{-1}[\{\sigma\}]$. We know from Lemma 6.13 what is the isotropy group $\mathcal{E}_{Q_0}$; we also know that the orbit $\mathcal{O}_{Q_0}$ of Q_0 has cardinality 2^{n-c+1}. Concerning the action of $\mathcal{E}$ on $\Pi_{\tilde{B}}^{-1}[\{\sigma\}]$, there are two possibilities:

(1) There exists $\varepsilon \in \mathcal{E}$ with $z_0^{[\varepsilon]} = -z_0$. In this case the orbit $\mathcal{O}_{z_0}$ equals $\Pi^{-1}[\mathcal{O}_{Q_0}]$ and has cardinality 2^{n-c+2}. The isotropy group $\mathcal{E}_{z_0}$ has index 2 in $\mathcal{E}_{Q_0}$. We set $c_{\mathrm{anti}}(z_0) = 1$.

(2) There exists no $\varepsilon \in \mathcal{E}$ with $z_0^{[\varepsilon]} = -z_0$. In this case the orbits $\mathcal{O}_{z_0}$ and $\mathcal{O}_{-z_0}$ are disjoint, each with cardinality 2^{n-c+1} and with union $\Pi^{-1}[\mathcal{O}_{Q_0}]$. The isotropy group $\mathcal{E}_{z_0}$ equals $\mathcal{E}_{Q_0}$. We say the orbit *splits* and set $c_{\mathrm{anti}}(z_0) = 0$.

ma:isotropyspin}

Lemma 6.14. *Let* $z_0 \in \tilde{B}^+_{n+1}$, $\sigma = \tilde{\Pi}(z_0) \in S_{n+1}$ *and* $c = \mathrm{nc}(\sigma)$. *The order of the isotropy group of* z_0 *is* $|\mathcal{E}_{z_0}| = 2^{c-1-c_{\mathrm{anti}}(z_0)}$.

Proof. This follows directly from Lemma 6.13 and the remarks above. $\square$

Examples will be given in the next sections. Now we are ready to prove Theorem 2.

Proof of Theorem 2. Lemma 6.14 gives us the desired formula for $|\mathcal{E}_{\acute\sigma}|$. As above, set $B = \mathrm{Block}(\sigma)$. The group $\mathcal{E}$ acts on $\{\pm 1\}^{[\![n]\!]\smallsetminus B}$. Indeed, given $\varepsilon \in \mathcal{E}$ and $\tilde\varepsilon \in \{\pm 1\}^{[\![n]\!]\smallsetminus B}$, define $\varepsilon\tilde\varepsilon \in \{\pm 1\}^{[\![n]\!]\smallsetminus B}$ by $(\varepsilon\tilde\varepsilon)(i) = \varepsilon(i)\tilde\varepsilon(i)$ for all $i \in [\![n]\!] \smallsetminus B$. The isotropy group of this action is $\{\pm 1\}^B \subset \mathcal{E}$, of cardinality 2^b. Thus, the number of distinct elements $\tilde\varepsilon \in \{\pm 1\}^{[\![n]\!]\smallsetminus B}$ with $\tilde{P}(\tilde\varepsilon) = \acute\sigma$ equals $e = 2^{-b}\,|\mathcal{E}_{\acute\sigma}|$.

For other values of $z_0 \in \Theta^{-1}[\{\sigma\}]$, if there exists $\varepsilon_0 \in \mathcal{E}$, $z_0 = \acute\sigma^{[\varepsilon_0]}$, then, for all $\varepsilon \in \mathcal{E}$, $z_0 = \acute\sigma^{[\varepsilon]}$ if and only if $\varepsilon \in \varepsilon_0\mathcal{E}_{\acute\sigma}$. Since $|\varepsilon_0\mathcal{E}_{\acute\sigma}| = |\mathcal{E}_{\acute\sigma}|$, this proves the formula for $N_{\mathrm{thin}}(z_0)$ in this case. If no such ε_0 exists we have $N_{\mathrm{thin}}(z_0) = 0$, completing the proof. $\square$

{remark:canti}

Remark 6.15. Determining the value of $c_{\mathrm{anti}}(z)$ for $z \in \tilde{B}^+_{n+1}$ appears to be a question worthy of further consideration. As we shall see in the examples, the value of c_{anti} is not a function of the permutation $\sigma = \Theta(z)$. A simple observation is that $\Re(z) \neq 0$ implies $c_{\mathrm{anti}}(z) = 0$. $\diamond$

{section:CW}

7. A CW complex

In this section we construct a finite CW complex which is homotopically equivalent to B_σ: our main result is Proposition 7.2. This is obtained from the partial order between labels, Lemma 6.3 above together with a topological Lemma 7.1 below.

Here $\mathbb{S}^{k-1}_r$ denotes the sphere of radius r, $\mathbb{B}^k_r$ denotes the open ball, $\mathbb{D}^k_r$ denotes the compact disk and $\mathbb{K}^k_{r_0,r_1}$ denotes the corona:

$$\mathbb{S}^{k-1}_r = \{v \in \mathbb{R}^k \mid |v| = r\}, \qquad \mathbb{B}^k_r = \{v \in \mathbb{R}^k \mid |v| < r\},$$
$$\mathbb{D}^k_r = \{v \in \mathbb{R}^k \mid |v| \le r\}, \qquad \mathbb{K}^k_{r_0,r_1} = \{v \in \mathbb{R}^k \mid r_0 \le |v| \le r_1\};$$

also, $\mathbb{D}^k = \mathbb{D}^k_1$. For a CW complex X, let $X^{[j]} \subseteq X$ denote the skeleton of dimension j, that is, the union of cells of dimension at most j.

ma:topolemma}

Lemma 7.1. *Let $M_0 \subset M_1$ be smooth manifolds of dimension ℓ. Assume that $N_1 = M_1 \smallsetminus M_0 \subset M_1$ is a smooth submanifold of codimension k, $0 < k \le \ell$, and that N_1 is diffeomorphic to $\mathbb{R}^{\ell-k}$. Assume that X_0 is a finite CW complex and that $i_0 : X_0 \to M_0$ is a homotopy equivalence.*

There exists a map $\beta : \mathbb{S}^{k-1} \to X_0^{[k-1]}$ with the following properties. Let X_1 be obtained from X_0 by attaching a cell C_1 of dimension k with glueing map β. There exists a map $i_1 : X_1 \to M_1$ with $i_1|_{X_0} = i_0$ such that $i_1 : X_1 \to M_1$ is a homotopy equivalence.

The maps i_0 and i_1 can be taken to be inclusions in many examples but are not required to be so. The proof provides us with a construction of β, of the CW complex X_1 and of the map $i_1 : X_1 \to M_1$.

Proof of Lemma 7.1. By hypothesis, the map $i_0 : X_0 \to M_0$ is a homotopy equivalence. Thus, there exist a continuous map $p_0 : M_0 \to X_0$ and two homotopies $H_0 : [0,1] \times M \to M$ and $\tilde{H}_0 : [0,1] \times X \to X$ with

$$H_0(0,z) = z, \quad H_0(1,z) = i_0(p_0(z)), \quad \tilde{H}_0(0,x) = x, \quad \tilde{H}_0(1,x) = p_0(i_0(x))$$

for all $z \in M_0$ and $x \in X_0$. Consider a tubular neighborhood of N_1 disjoint from the compact set $i_0[X_0]$. Since N_1 is diffeomorphic to $\mathbb{R}^{\ell-k}$, the tubular neighborhood may be assumed to be a smooth injective map $\Phi : \mathbb{D}^k_{\frac{1}{2}} \times \mathbb{R}^{\ell-k} \to M_1$ with $\Phi[\{0\} \times \mathbb{R}^{\ell-k}] = N_1$. Let $\alpha_1 : \mathbb{D}^k_{\frac{1}{2}} \to M_1$, $\alpha_1(x) = \Phi(x,0)$, $z_1 = \alpha_1(0)$.

Consider the restriction $\beta_1 = \alpha_1|_{\mathbb{S}^{k-1}_{\frac{1}{2}}}$. We ignore the radius to write $\beta_1 : \mathbb{S}^{k-1} \to \tilde{M}_0$. Define $\beta_2 = p_0 \circ \beta_1 : \mathbb{S}^{k-1} \to X_0$: notice that β_1 and $i_0 \circ \beta_2$ are homotopic in $\tilde{M}_0$, with homotopy $H_0(\cdot, \beta_1(\cdot))$. Also, there exists $\beta : \mathbb{S}^{k-1} \to X_0^{[k-1]}$ such that β_2 and β are homotopic in X_0; let $H_X : [0,1] \times \mathbb{S}^{k-1} \to X_0$ be such a homotopy. Thus, β_1 and $i_0 \circ \beta$ are homotopic in $\tilde{M}_0$. This is the desired glueing map $\beta : \mathbb{S}^{k-1} \to X_0^{[k-1]}$.

The construction of X_1 is a forced move; we proceed to construct $i_1 : X_1 \to M_1$. Let $\alpha_2 : \mathbb{K}^k_{\frac{1}{2},1} \to \tilde{M}_0$ satisfy $\alpha_2|_{\mathbb{S}^{k-1}_{\frac{1}{2}}} = \beta_1$, $\alpha_2|_{\mathbb{S}^{k-1}_{\frac{3}{4}}} = i_0 \circ \beta_2$ and $\alpha_2|_{\mathbb{S}^{k-1}_1} = i_0 \circ \beta$. More precisely, for $r \in [\frac{1}{2}, 1]$ and $u \in \mathbb{S}^{k-1}$, set

$$\alpha_2(ru) = \begin{cases} H_0(4r-2, \beta_1(u)), & r \in [\frac{1}{2}, \frac{3}{4}], \\ i_0(H_X(4r-3, u)), & r \in [\frac{3}{4}, 1]. \end{cases}$$

Define $\alpha : \mathbb{D}^k \to \tilde{M}_1$ by $\alpha|_{\mathbb{D}^k_{\frac{1}{2}}} = \alpha_1$ and $\alpha|_{\mathbb{K}^k_{\frac{1}{2},1}} = \alpha_2$. Define i_1 by $i_1|_{X_0} = i_0$ and by $i_1(x) = \alpha(x)$ for $x \in C_1 = \mathbb{D}^k$.

We need to prove that i_1 is a homotopy equivalence. We could at this point construct $p_1 : M_1 \to X_1$ and homotopies H_1 and $\tilde{H}_1$. Since that construction is rather cumbersome, we prefer to proceed in a slightly different way: we first prove that for any $j \geq 0$ the map $\pi_j(i_1) : \pi_j(X_1) \to \pi_j(M_1)$ is a bijection.

We first prove the surjectivity of $\pi_j(i_1)$. Let $\gamma_M : \mathbb{S}^j \to M_1$: we want to prove that there exists $\gamma_X : \mathbb{S}^j \to X_1$ such that γ_M and $i_1 \circ \gamma_X$ are homotopic. We may assume that γ_M is smooth and transversal to N_1: let $N_S = \gamma_M^{-1}[N_1] \subset \mathbb{S}^j$, a smooth submanifold of codimension k. By transversality, there exist $\epsilon \in (0, \frac{1}{2})$ and a tubular neighborhood $\Psi : \mathbb{D}^k_\epsilon \times N_S \to \mathbb{S}^j$ with $\Psi(0, s) = s$ (for all $s \in N_S$). We may furthermore assume that there exists a smooth function $f_0 : \mathbb{D}^k_\epsilon \times N_S \to \mathbb{R}^{\ell-k}$ such that $\gamma_M(\Psi(v, s)) = \Phi(v, f_0(v, s))$ (here Φ is the tubular neighborhood of N_1 described above).

Multiplication of f_0 by a bump function takes us from γ_M to a homotopic function $\gamma_{M,1}$ such that γ_M and $\gamma_{M,1}$ coincide in $\mathbb{S}^j \smallsetminus \Psi[\mathbb{B}^k_\epsilon \times N_S]$ and $\gamma_{M,1}(\Psi(v, s)) = \alpha_1(v)$ for all $s \in N_S$ and $v \in \mathbb{D}^k_{\epsilon/2}$. Another homotopy takes us to $\gamma_{M,2}$ such that $\gamma_{M,1}$ and γ_{M_2} coincide in $\mathbb{S}^j \smallsetminus \Psi[\mathbb{B}^k_{\epsilon/2} \times N_S]$ and, if $s \in N_S$ and $v \in \mathbb{D}^k_{\epsilon/2}$ then

$$\gamma_{M,2}(\Psi(v, s)) = \begin{cases} \alpha_1(2v/\epsilon), & |v| \leq \epsilon/4; \\ \alpha_1(v/(2|v|)), & \epsilon/4 \leq |v| \leq 3\epsilon/8; \\ \alpha_1((2 - (3\epsilon/2) - (4(1-\epsilon)/\epsilon)|v|)v), & 3\epsilon/8 \leq |v| \leq \epsilon/2. \end{cases}$$

We now compose with H_0 to obtain $\gamma_{M,3}$ as follows:

$$\gamma_{M,3}(s) = \begin{cases} (i_0 \circ p_0 \circ \gamma_{M,2})(s), & s \notin \Psi[\mathbb{B}^k_{3\epsilon/8} \times N_S], \\ H_0(8(|v| - (\epsilon/4))/\epsilon, \gamma_{M,2}(s)), & s = \Psi(v, s_0),\ |v| \in [\epsilon/4, 3\epsilon/8], \\ \gamma_{M,2}(s), & s \in \Psi[\mathbb{B}^k_{\epsilon/4} \times N_S]. \end{cases}$$

Notice that for $s \in N_S$ and $|v| \leq 3\epsilon/8$ we have $\gamma_{M,3}(\Psi(v, s)) = \alpha(2v/\epsilon)$. For $s \in \mathbb{S}^j \smallsetminus \Psi[\mathbb{B}^k_{3\epsilon/8} \times N_S]$ we have $\gamma_{M,3}(s) = (i_0 \circ p_0 \circ \gamma_{M,2})(s)$. A small adjustment using H_X takes us to $\gamma_{M,4} = i_1 \circ \gamma_{X,4}$, completing the proof of surjectivity.

The proof of injectivity of $\pi_j(i_1)$ is similar, and will be presented in a less detail. Consider $\gamma_X : \mathbb{S}^j \to X_1$ and assume that $\pi_j(i_1)(\gamma_X) = 0 \in \pi_j(M_1)$. We may assume that γ_X is smooth in the interior of the new cell C_1 and that the center $0_{C_1} \in C_1$ is a regular value. By hypothesis, there exists $\Gamma_M : \mathbb{D}^{j+1} \to M_1$ such that $\Gamma_M|_{\mathbb{S}^j} = i_1 \circ \gamma_X$. Again, we may assume that Γ_M is smooth in a neighborhood of $\Gamma_M^{-1}[0_{C_1}]$ and that 0_{C_1} is a regular value. Let $N_D = \Gamma_M^{-1}[0_{C_1}] \subset \mathbb{D}^{j+1}$: this is a smooth submanifold of codimension k with boundary $\partial N_D \subset \mathbb{S}^j$, also a smooth submanifold of codimension k. As above, pulling back Φ gives us a tubular neighborhood Ψ. Again as above, we construct $\Gamma_{M,*}$ of the form $\Gamma_{M,*} = i_1 \circ \Gamma_{X,*}$ where $\Gamma_{X,*} : \mathbb{D}^{j+1} \to X_1$ satisfies $\Gamma_{X,*}|_{\mathbb{S}^j} = \gamma_X$. We thus have $[\gamma_X] = 0 \in \pi_j(X_1)$, completing the proof of injectivity.

At this point we know that $i_1 : X_1 \to M_1$ is such that $\pi_j(i_1)$ is bijective for all j. In other words, i_1 is a weak homotopy equivalence. The set X_1 is a CW complex and M_1 is a manifold and therefore homeomorphic to a CW complex. By Whitehead's Theorem, the map i_1 is a homotopy equivalence, as desired. $\square$

ma:CWcomplex} In the situation of Lemma 7.1, both pairs (M_1, M_0) and (X_1, X_0) are ANRs.

Proposition 7.2. *The set B_σ is homotopically equivalent to a finite CW complex, with one cell of dimension d for each label of codimension d.*

Proof. Labels of codimension 0 are maximal elements under the partial order $\succeq$. Let $B_{\sigma;0} \subseteq B_\sigma$ be the union of the open, disjoint, contractible sets B_ξ for ξ of codimension 0. The set $B_{\sigma;0}$ is homotopically equivalent to a finite set with one vertex per label, which is of course a CW complex of dimension 0. This is the basis of a recursive construction.

We can list the set of ξ-labels of codimension $d > 0$ as $(\xi_i)_{1 \le i \le N_\xi}$ in such a way that $\xi_i \succeq \xi_j$ implies $i \le j$. Define recursively the subsets $B_{\sigma;i} = B_{\sigma;i-1} \cup B_{\xi_i} \subseteq B_\sigma$. The partial order $\succeq$ guarantees that each subset $B_{\sigma;i} \subset B_\sigma$ is an open subset: in other words, the sequence

$$B_{\sigma;0} \subset B_{\sigma;1} \subset \cdots \subset B_{\sigma;N_\xi - 1} \subset B_{\sigma;N_\xi} \tag{27}$$

is a filtration. We may therefore apply Lemma 7.1 to the pair $B_{\sigma;i-1} \subset B_{\sigma;i}$, completing the recursive construction and the proof. $\square$

The filtration in Equation 27 has the property that, for all i, the pair $(B_{\sigma;i}, B_{\sigma;i-1})$ is an ANR. This is crucial for Lemma 7.2.

The actual construction of the CW complex is not as easy as might perhaps be desired. In the next sections we describe the glueing map for low dimensional strata.

ection:codimone}

8. Strata of codimension one

We already identified the labels and strata of codimension 0. In this section we describe the strata of codimensions 1 and their glueing instructions. This allows us to compute the connected components of B_σ or B_z. We first make some general remarks concerning strata of positive codimension.

As usual, assume $\sigma \in S_{n+1}$ and a reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$ to be fixed. As we saw, a matrix $L \in B_\sigma$ can be identified with a sequence $(z_k)_{0 \le k \le \ell}$ of elements of Spin_{n+1} with $z_0 = 1$, $z_k = z_{k-1}\alpha_{i_k}(\theta_k)$, $\theta_k \in (0,\pi)$, $z_\ell \in \mathrm{Bru}_{\acute\sigma} \cap (\grave\eta \,\mathrm{Bru}_\eta)$. The values of θ_k are smooth functions of L. Assume $L \in B_\xi \subset B_\sigma$ where ξ is a label of positive codimension d. We can construct a transversal section to B_ξ by keeping fixed the values of θ_k if either $\xi(k) \neq 1$ or $\varrho_k \ge \varrho_{k-1}$. There are d values of k for which $\xi(k) = 1$ and $\varrho_k < \varrho_{k-1}$: for these values of k we allow the coordinates θ_k to vary freely (and independently) in a small neighborhood of their original values. We must then determine the labels of the perturbed strata. As in the proof of Lemma 7.1, an understanding of a transversal section yields a description of the boundary map.

{lemma:codimone}

Lemma 8.1. *If k_1, k_2 satisfy*

$$i_{k_1} = i_{k_2}, \qquad \forall k, (k_1 < k < k_2) \to (i_k \neq i_{k_1}). \tag{28}$$

then any function $\xi : [\![\ell]\!] \to \{0,1,2\}$ with $\xi^{-1}[\{1\}] = \{k_1,k_2\}$ is a label of codimension 1. Conversely, if ξ is a label of codimension $d = 1$ then $\xi^{-1}[\{1\}] = \{k_1,k_2\}$ where $k_1 < k_2$ satisfy the condition in Equation (28). Then there are precisely two labels $\tilde\xi$ of codimension 0 with $\xi \preceq \tilde\xi$. We can call them ξ_0, ξ_2 with $\xi_i(k_1) = i$. For $k \notin \{k_1,k_2\}$ we have $\xi_0(k) = \xi_2(k) = \xi(k)$. The set $B_\xi \subset B_\sigma$ is a submanifold of codimension one with B_{ξ_0} on one side and B_{ξ_2} on the other side. In the CW complex, ξ is represented by an edge from ξ_0 to ξ_2.

Proof. The first two claims follow directly from the definition of labels. Let ξ be a label of codimension $d = 1$, with $k_1 < k_2$ as above. We then have

$$\rho_k = \begin{cases} \eta a_{i_{k_1}}, & k_1 \le k < k_2, \\ \eta, & \text{otherwise.} \end{cases}$$

If $\tilde\xi \succ \xi$ we must have $\tilde\rho_k = \eta$ for $k < k_1$ or $k \ge k_2$: we thus also have $\tilde\varrho_k = \varrho_k$ and therefore $\tilde\xi(k) = \xi(k)$ for $k < k_1$ or $k > k_2$. For $k_1 \le k < k_2$ we must have $\tilde\rho_k \in \{\eta, \eta a_{i_{k_1}}\}$. If $k_1 \le k-1 < k < k_2$ we have either $\rho_k = \rho_{k-1}$ or $\rho_k = \rho_{k-1}a_{i_k}$: the second case contradicts the previous remarks. We thus have $\tilde\rho_k = \eta$ for all k. In particular, $\mathrm{codim}(\tilde\xi) = 0$.

If $w_0, w_1 \in \tilde B^+_{n+1}$, $w_0 < w_1$ and $\Pi(w_0) = \eta a_{i_{k_1}}$ then either $w_1 = w_0 \acute a_{i_{k_1}}$ or $w_1 = w_0 \grave a_{i_{k_1}}$. Thus, if $\tilde\xi \succ \xi$ there exists $\tilde\varepsilon : \{k_1, \ldots, k_2 - 1\} \to \{\pm 1\}$ such that, for all k, $k_1 \leq k < k_2$ implies $\tilde{\varrho}_k = \varrho_k(\acute{a}_{i_{k_1}})^{\tilde{\varepsilon}(k)}$. For $k_1 < k < k_2$, we have

$$\begin{aligned}\tilde{\varrho}_k &= \varrho_k(\acute{a}_{i_{k_1}})^{\tilde{\varepsilon}(k)} = \varrho_{k-1}(\acute{a}_{i_k})^{\xi(k)}(\acute{a}_{i_{k_1}})^{\tilde{\varepsilon}(k)} \\ &= \tilde{\varrho}_{k-1}(\acute{a}_{i_k})^{\tilde{\xi}(k)} = \varrho_{k-1}(\acute{a}_{i_{k_1}})^{\tilde{\varepsilon}(k-1)}(\acute{a}_{i_k})^{\tilde{\xi}(k)}\end{aligned}$$

and therefore $(\acute{a}_{i_k})^{\xi(k)}(\acute{a}_{i_{k_1}})^{\tilde{\varepsilon}(k)} = (\acute{a}_{i_{k_1}})^{\tilde{\varepsilon}(k-1)}(\acute{a}_{i_k})^{\tilde{\xi}(k)}$. If $|i_k - i_{k_1}| = 1$ and $\xi(k) = 2$ this implies $\tilde{\xi}(k) = \xi(k)$ and $\tilde{\varepsilon}(k) = -\tilde{\varepsilon}(k-1)$. Otherwise, this implies $\tilde{\xi}(k) = \xi(k)$ and $\tilde{\varepsilon}(k) = \tilde{\varepsilon}(k-1)$. In either case, this implies $\tilde{\xi}(k) = \xi(k)$ for all $k \notin \{k_1, k_2\}$, as desired. Furthermore, a choice of $\tilde{\xi}(k_1)$ uniquely determines $\tilde{\xi}(k)$ for $k_1 < k < k_2$. Similarly, we have

$$\tilde{\varrho}_{k_2} = \varrho_{k_2} = \varrho_{k_2-1}\acute{a}_{i_{k_1}} = \tilde{\varrho}_{k_2-1}(\acute{a}_{i_{k_2}})^{\tilde{\xi}(k_2)} = \varrho_{k_2-1}(\acute{a}_{i_{k_1}})^{\tilde{\varepsilon}(k_2-1)}(\acute{a}_{i_{k_1}})^{\tilde{\xi}(k_2)}$$

and therefore $\tilde{\xi}(k_2) = 1 - \tilde{\varepsilon}(k_2 - 1)$, completing the proof that there are exactly two labels $\tilde{\xi}$ with $\tilde{\xi} \succ \xi$. The other claims follow by construction. ☐

ark:codimone}

Remark 8.2. Notice that if ξ is a label of codimension one there are two other labels $\tilde{\xi}_0, \tilde{\xi}_1$ which are also of codimension 0 and satisfy $\tilde{\xi}_i(k_1) = i$ and $\tilde{\xi}_0(k) = \tilde{\xi}_2(k) = \xi(k)$ for $k \notin \{k_1, k_2\}$. A straightforward computation gives us $P(\tilde{\xi}_0) \neq P(\xi) \neq P(\tilde{\xi}_2)$, $P(\tilde{\xi}_0) = -P(\tilde{\xi}_2)$. ◇

We may want to translate from ξ- to ε-labels. This requires working with the variables $\tilde{\theta}_k$ instead of θ_k. Recall that $\tilde{\theta}_k$ is a smooth function of L when L is restricted to a fixed stratum B_ε but is not a continuous function of $L \in B_\sigma$. The aim of the next lemma is to discuss this change of variables in a simple situation.

{lemma:flop}

Lemma 8.3. *Let $i_1, \ldots i_\ell \in [\![n]\!]$, $i_1 = i_\ell$. Let $\theta_1, \ldots, \theta_\ell \in \mathbb{R}$. We have*

$$\begin{gathered}\alpha_{i_1}(\theta_1)\cdots\alpha_{i_\ell}(\theta_\ell) = \alpha_{i_1}(\theta_1+\pi)\alpha_{i_2}(\hat{\theta}_2)\cdots\alpha_{i_{\ell-1}}(\hat{\theta}_{\ell-1})\alpha_{i_\ell}(\theta_\ell-\pi), \\ \hat{\theta}_{i_k} = (-1)^{[|i_k-i_1=1|]}\theta_{i_k}.\end{gathered}$$

Proof. Notice that $(\hat{a}_{i_1})^{-1}\mathfrak{a}_j\hat{a}_{i_1} = (-1)^{[|j-i_1=1|]}\mathfrak{a}_j$ and therefore

$$(\hat{a}_{i_1})^{-1}\alpha_j(\theta)\hat{a}_{i_1} = \alpha_j((-1)^{[|j-i_1=1|]}\theta).$$

The result follows by inserting $1 = \hat{a}_{i_1}(\hat{a}_{i_1})^{-1}$ between every two consecutive terms of the left hand side. ☐

Consider a label ε of codimension one. Let ξ be the corresponding ξ-label: let $k_1 < k_2$ be as in Lemma 8.1. Clearly, $k_1 < k_2$ are the two entries of ξ of absolute value 2. Let $\varepsilon_+, \varepsilon_-$ be labels of codimension 0 defined by

$$\varepsilon_+(k) = \operatorname{sign}(\varepsilon(k)), \qquad \varepsilon_-(k) = \begin{cases} -\varepsilon_+(k), & k \in \{k_1, k_2\}, \\ -\varepsilon_+(k), & |i_k - i_{k_1}| = 1,\ k_1 < k < k_2, \\ \varepsilon_+(k), & \text{otherwise.}\end{cases}$$

dimonedirect}

Lemma 8.4. *The labels $\varepsilon_\pm$ correspond to the labels ξ_0, ξ_2.*

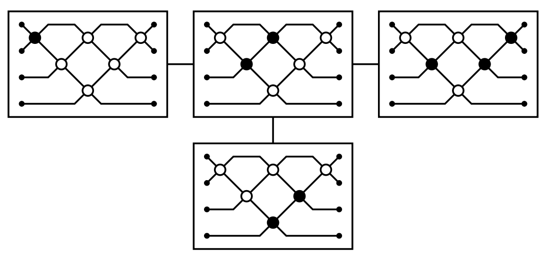
Proof. This is a straightforward computation using Lemma 8.3. □

A ε-label of codimension 0 can be represented over a diagram for σ by indicating a sign at each intersection. The edges are then constructed as follows. A bounded connected component of the complement of the diagram has vertices k_1 and k_2 on row i_{k_1} plus all vertices k with $k_1 < k < k_2$ and $|i_k - i_{k_1}| = 1$. If k_1 and k_2 have opposite signs we can click on that connecteed component, with the effect of changing all signs on its boundary.

example:4321-a1}

Example 8.5. Figure 4 shows an example of the construction above. We take $n = 3$ and $\sigma = \eta = a_1 a_2 a_1 a_3 a_2 a_1$, the top permutation (with $\ell = 6$). As an example, take $z_0 = \grave{a}_1 \acute{a}_2 \acute{a}_1 \acute{a}_3 \acute{a}_2 \acute{a}_1 = (-\hat{a}_1)\acute{\eta} = \acute{\eta}(-\hat{a}_3)$:

$$\acute{\eta} = \frac{-1 + \hat{a}_2 + \hat{a}_1\hat{a}_3 - \hat{a}_1\hat{a}_2\hat{a}_3}{2}, \qquad z_0 = \frac{\hat{a}_1 - \hat{a}_1\hat{a}_2 + \hat{a}_3 - \hat{a}_2\hat{a}_3}{2}.$$



{fig:4321-a1} FIGURE 4. The stratification of $B_{z_0} \subset B_\sigma$.

A case by case verification shows that B_{z_0} has 4 strata (or labels) of codimension 0, 3 strata of codimension 1 and no strata of codimension higher that 1. It follows that B_{z_0} is homotopically equivalent to the graph in Figure 4 and therefore contractible. In the figure, black indicates $\varepsilon(k) = -1$ and white indicates $\varepsilon(k) = +1$. ◇

ection:codimtwo}

9. Strata of codimension two

Labels and strata of codimension 2 also admit a relatively simple description. Understanding them allows us to compute the fundamental group of each connected component.

We prefer to break into cases. A ξ-label has the same type as its corresponding ε-label. A label ε of codimension 2 is of type I if and only if it fits at least one of the two descriptions below:

(1) There exist $k_1 < k_2 < k_3 < k_4$ with

$$\varepsilon(k_1) = \varepsilon(k_3) = -2, \qquad \varepsilon(k_2) = \varepsilon(k_4) = +2.$$

(2) There exist distinct k_1, k_2, k_3, k_4 with

$$\varepsilon(k_1) = \varepsilon(k_3) = -2, \qquad \varepsilon(k_2) = \varepsilon(k_4) = +2.$$
$$i_{k_1} = i_{k_2}, \qquad i_{k_3} = i_{k_4}, \qquad |i_{k_1} - i_{k_3}| > 1.$$

{lemma:typeI} We will later define type II; we first discuss labels of type I.

Lemma 9.1. *Given a label ε of codimension 2 and type I, there exist exactly four labels $\tilde{\varepsilon}$ of codimension 0 and four of codimension 1 such that $\tilde{\varepsilon} \succ \varepsilon$. Figure 5 shows what every transversal section to B_ε looks like. In the CW complex, ε corresponds to a quadrilateral.*

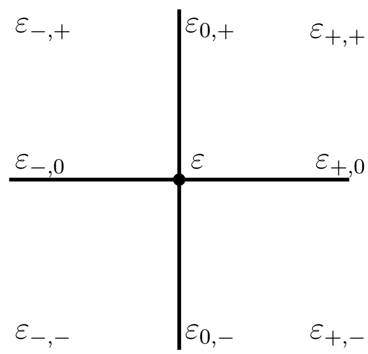


FIGURE 5. A transversal section to a stratum of codimension 2, type I.

{fig:codim2I}

Proof. Given a label ε of codimension 2 and type I, we define labels of codimension 1:

$$\varepsilon_{+,0}(k) = \begin{cases} \operatorname{sign}(\varepsilon(k)), & k \in \{k_1, k_2\}, \\ \varepsilon(k), & \text{otherwise}; \end{cases}$$

$$\varepsilon_{-,0}(k) = \begin{cases} -\operatorname{sign}(\varepsilon(k)), & k \in \{k_1, k_2\}, \\ (-1)^{[|i_k - i_{k_1}|=1]}\varepsilon(k), & k_1 < k < k_2, \\ \varepsilon(k), & \text{otherwise}; \end{cases}$$

$$\varepsilon_{0,+}(k) = \begin{cases} \operatorname{sign}(\varepsilon(k)), & k \in \{k_3, k_4\}, \\ \varepsilon(k), & \text{otherwise}; \end{cases}$$

$$\varepsilon_{0,-}(k) = \begin{cases} -\operatorname{sign}(\varepsilon(k)), & k \in \{k_3, k_4\}, \\ (-1)^{[|i_k - i_{k_3}|=1]}\varepsilon(k), & k_3 < k < k_4, \\ \varepsilon(k), & \text{otherwise}. \end{cases}$$

In the notation of codimension 1, define labels of codimension 0: $\varepsilon_{+,\pm} = (\varepsilon_{0,\pm})_+$, $\varepsilon_{-,\pm} = (\varepsilon_{0,\pm})_-$. Alternatively, we have $\varepsilon_{\pm,+} = (\varepsilon_{\pm,0})_+$, $\varepsilon_{\pm,-} = (\varepsilon_{\pm,0})_-$. It follows easily from Lemma 8.3 that $P(\varepsilon) = P(\varepsilon_{0,\pm}) = P(\varepsilon_{\pm,0}) = P(\varepsilon_{\pm,\pm})$.

The translation from ε- to ξ-labels is easy. Indeed, we have $\xi(k_1) = \xi(k_2) = \xi(k_3) = \xi(k_4) = 1$; for other values of k, $\xi(k) \in \{0,2\}$. We construct labels ξ_{j_1,j_2}, $j_1, j_2 \in \{0,1,2\}$, $\xi_{1,1} = \xi$. The labels are characterized by $\xi_{j_1,j_2}(k_1) = j_1$, $\xi_{j_1,j_2}(k_3) = j_2$, $k \notin \{k_1,k_2,k_3,k_4\} \to \xi_{j_1,j_2}(k) = \xi(k)$ and $P(\xi_{j_1,j_2}) = P(\xi)$. Given $L \in B_\xi = B_\varepsilon$, obtain $\theta_1, \dots, \theta_\ell$: a transversal section to B_ξ through L is given by

$$\begin{aligned} L(x_1,x_2) &= \mathbf{Q}(q_\ell^{-1} z(x_1,x_2)), \\ z(x_1,x_2) &= \alpha_{i_1}(\theta_1) \cdots \alpha_{i_{k_1}}(\theta_{k_1} + x_1) \cdots \alpha_{i_{k_3}}(\theta_{k_3} + x_2) \cdots \alpha_{i_\ell}(\theta_\ell) \end{aligned}$$

where $(x_1,x_2) \in U \subset \mathbb{R}^2$, U being a small open neighborhood of $0 \in \mathbb{R}^2$. Given x_1 (near 0), there exist unique $g(x_1) \in \mathbb{R}$ (near 0) and $\varrho_\star \in \tilde{B}^+_{n+1}$ such that $z_{k_3}(x_1, g(x_1)) \in \grave{\eta}\,\mathrm{Bru}_{\varrho_\star}$. The function g is continuous in a small neighborhood of 0. Near the origin we have $L(x_1,x_2) \in \xi_{1+\mathrm{sign}(x_1),1+\mathrm{sign}(x_2-g(x_1))}$. The rest of the proof is similar to that of Lemma 8.1. $\square$

A label ε of codimension 2 is of type II if and only if there exist integers $k_1 < k_3 < k_4 < k_2$ such that:

$$\varepsilon(k_1) = \varepsilon(k_3) = -2, \qquad \varepsilon(k_2) = \varepsilon(k_4) = +2, \qquad |i_{k_1} - i_{k_3}| = 1. \tag{29}$$

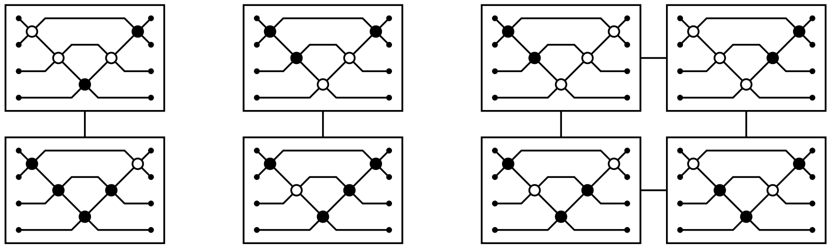
Clearly, a label of codimension 2 is either of type I or of type II. A label of codimension 2 type II is of subtype II-j if there are precisely j values of k with
{example:4231} $k_1 \le k \le k_2$ and $i_k = i_{k_1}$.

Example 9.2. Consider now $\sigma = (4,2,3,1) = a_1a_2a_3a_2a_1$. A simple computation gives $\acute{\sigma} = (\hat{a}_2 + \hat{a}_1\hat{a}_3)/\sqrt{2}$ and

$$\acute{\sigma}\,\mathrm{Quat}_4 = \left\{ \frac{\pm 1 \pm \hat{a}_1\hat{a}_2\hat{a}_3}{\sqrt{2}}, \frac{\pm\hat{a}_1 \pm \hat{a}_2\hat{a}_3}{\sqrt{2}}, \frac{\pm\hat{a}_2 \pm \hat{a}_1\hat{a}_3}{\sqrt{2}}, \frac{\pm\hat{a}_3 \pm \hat{a}_1\hat{a}_2}{\sqrt{2}} \right\}.$$

The action of $\mathcal{E}$ on $\acute{\sigma}\,\mathrm{Quat}_4$ has 5 orbits. The orbit $\mathcal{O}_{\acute{\sigma}} = \{(\pm\hat{a}_2 \pm \hat{a}_1\hat{a}_3)/\sqrt{2}\}$ has size 4, and for each z we have $N(z) = N_{\mathrm{thin}}(z) = 2$ so that B_z has two thin (and therefore contractible) connected components. The orbits $\mathcal{O}_{\hat{a}_1\acute{\sigma}} = \{(\pm\hat{a}_3 \pm \hat{a}_1\hat{a}_2)/\sqrt{2}\}$ and $\mathcal{O}_{\hat{a}_3\acute{\sigma}} = \{(\pm\hat{a}_1 \pm \hat{a}_2\hat{a}_3)/\sqrt{2}\}$ both have size 4. For each z in one of these two orbits we have $N(z) = 2$ and $N_{\mathrm{thin}}(z) = 0$. Figure 6 shows the stratification of B_z for one representative of each orbit: it follows from the previous section that such sets B_z are contractible. The orbit $\mathcal{O}_{\hat{a}_2\acute{\sigma}} = \{(-1 \pm \hat{a}_1\hat{a}_2\hat{a}_3)/\sqrt{2}\}$ has size 2. For z in this orbit we have $N(z) = 0$, so that the corresponding sets B_z are empty.

Finally, the orbit $\mathcal{O}_{-\hat{a}_2\acute{\sigma}} = \{(1 \pm \hat{a}_1\hat{a}_2\hat{a}_3)/\sqrt{2}\}$ also has size 2: consider $z = -\hat{a}_2\acute{\sigma} = \acute{a}_1\acute{a}_2\acute{a}_3\grave{a}_2\grave{a}_1 = (1 + \hat{a}_1\hat{a}_2\hat{a}_3)/\sqrt{2}$. Figure 6 also shows the stratification of



{fig:4231x} FIGURE 6. The stratifications of $B_{\hat{a}_1\acute{\sigma}}$, $B_{\hat{a}_3\acute{\sigma}}$, and $B_{-\hat{a}_2\acute{\sigma}}$ for $\sigma = a_1a_2a_3a_2a_1$.

B_z. A straightforward computation verifies that the set B_z consists of matrices of the form:

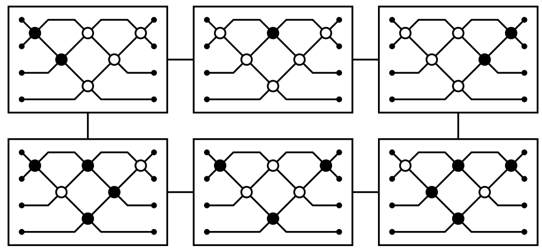
$$L = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix}, \qquad \begin{aligned} & l_{41} > \max\{0, l_{21}l_{42}, l_{31}l_{43}\}, \\ & l_{32} = \frac{l_{31}l_{42}}{l_{41}}. \end{aligned} \tag{30}$$

The above description makes it clear that B_z is contractible, but we want to explore the decomposition into strata. The set B_z contains 4 open strata, 4 strata of codimension 1 and one stratum of codimension 2, with label $\varepsilon = (-2,-2,+1,+2,+2)$ and corresponding ξ-label $\xi = (1,1,0,1,1)$. This is a label of subtype II-2. The set B_ξ is the set of matrices of the form above with $l_{32} = l_{42} = l_{43} = 0$. The open strata are characterized by the signs of l_{42} and l_{43}. For instance, $B_{-1,-1,+1,+1,+1}$ is the set of matrices L of the form described in Equation (30) with $l_{42} > 0$, $l_{43} > 0$. Similarly, $B_{-1,+1,-1,-1,+1}$ is the corresponding set with $l_{42} > 0$, $l_{43} < 0$. Thus, the nine strata form the same configuration as shown in Figure 5, merely changing names. In our CW complex, the label $\varepsilon = (-2,-2,+1,+2,+2)$ corresponds to a square, glued along the four edges (corresponding to labels of codimension 1) in the obvious way. Thus, B_z is also
example:4321} contractible. Summing up, B_σ has 18 connected components, all contractible. $\diamond$

Example 9.3. Set $\sigma = \eta = a_1a_2a_1a_3a_2a_1$. We have

$$\acute{\eta} = \frac{-1 + \hat{a}_2 + \hat{a}_1\hat{a}_3 - \hat{a}_1\hat{a}_2\hat{a}_3}{2}, \qquad \grave{\eta} = \frac{-1 - \hat{a}_2 + \hat{a}_1\hat{a}_3 + \hat{a}_2\hat{a}_2\hat{a}_3}{2},$$
$$\acute{\eta}\,\mathrm{Quat}_4 = \left\{ \frac{\pm 1 \pm \hat{a}_2 \pm \hat{a}_1\hat{a}_3 \pm \hat{a}_1\hat{a}_2\hat{a}_3}{2}, \frac{\pm\hat{a}_1 \pm \hat{a}_1\hat{a}_2 \pm \hat{a}_3 \pm \hat{a}_2\hat{a}_3}{2} \right\}$$

where we must take an even number of '$-$' signs (so that the above set has 16 elements). There are three $\mathcal{E}$-orbits, determined by real part (two of size 4, one of size 8). If $\Re(z) = -\frac{1}{2}$, the set B_z has two thin connected components (and no thick ones).



{fig:4321-a2} FIGURE 7. The stratification of $B_{-\hat{a}_2\acute{\eta}}$.

We already discussed in Example 8.5 the stratification of $B_{-\hat{a}_1\acute{\eta}}$, which is contractible. Figure 7 shows the stratifications of B_{z_1} where

$$z_1 = -\hat{a}_2\acute{\eta} = -\acute{\eta}\hat{a}_2 = \frac{1+\hat{a}_2+\hat{a}_1\hat{a}_3+\hat{a}_1\hat{a}_2\hat{a}_3}{2}.$$

In B_{z_1} there are 6 labels of codimension 0, 6 labels of codimension 1 and exactly one label of codimension 2: the ε-label is $\varepsilon = (-2,-2,+1,+1,+2,+2)$ and the corresponding ξ-label is $\xi = (1,1,0,0,1,1)$. The label ε is of Subtype II.3. In order to study a transversal section to $B_\varepsilon = B_\xi$, we take

$$z_1 = \alpha_1\left(\frac{\pi}{2}+x_1\right), \quad z_2 = z_1\alpha_2\left(\frac{\pi}{2}+x_2\right), \quad z_3 = z_2\alpha_1\left(\frac{\pi}{4}\right).$$

We perform the computations in the orthogonal group. In order to determine the position of a point in the strata above, we must study the signs of

$$(z_1)_{1,1} = -\sin(x_1), \quad \det\begin{pmatrix}(z_2)_{1,1} & (z_2)_{1,2}\\ (z_2)_{2,1} & (z_2)_{2,2}\end{pmatrix} = -\sin(x_2),$$
$$(z_3)_{1,1} = -\frac{\sqrt{2}}{2}(\sin(x_1)-\cos(x_1)\sin(x_2)).$$

These three expressions have pairwise linearly independent deritatives in the origin. The transversal section is shown in Figure 8. Thus, in the CW complex shown in Figure 7, the cell of dimension 2 glues in the obvious way. The set $B_{-\hat{a}_1\acute{\eta}}$ is therefore contractible.

Summing up, the set B_η has 20 connected components, all contractible. The total number of connected components of B_η was first calculated by the third and the fourth authors jointly with Vl. Kostov using ad hoc methods back in 1987
xample:bcbabdcb} (unpublished); see also [4, 12]. $\diamond$

Example 9.4. Consider $\sigma = a_2a_3a_2a_1a_2a_4a_3a_2$ and the labels

$$\xi_0 = (1,1,0,0,2,0,1,1), \qquad \xi_1 = (1,0,1,0,1,0,0,1).$$

We claim that

$$\overline{B_{\xi_1}} \cap B_{\xi_0} \neq \emptyset, \qquad B_{\xi_0} \not\subseteq \overline{B_{\xi_1}}.$$

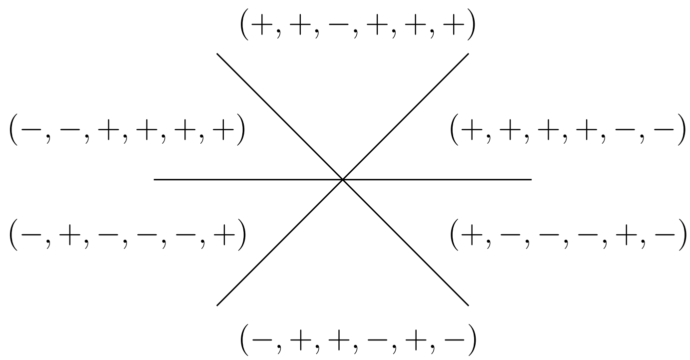


FIGURE 8. A transversal section to a stratum of codimension 2, subtype II-3.

ig:codim2II3}

The fact that inclusion does not hold follows from the fact that B_{ξ_0} and B_{ξ_1} are disjoint smooth submanifolds of the same dimension. Consider

$$z_8 = \alpha_2(\theta_1)\alpha_3(\theta_2)\alpha_2(\theta_3)\alpha_1(\theta_4)\alpha_2(\theta_5)\alpha_4(\theta_6)\alpha_3(\theta_7)\alpha_2(\theta_8) \in \mathrm{Bru}_{\acute{\sigma}}$$

and corresponding L. Take $\theta_1 = \pi/2$, $\theta_3 = \theta_4 = \theta_6 = \theta_7 = \theta_8 = \pi/4$. For $\theta_2 = \pi/2$ and $\theta_5 = \pi - \arctan(\sqrt{2})$ we have $L \in B_{\xi_0}$. For $\theta_2 \in (\pi/3, \pi/2)$ and $\theta_5 = \pi - \arctan(\sqrt{2})$ we have $L \in B_{\xi_1}$.

On the other hand, take $\theta_2 = \pi/2$, $\theta_5 = 3\pi/4$: we have $L \in B_{\xi_0}$. Consider the transversal section

$$z(x_1, x_2) = \alpha_2(\theta_1 + x_1)\alpha_3(\theta_2 + x_2)\alpha_2(\theta_3)\alpha_1(\theta_4)\alpha_2(\theta_5)\alpha_4(\theta_6)\alpha_3(\theta_7)\alpha_2(\theta_8);$$

We have ... ◇

We are ready to describe the situation of strata of codimension 2, subtype II-j, $j \leq 3$.

a:subtypeII3}

Lemma 9.5. *Consider σ and a reduced word fixed. Let ξ be a label of codimension 2, subtype II-j, $j \in \{2, 3\}$. The set of labels $\tilde{\xi}$ with $\xi \prec \tilde{\xi}$ has precisely $4j$ elements: $2j$ labels of codimension zero and $2j$ labels of codimension one. Combinatorially, these $4j$ labels form a square if $j = 2$ or a hexagon if $j = 3$. Any smooth transversal cross section to $B_\xi \subset B_\sigma$ meets the $4j$ lower codimensional strata as in Figure 5 (if $j = 2$) or as in Figure 8 (if $j = 3$). In the CW complex, the 2-cell corresponding to ξ is glued along the square or hexagon in the obvious way.*

Proof. To be written. □

10. EXAMPLES

Here we study in details the low-dimensional cases $n = 2, 3$ and 4.

{section:n2}

10.1. **The case $n = 2$.** Set $a_1 = (1,2)$, $a_2 = (2,3)$. There is an identification $\mathrm{Cl}_3^0 = \mathbb{H}$ (the quaternions) given by $\hat{a}_1 = \mathbf{k}$, $\hat{a}_2 = \mathbf{i}$. We have

$$\mathrm{Quat}_3 = \{\pm 1, \pm\hat{a}_1, \pm\hat{a}_2, \pm\hat{a}_1\hat{a}_2\} = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\} = Q_8,$$

$$\tilde{B}_3^+ = \left\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \frac{\pm 1 \pm \mathbf{i}}{\sqrt{2}}, \frac{\pm 1 \pm \mathbf{j}}{\sqrt{2}}, \ldots, \frac{\pm\mathbf{j} \pm \mathbf{k}}{\sqrt{2}}, \frac{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}}{2}\right\}.$$

As a first example, take $\sigma = \eta = a_1 a_2 a_1 = (3,2,1) \in S_3$; we have

$$\acute{\eta}\,\mathrm{Quat}_3 = \Pi^{-1}[\{\eta\}] = \left\{\frac{\pm 1 \pm \mathbf{j}}{\sqrt{2}}, \frac{\pm\mathbf{i} \pm \mathbf{k}}{\sqrt{2}}\right\}.$$

The map $P : \{\pm 1\}^{[\![3]\!]} \to \Pi^{-1}[\{\eta\}]$ satisfies

$$P(+,+,+) = \acute{a}_1\acute{a}_2\acute{a}_1 = \frac{(1+\mathbf{k})(1+\mathbf{i})(1+\mathbf{k})}{2\sqrt{2}} = \frac{\mathbf{i}+\mathbf{k}}{\sqrt{2}},$$

$$P(+,-,+) = \frac{-\mathbf{i}+\mathbf{k}}{\sqrt{2}}, \quad P(-,+,-) = \frac{\mathbf{i}-\mathbf{k}}{\sqrt{2}}, \quad P(-,-,-) = \frac{-\mathbf{i}-\mathbf{k}}{\sqrt{2}},$$

$$P(-,+,+) = P(+,-,-) = \frac{1-\mathbf{j}}{\sqrt{2}}, \quad P(-,-,+) = P(+,+,-) = \frac{1+\mathbf{j}}{\sqrt{2}}.$$

We thus have

$$N\left(\frac{1 \pm \mathbf{j}}{\sqrt{2}}\right) = 2, \quad N\left(\frac{\pm\mathbf{i} \pm \mathbf{k}}{\sqrt{2}}\right) = 1, \quad N\left(\frac{-1 \pm \mathbf{j}}{\sqrt{2}}\right) = 0.$$

In the notation of Theorem 1, we have $\ell = 3$, $B = \emptyset$, $b = 0$ and $\tilde{H}_B = \tilde{B}_{n+1}^+$. Theorem 1, Lemmas 6.7 and 6.8 thus predict that for any $z \in \Pi^{-1}[\{\eta\}]$ we have $N(z) + N(-z) = 2$ and $N(z) - N(-z) = 2\sqrt{2}\,\Re(z)$. In the notation of Theorem 2, we have $c = 2$ (in cycle notation, $\eta = (13)(2)$) and $c_{\mathrm{anti}} = 1$ ($(\acute{\eta})^{[(-,-)]} = -\acute{\eta}$). Theorem 2 thus predicts $|\mathcal{E}_{\eta}| = 1$. Indeed, we have $N_{\mathrm{thin}}((\pm\mathbf{i} \pm \mathbf{k})/\sqrt{2}) = 1$, $N_{\mathrm{thin}}((\pm 1 \pm \mathbf{j})/\sqrt{2}) = 0$, also consistent with the numbers above.

As a second example, consider now $\sigma = a_2 = (2,3) \in S_3$ and

$$\Pi^{-1}[\{a_2\}] = \left\{\frac{\pm 1 \pm \mathbf{i}}{\sqrt{2}}, \frac{\pm\mathbf{j} \pm \mathbf{k}}{\sqrt{2}}\right\}.$$

The image of the map P consists of 2 points only: $P(+) = \acute{a}_2 = (1+\mathbf{i})/\sqrt{2}$ and $P(-) = \grave{a}_2 = (1-\mathbf{i})/\sqrt{2}$. We thus have $N((1 \pm \mathbf{i})/\sqrt{2}) = 1$ and otherwise $N(z) = 0$ for $z \in \Pi^{-1}[\{a_2\}]$. We have $\ell = 1$, $B = \{1\}$ and $b = 1$. Theorem 1 predicts that for $z \in \tilde{H}_B \cap \Pi^{-1}[\{a_2\}] = \{(\pm 1 \pm \mathbf{i})/\sqrt{2}\}$ we have $N(z) + N(-z) = 1$ and $N(z) - N(-z) = \sqrt{2}\,\Re(z)$, again consistent with the numbers above.

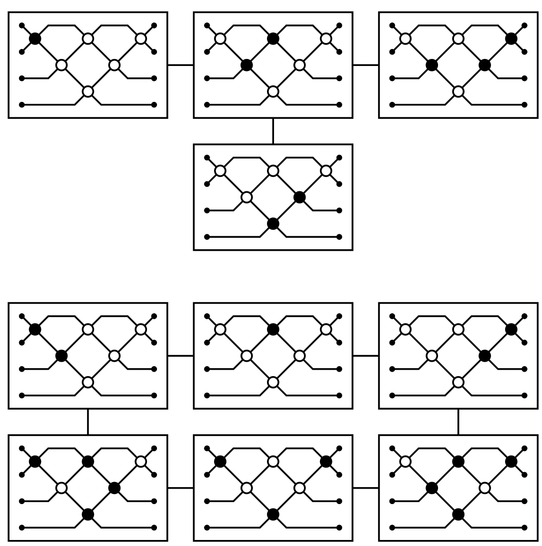
{section:n3}

10.2. **The case** $n = 3$. Set $\sigma = \eta = a_1a_2a_1a_3a_2a_1$. We have $\ell = 6$, $b = |\operatorname{Block}(\eta)| = 0$, $c = \operatorname{nc}(\eta) = 2$ and

$$\acute{\eta} = \frac{-1 + \hat{a}_2 + \hat{a}_1\hat{a}_3 - \hat{a}_1\hat{a}_2\hat{a}_3}{2}, \qquad \grave{\eta} = \frac{-1 - \hat{a}_2 + \hat{a}_1\hat{a}_3 + \hat{a}_2\hat{a}_2\hat{a}_3}{2},$$

$$\Pi^{-1}[\{\eta\}] = \left\{ \frac{\pm 1 \pm \hat{a}_2 \pm \hat{a}_1\hat{a}_3 \pm \hat{a}_1\hat{a}_2\hat{a}_3}{2}, \frac{\pm\hat{a}_1 \pm \hat{a}_1\hat{a}_2 \pm \hat{a}_3 \pm \hat{a}_2\hat{a}_3}{2} \right\}$$

where we must take an even number of '$-$' signs (so that $|\Pi^{-1}[\{\eta\}]| = 16$). If $z \in \Pi^{-1}[\{\eta\}]$ and $\Re(z) \neq 0$ then (from Remark 6.15) $c_{\text{anti}}(z) = 0$; if $\Re(z) = 0$ then (from a direct computation) $c_{\text{anti}}(z) = 1$. It follows from Lemma 6.14 that there are three $\mathcal{E}$-orbits, determined by real part (two of size 4, one of size 8). It follows from Theorem 1 that $N(z) = 4 + 4\Re(z)$ and from Theorem 2 that $N_{\text{thin}}(z) = 2$ if $\Re(z) = -\frac{1}{2}$ and $N_{\text{thin}}(z) = 0$ otherwise. Thus, if $\Re(z) = -\frac{1}{2}$, the set B_z has two thin connected components (and no thick ones).



{fig:4321x}

FIGURE 9. The stratifications of $B_{-\hat{a}_1\acute{\eta}}$ and $B_{-\hat{a}_2\acute{\eta}}$.

Figure 9 shows the stratifications of B_{z_0} and B_{z_1}, where

$$z_0 = -\hat{a}_1\acute{\eta} = -\acute{\eta}\hat{a}_3 = \frac{\hat{a}_1 - \hat{a}_1\hat{a}_2 + \hat{a}_3 - \hat{a}_2\hat{a}_3}{2},$$

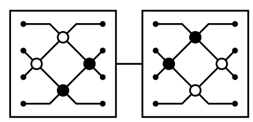
$$z_1 = -\hat{a}_2\acute{\eta} = -\acute{\eta}\hat{a}_2 = \frac{1 + \hat{a}_2 + \hat{a}_1\hat{a}_3 + \hat{a}_1\hat{a}_2\hat{a}_3}{2}.$$

Notice that z_0 and z_1 are representatives of the two remaining orbits.

As computed above, we know the numbers of labels of codimension 0: there are 4 with $P(\varepsilon) = z_0$ and 6 with $P(\varepsilon) = z_1$. They are listed in Figure 9: each such label is represented by indicating over a diagram for η the value of ε at each intersection. We follow the convention that white stands for $+1$ and black for -1. Each small box therefore represents a stratum of codimension 0.

A label ε of codimension 1 is indicated by an edge between the boxes for ε_+ and ε_-. Recall that B_ε is a two-sided contractible hypersurface in B_η, with $B_{\varepsilon_\pm}$ on the two sides. There are no labels of codimension 2 in B_{z_0}. There is exactly one label of codimension 2 in B_{z_1}: $(-2, -2, +1, +1, +2, +2)$.

The stratification shows that both B_{z_0} and B_{z_1} consist of a single thick connected component (and no thin ones). The total number of connected components of B_η is therefore 20, comp. [12]. (This answer was first calculated by the third and the fourth authors jointly with Vl. Kostov using ad hoc methods back in 1987, unpublished.) It follows from the stratifications that B_{z_0} is contractible and B_{z_1} is . . . (What? Also contractible?).



{fig:3412x} FIGURE 10. The stratification of $B_{-\acute{\sigma}\hat{a}_3}$, $\sigma = a_2a_1a_3a_2$.

As another example, set $\sigma = a_2a_1a_3a_2$. We have $\ell = 4$, $b = |\operatorname{Block}(\sigma)| = 0$, $c = \operatorname{nc}(\sigma) = 2$ and

$$\acute{\sigma} = -\grave{\sigma} = \frac{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 - \hat{a}_1\hat{a}_2\hat{a}_3}{2},$$
$$\Pi^{-1}[\{\sigma\}] = \left\{ \frac{\pm\hat{a}_1 \pm \hat{a}_2 \pm \hat{a}_3 \pm \hat{a}_1\hat{a}_2\hat{a}_3}{2}, \frac{\pm 1 \pm \hat{a}_1\hat{a}_2 \pm \hat{a}_1\hat{a}_3 \pm \hat{a}_2\hat{a}_3}{2} \right\}$$

where we must take an odd number of '$-$' signs. It follows from Theorem 1 that $N(z) = 1 + 2\Re(z)$. If $\Re(z) = -\frac{1}{2}$ we have $N(z) = 0$ and therefore $B_z = \emptyset$. If $\Re(z) = 0$ we have $N(z) = 1$ and B_z must therefore consist of a single thin component. This is confirmed by Theorem 2 and $c_{\text{anti}}(\acute{\sigma}) = 1$. Finally, if $\Re(z) = \frac{1}{2}$ we have $N(z) = 2$ and B_z has a unique thick connected component. Figure 10 shows the stratification of $B_{-\acute{\sigma}\hat{a}_3}$, $-\acute{\sigma}\hat{a}_3 = (1 - \hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_3 - \hat{a}_2\hat{a}_3)/2$. The total number of connected components of B_σ is therefore 12, and they are all contractible.

{section:n4}

10.3. **The case** $n = 4$. Set $\sigma = \eta = a_1a_2a_1a_3a_2a_1a_4a_3a_2a_1$. We have $\ell = 10$, $b = |\operatorname{Block}(\eta)| = 0$, $c = \operatorname{nc}(\eta) = 3$ and

$$\acute{\eta} = -\grave{\eta} = \frac{-\hat{a}_1 - \hat{a}_1\hat{a}_2\hat{a}_3 - \hat{a}_4 - \hat{a}_2\hat{a}_3\hat{a}_4}{2},$$

It follows from Theorem 1 that $N(z) = 32 + 16\Re(z)$. In this example, it turns out that $\Pi^{-1}[\{\eta\}]$ contains 4 elements with $\Re(z) = \frac{1}{2}$, 4 elements with $\Re(z) = -\frac{1}{2}$

and 24 elements with $\Re(z) = 0$. It turns out that, for $z \in \Pi^{-1}[\{\eta\}]$, $c_{\mathrm{anti}}(z) = 1$ if and only if $\Re(z) = 0$. The set $\Pi^{-1}[\{\eta\}]$ thus has 5 orbits, of sizes $8, 4, 4, 8, 8$, shown below.

$$\begin{aligned}
\mathcal{O}_{\acute{\eta}} &= \left\{ \frac{\pm\hat{a}_1 \pm \hat{a}_1\hat{a}_2\hat{a}_3 \pm \hat{a}_4 \pm \hat{a}_2\hat{a}_3\hat{a}_4}{2} \right\}, \quad N(z) = 32, \quad N_{\mathrm{thin}}(z) = 2, \\
\mathcal{O}_{\hat{a}_1\acute{\eta}} &= \left\{ \frac{1 \pm \hat{a}_2\hat{a}_3 \pm \hat{a}_1\hat{a}_4 \pm \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4}{2} \right\}, \quad N(z) = 40, \quad N_{\mathrm{thin}}(z) = 0, \\
\mathcal{O}_{-\hat{a}_1\acute{\eta}} &= \left\{ \frac{-1 \pm \hat{a}_2\hat{a}_3 \pm \hat{a}_1\hat{a}_4 \pm \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4}{2} \right\}, \quad N(z) = 24, \quad N_{\mathrm{thin}}(z) = 0, \\
\mathcal{O}_{\hat{a}_2\acute{\eta}} &= \left\{ \frac{\pm\hat{a}_1\hat{a}_2 \pm \hat{a}_1\hat{a}_3 \pm \hat{a}_2\hat{a}_4 \pm \hat{a}_1\hat{a}_4}{2} \right\}, \quad N(z) = 32, \quad N_{\mathrm{thin}}(z) = 0, \\
\mathcal{O}_{\hat{a}_1\hat{a}_2\acute{\eta}} &= \left\{ \frac{\pm\hat{a}_2 \pm \hat{a}_3 \pm \hat{a}_1\hat{a}_2\hat{a}_4 \pm \hat{a}_1\hat{a}_3\hat{a}_4}{2} \right\}, \quad N(z) = 32, \quad N_{\mathrm{thin}}(z) = 0,
\end{aligned}$$

The action of $\mathcal{E}$ splits the set $\acute{\eta}\,\mathrm{Quat}_{n+1}$ into 5 orbits. In the expressions in the Clifford algebra, we must always have an even number of '$-$' signs.

In order to count connected components and obtain further information about the topology of the sets B_z, $z \in \acute{\eta}\,\mathrm{Quat}_{n+1}$, we can pick one representative from each orbit and draw the strata. As a sample, we do this in Figure 11 for $z = -\hat{a}_1\acute{\eta} = -\acute{\eta}\hat{a}_4$. In this case, there are exactly two labels of codimension 2:

$$\begin{gathered}
(+1, -2, -2, +1, +1, +2, +1, +1, +2, +1), \\
(-1, -2, -2, +1, -1, +2, -1, +1, +2, -1);
\end{gathered}$$

there are no labels of higher dimension. It follows that $B_{-\hat{a}_1\acute{\eta}}$ is homotopically equivalent to the disjoint union of two points. In other words, the connected components are contractible.

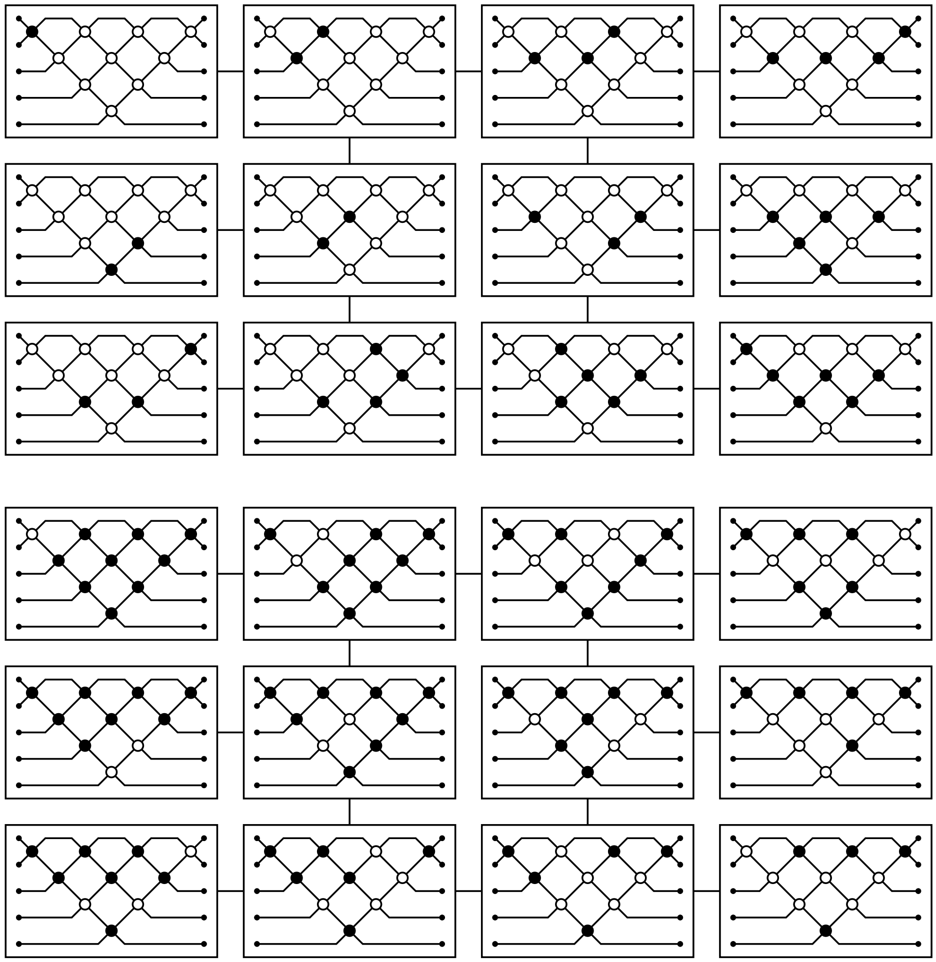
Remark 10.1. We know that the thick parts are otherwise connected. This is sufficient to reproduce the counting of 52 components.

Note to authors: we should complete the verification that all connected components of B_η are contractible. $\diamond$

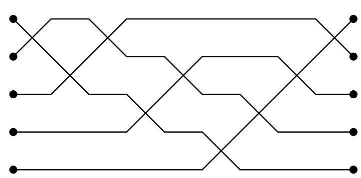
Consider now $\sigma = (5, 4, 2, 3, 1) = a_1a_2a_1a_3a_2a_1a_4a_3a_2a_1$; Figure 12 shows this reduced word as a diagram. In the notation of cycles, $\sigma = (15)(243)$; we therefore have $n = 4$, $\ell = 9$, $c = 2$ and $b = 0$. Theorem 1 tells us that, for $z \in \acute{\sigma}\,\mathrm{Quat}_5$, we have $N(z) = 16 + 8\sqrt{2}\Re(z)$. We have

$$\acute{\sigma} = \frac{-\hat{a}_1 + \hat{a}_1\hat{a}_2 + \hat{a}_1\hat{a}_3 - \hat{a}_1\hat{a}_2\hat{a}_3 - \hat{a}_4 + \hat{a}_2\hat{a}_4 + \hat{a}_3\hat{a}_4 - \hat{a}_2\hat{a}_3\hat{a}_4}{2\sqrt{2}}$$

and $\Re(\pm\hat{a}_1\acute{\sigma}) = \pm\sqrt{2}/4$.



{fig:54321x} FIGURE 11. The stratification of $B_{-\hat{a}_1\acute{\eta}}$.



{fig:54231} FIGURE 12. The permutation $\sigma \in S_5$.

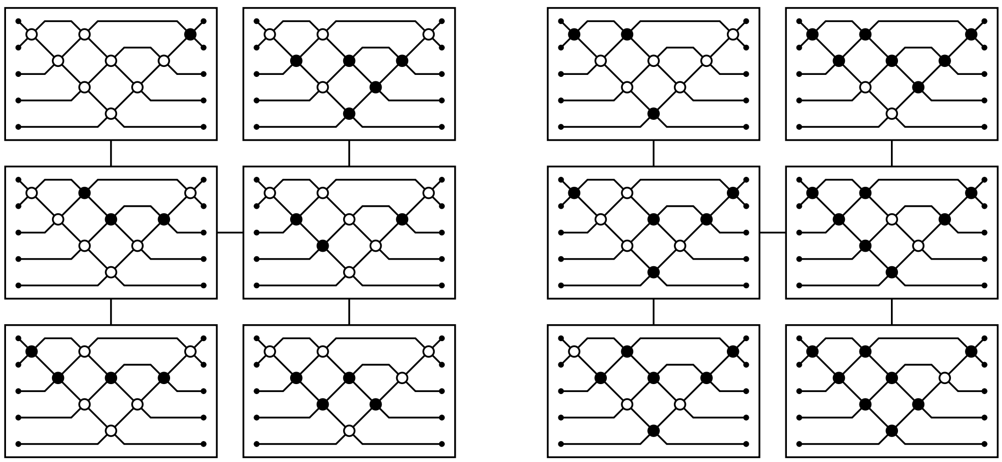
It turns out that $\Re(z) = 0$ implies $c_{\text{anti}} = 1$: the set $\acute{\sigma}\,\mathrm{Quat}_5$ thus has 3 orbits under $\mathcal{E}$, of sizes 16, 8 and 8:

$$\mathcal{O}_{\acute{\sigma}}, \quad \Re(z) = 0, \quad N(z) = 16, \quad N_{\text{thin}}(z) = 1,$$

$$\mathcal{O}_{\hat{a}_1\acute{\sigma}}, \quad \Re(z) = \frac{\sqrt{2}}{4}, \quad N(z) = 20, \quad N_{\text{thin}}(z) = 0,$$

$$\mathcal{O}_{-\hat{a}_1\acute{\sigma}}, \quad \Re(z) = -\frac{\sqrt{2}}{4}, \quad N(z) = 12, \quad N_{\text{thin}}(z) = 0.$$

The thick part of $B_{\acute{\sigma}}$ is connected, so that $B_{\acute{\sigma}}$ has two connected components. The set $B_{\hat{a}_1\acute{\sigma}}$ is also connected, but $B_{-\hat{a}_1\acute{\sigma}}$ has two connected components. In Figure 13 we show the two connected components of B_z for $z = -\acute{\sigma}\hat{a}_1$. There are no labels of codimension 2 or higher and therefore these connected components are contractible. Notice that an involution takes one component of B_z to the other. The total number of connected components of B_σ is therefore 56.



{fig:54231x}
remark:54231}

FIGURE 13. The twelve sets $B_\varepsilon \subset B_z$.

Remark 10.2. Note to authors: the hand written notes contain a sketch of proof that all connected components are contractible. What is missing is a solid proof that cells of dimension 3 are glued to the CW complex in the expected way. $\diamond$

11. Outlook

1. To finish the introduction, let us mention that we hope to extend our methods to the case of pairwise intersections of (big) Bruhat cells over $\mathbb{C}$ whose cohomology has an important representation-theoretical interpretation, see e.g. [7], sec. 1.5.

2. What about the Deodhar decomposition?

12. Appendix 1. On the number of connected components of Bru_σ.

As we mentioned in the introduction the number of connected components of B_η equals $3\cdot 2^n$ for all $n \geq 5$. Our next result shows that this formula gives a lower bound for almost all permutations $\sigma \in S_{n+1}$ if n is large. Recall the following notion from the introduction.

Definition 12.1. A permutation $\sigma \in S_{n+1}$ *blocks at* k, $k \in [\![n]\!] = \{1, 2, \ldots, n\}$, if and only if for all j, $j \leq k$ implies $j^\sigma \leq k$. Given σ, let $\mathrm{Block}(\sigma) \subset [\![n]\!]$ be the set of values of k such that σ blocks at k. A pair $(i_0, i_1) \in [\![n+1]\!]^2$ is an *inversion* of $\sigma \in S_{n+1}$ if $i_0 < i_1$ and $i_0^\sigma > i_1^\sigma$; also, $\mathrm{inv}(\sigma) \in \mathbb{N}$ denotes the number of inversions of σ.

theo:lowerbound}

Theorem 3. *If, for a permutation* $\sigma \in S_{n+1}$, $\mathrm{Block}(\sigma) = \emptyset$ *and* $\mathrm{inv}(\sigma) > 2n + 2$ *then* Bru_σ *has at least* $3 \cdot 2^n$ *connected components.*

Notice that, for n large, almost every permutation $\sigma \in S_{n+1}$ satisfies the assumptions $\mathrm{Block}(\sigma) = \emptyset$ and $\mathrm{inv}(\sigma) > 2n + 2$ of Theorem 3. There are reasons to believe that the number of connected components of Bru_σ is exactly $3 \cdot 2^n$ for most permutations.

Recall that $\varepsilon \in \{\pm 1\}^\ell$ is thin if and only if $i_{j_0} = i_{j_1}$ implies $\varepsilon(j_0) = \varepsilon(j_1)$ (a reduced word $\sigma = a_{i_1} \cdots a_{i_\ell} \in S_{n+1}$ is assumed to be fixed). Equivalently, $\varepsilon \in \{\pm 1\}^\ell$ is thin if and only if it can be written as $\varepsilon = \tilde{\varepsilon} \circ i$ for some $\tilde{\varepsilon} \in \{\pm 1\}^n$. If $\varepsilon \in \{\pm 1\}^\ell$ is thin and $z = P(\varepsilon)$ then B_ε is a contractible connected component of Bru_z, $z = P(\varepsilon)$. Indeed, if $\varepsilon(j) = +1$ for all j then $B_\varepsilon = \mathrm{Pos}_\sigma$. We follow here the notation of [5]; Pos_η is the open semigroup of totally positive matrices [3] and, for other $\sigma \in S_{n+1}$, $\mathrm{Pos}_\sigma = \mathrm{Bru}_\sigma \cap \overline{\mathrm{Pos}_\eta}$ is a contractible submanifold of dimension $\mathrm{inv}(\sigma)$. The other sets B_ε, ε thin, are obtained by conjugation by a diagonal matrix E with $\varepsilon(j) = E_{j,j} E_{j+1,j+1}$ (this is discussed in detail in Section **??**). Such connected components $B_\varepsilon \subset \mathrm{Bru}_z$ are *thin*; the others are *thick*.

{lemma:countBz}

Lemma 12.2. *The number of thin connected components of* Bru_z *is* $N_{\mathrm{thin}}(z)$. *If* $N_{\mathrm{thin}}(z) = N(z)$ *then* Bru_z *admits no thick connected component. On the other hand, if* $N_{\mathrm{thin}}(z) < N(z)$ *then* Bru_z *admits at least one connected component.*

Proof. If ε is a thick solution of $P(\varepsilon) = z$ then B_ε is contained in a thick connected component of Bru_z. $\square$

Lemma 12.2 makes it clear that giving an estimate of $N(z)$ and $N_{\mathrm{thin}}(z)$, as in Theorems 1 and 2, is relevant for determining the number of connected components of Bru_z and therefore of Bru_σ. We have not, however, used either Theorem 1 or 2 up to this point. For the rest of this section we change point of view: the proof of the following lemma assumes Theorems 1 and 2.

mma:thickexists}

Lemma 12.3. *Consider* $\sigma \in S_{n+1}$. *If* $\mathrm{Block}(\sigma) = \emptyset$ *and* $\ell = \ell(\sigma) \geq 2n + 2$ *then, for all* $z \in \Pi^{-1}[\{\sigma\}]$, *the set* Bru_z *has at least one thick connected component.*

Proof. We have $c \leq n$ and therefore, from Theorem 2, $N_{\mathrm{thin}}(z) \leq 2^{n-1}$ for all $z \in \Pi^{-1}[\{\sigma\}]$. We also have $|\Re(z)| < 1$ and therefore, from Theorem 1, $N(z) > 2^{\ell-n-1} - 2^{\frac{\ell}{2}-1} \geq 2^{n+1} - 2^n$. Thus, for all $z \in \Pi^{-1}[\{\sigma\}]$, we have $N(z) > N_{\mathrm{thin}}(z)$. Lemma 12.2 completes the proof. $\square$

We are ready to prove Theorem 3 (again assuming Theorems 1 and 2).

Proof of Theorem 3. Since $b = 0$, the total number of thin connected components is 2^n. There are 2^{n+1} values of $z \in \Pi^{-1}[\{\sigma\}]$: from Lemma 12.3, for each z there exists at least one thick connected component. □

13. Appendix 2. Tables for SL_4/B and SL_5/B

1 2

permutation	# inversions	# connected components
1234	0	1
1243	1	2
1324	1	2
2134	1	2
1423	2	4
2143	2	4
1342	2	4
3124	2	4
2314	2	4
1432	3	6
4123	3	8
2413	3	8
3142	3	8
3214	3	6
2341	3	8
4132	4	12
4213	4	12
2431	4	12
3412	4	12
3241	4	12
4312	5	16
4231	5	18
3421	5	16
4321	6	20

permutation	# inversions	# connected components
12345	0	1
12354	1	2
12435	1	2
13245	1	2
21345	1	2
12534	2	4
12453	2	4
13254	2	4
21354	2	4
14235	2	4
21435	2	4
13425	2	4
31245	2	4
23145	2	4
15234	3	8
12543	3	6
13524	3	8
21534	3	8
14253	3	8
21453	3	8
13452	3	8
31254	3	8
23154	3	8
14325	3	6
41235	3	8
24135	3	8

ig:tables1-2}

Figure 14. The number of connected components in the intersections of big Bruhat cells in SL_4/B (left) and the beginning of the table for SL_5/B (right).

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3 4

permutation	# inversions	# connected components
31425	3	8
32145	3	6
23415	3	8
51234	4	16
15243	4	12
15324	4	12
25134	4	16
14523	4	12
21543	4	12
13542	4	12
31524	4	16
23514	4	16
14352	4	12
41253	4	16
24153	4	16
31452	4	16
32154	4	12
23451	4	16
41325	4	12
42135	4	12
24315	4	12
34125	4	12
32415	4	12
51243	5	24
51324	5	24
52134	5	24
15423	5	16

permutation	# inversions	# connected components
25143	5	24
15342	5	18
35124	5	24
25314	5	24
14532	5	16
41523	5	24
24513	5	24
31542	5	24
32514	5	24
23541	5	24
41352	5	24
42153	5	24
24351	5	24
34152	5	24
32451	5	24
43125	5	16
42315	5	18
34215	5	16
51423	6	32
52143	6	36
51342	6	36
53124	6	32
52314	6	36
15432	6	20
45123	6	32
25413	6	32
35142	6	36

{fig:tables3-4} FIGURE 15. Continuation of the table for SL_5/B.

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5

permutation	# inversions	# connected components
35214	6	32
25341	6	36
41532	6	32
42513	6	36
24531	6	32
34512	6	32
32541	6	36
43152	6	32
42351	6	36
34251	6	32
43215	6	20
51432	7	40
54123	7	40
52413	7	48
53142	7	48
53214	7	40
52341	7	54
45132	7	40
45213	7	40
25431	7	40
35412	7	40
35241	7	48
43512	7	40
42531	7	48
34521	7	40
43251	7	40

6

permutation	# inversions	# connected components
54132	8	40
54213	8	48
52431	8	60
53412	8	48
53241	8	60
45312	8	44
45231	8	48
35421	8	48
43521	8	48
54312	9	48
54231	9	56
53421	9	52
45321	9	48
54321	10	52

ig:tables5-6}

FIGURE 16. End of the table for SL_5 /B.

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