# HYPERBOLIC POLYNOMIALS AND SPECTRAL ORDER

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**Abstract.** The spectral order on $\mathbb{R}$ induces a partial ordering on the manifold $\mathcal{H}_n$ of monic hyperbolic polynomials of degree $n$. We show that the semigroup generated by differential operators of the form $\left(1-\lambda\frac{d}{dx}\right)e^{\lambda\frac{d}{dx}}$, $\lambda \in \mathbb{R}$, acts on $\mathcal{H}_n$ in an order-preserving fashion. We also show that polynomials in $\mathcal{H}_n$ are global minima of their respective orbits and we conjecture that a similar result holds even for complex polynomials. Finally, we show that only those pencils of polynomials in $\mathcal{H}_n$ which are of logarithmic derivative type satisfy a certain local minimum property for the spectral order.

***Polynômes hyperboliques et ordre spectral***

**Résumé.** L'ordre spectral sur $\mathbb{R}$ induit un ordre partiel sur la variété $\mathcal{H}_n$ des polynômes hyperboliques dont le coefficient dominant est égal à un. On montre que cet ordre est préservé par l'action sur $\mathcal{H}_n$ du semigroupe engendré par les opérateurs différentiels du type $\left(1-\lambda\frac{d}{dx}\right)e^{\lambda\frac{d}{dx}}$, $\lambda \in \mathbb{R}$. On démontre aussi que tout polynôme de $\mathcal{H}_n$ est le minimum global de son orbite et on propose une conjecture selon laquelle un résultat similaire serait valable dans le cas des polynômes à coefficients complexes. On montre enfin que de tous les faisceaux de polynômes dans $\mathcal{H}_n$, seulement ceux qui sont associés aux dérivées logarithmiques satisfont une certaine propriété de minimum local pour l'ordre spectral.

## Introduction and Main Results

Given a complex polynomial $P$ of degree $n$ we define $Z(P)$ to be the unordered $n$-tuple consisting of the zeros of $P$, where each zero occurs as many times as its multiplicity. We denote by $\Re Z(P)$ the (unordered) $n$-tuple consisting of the real parts of the points in $Z(P)$. The polynomial $P$ is said to be *hyperbolic* if all its zeros are real. Note that in this case $\Re Z(P) = Z(P)$. A hyperbolic polynomial whose zeros are simple is called *strictly hyperbolic.*

The main purpose of this paper is to study the behaviour of the $n$-tuple $Z(P)$ under the action of certain semigroups of differential operators. For this we shall use the following fundamental result from the theory of stochastic majorizations:

**Theorem 1.** *Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two unordered $n$-tuples of vectors in $\mathbb{R}^k$. The following conditions are equivalent:*

1. *For any convex function $f : \mathbb{R}^k \to \mathbb{R}$ one has $\sum_{i=1}^n f(x_i) \le \sum_{i=1}^n f(y_i)$.*
2. *There exists a doubly-stochastic $n \times n$-matrix $A$ such that $\tilde{X} = A\tilde{Y}$, where $\tilde{X}$ and $\tilde{Y}$ are $n \times k$-matrices obtained by some (and then any) ordering of the vectors in $X$ and $Y$.*

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If the conditions of Theorem 1 are satisfied then we say that $X$ is *majorized* by $Y$ or $X$ is *less than* $Y$ *in the spectral order*, and write $X \prec Y$. The theorem above is due to Hardy, Littlewood, and Pólya in the one-dimensional case ([HLP]) and to Sherman in the multivariate case ([S]). These cases are also known as classical and multivariate majorization, respectively. One can easily check that $X \prec Y$ implies that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$.

Let $\mathcal{H}_n$ denote the manifold of monic hyperbolic polynomials of degree $n$. We may view $(\mathcal{H}_n, \preccurlyeq)$ as a partially ordered set, where the ordering relation $\preccurlyeq$ is induced by the spectral order on $n$-tuples of real numbers (cf. Theorem 1). Thus, if $P, Q \in \mathcal{H}_n$ then $P \preccurlyeq Q$ if and only if $Z(P) \prec Z(Q)$. Note that although the spectral order is only a preordering on $n$-tuples of points in $\mathbb{R}$, it actually induces a partial ordering on $\mathcal{H}_n$.

Define the following semigroups of differential operators:

$$\mathcal{S} = \Big\langle 1 - \lambda \frac{d}{dx} \,\Big|\, \lambda \in \mathbb{R} \Big\rangle, \quad \tilde{\mathcal{S}} = \Big\langle \Big(1 - \lambda \frac{d}{dx}\Big) e^{\lambda \frac{d}{dx}} \,\Big|\, \lambda \in \mathbb{R} \Big\rangle.$$

Note that $\tilde{\mathcal{S}}$ is the subsemigroup of $\mathcal{S} \times \langle e^{\mu \frac{d}{dx}} \mid \mu \in \mathbb{R} \rangle$ consisting of operators that preserve the averages of the zeros of polynomials in $\mathcal{H}_n$. The operator $\big(1 - \lambda \frac{d}{dx}\big) e^{\lambda \frac{d}{dx}}$, $\lambda \in \mathbb{R}$, will be denoted by $D_\lambda$ throughout this paper. It follows from the well-known Hermite-Poulain theorem (see [O]) that the semigroups $\mathcal{S}$ and $\tilde{\mathcal{S}}$ act on $\mathcal{H}_n$. Our first main result asserts that in fact these semigroups act on $\mathcal{H}_n$ in an order-preserving fashion:

**Theorem 2.** *Let $P, Q \in \mathcal{H}_n$ be such that $P \preccurlyeq Q$. Then $P + \lambda P' \preccurlyeq Q + \lambda Q'$ for any $\lambda \in \mathbb{R}$.*

We point out an interesting consequence of Theorem 2:

**Corollary 1.** *If $P, Q \in \mathcal{H}_n$ are such that $P \preccurlyeq Q$ then $n^{-1}P' \preccurlyeq n^{-1}Q'$.*

The next theorem shows that any polynomial in $\mathcal{H}_n$ is the global minimum of its orbit under the action of the semigroup $\tilde{\mathcal{S}}$.

**Theorem 3.** *If $P \in \mathcal{H}_n$ then $P \preccurlyeq D_\lambda P$ for any $\lambda \in \mathbb{R}$.*

A well-known theorem of Obreschkoff (see [O]) states that if $P$ and $Q$ are real polynomials then the linear pencil of polynomials $P + \lambda Q$, $\lambda \in \mathbb{R}$, consists of hyperbolic polynomials if and only if $P$ and $Q$ are hyperbolic and those of their zeros which are not common separate each other. The following converse to Theorem 3 shows that real lines in $\mathcal{H}_n$ of the form $P + \lambda P'$ are characterized by a local minimum property with respect to the partial ordering $\preccurlyeq$ on $\mathcal{H}_n$.

**Theorem 4.** *Let $P \in \mathcal{H}_n$, $\lambda \in \mathbb{R}$, and let $Q$ be a complex polynomial of degree at most $n-1$. Set $R_\lambda(x) = P(x+\lambda) - \lambda Q(x+\lambda)$. If $R_\lambda \in \mathcal{H}_n$ and $R_0 \preccurlyeq R_\lambda$ for all small $\lambda \in \mathbb{R}$ then $Q = P'$.*

We also obtain a generalization of Theorem 3 which shows that real lines in $\mathcal{H}_n$ of the form $P + \lambda P'$ satisfy in fact a global monotony property:

**Theorem 5.** *If $P \in \mathcal{H}_n$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ are such that $\lambda_1 \lambda_2 \geq 0$ and $|\lambda_1| \leq |\lambda_2|$ then $D_{\lambda_1} P \preccurlyeq D_{\lambda_2} P$.*

Finally, we show that real lines in $\mathcal{H}_n$ of the form $P + \lambda P'$ satisfy an inequality *à la* Gårding (cf. [G]):

**Theorem 6.** *If $P \in \mathcal{H}_n$ then $\mathbb{R} \ni \lambda \mapsto \max Z(D_\lambda P)$ is a convex function with a global minimum at $\lambda = 0$.*

**Remark 1.** It follows from Theorem 6 that the so-called spread function $\mathbb{R} \ni \lambda \mapsto \max Z(D_\lambda P) - \min Z(D_\lambda P)$ is a convex function with a global minimum at $\lambda = 0$.

The structure of the paper is as follows: in §1 we sketch the proofs of our main results and in §2 we present further questions and conjectures. The complete proofs will appear elsewhere.

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## 1. Outline of the proofs

One of the key ingredients in the proofs of Theorems 2-6 is the following criterion for classical majorization due to Hardy, Littlewood, and Pólya (cf. [HLP]). We should mention that there are no known analogues of this criterion for multivariate majorization.

**Theorem 7.** *Let $X = (x_1 \leq x_2 \leq \ldots \leq x_n) \subset \mathbb{R}$ and $Y = (y_1 \leq y_2 \leq \ldots \leq y_n)$ be two $n$-tuples of real numbers. Then $X \prec Y$ if and only if the $x_i$'s and the $y_i$'s satisfy the following conditions:*

(1) $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$;
(2) $\sum_{i=0}^k x_{n-i} \leq \sum_{i=0}^k y_{n-i}$ *for* $0 \leq k \leq n-2$.

The proof of Theorem 2 is based on a several auxiliary results. Let us first make the following

**Definition 1.** Let $P(x) = \prod_{i=1}^n (x - x_i) \in \mathcal{H}_n$, $n \geq 2$, and $1 \leq k < l \leq n$. Assume that $x_i \leq x_{i+1}$, $1 \leq i \leq n-1$, and that $x_k \neq x_l$. Let further $t \in ]0, \frac{x_l - x_k}{2}]$ and define $Q \in \mathcal{H}_n$ to be the polynomial with zeros $y_i$, $1 \leq i \leq n$, where $y_k = x_k + t$, $y_l = x_l - t$, and $y_i = x_i$, $i \neq k, l$. The polynomial $Q$ is called the *contraction of $P$ of type $(k, l)$ and coefficient $t$*. The contraction is called *simple* if $l = k + 1$ and it is called *non-degenerate* if $t \neq \frac{x_l - x_k}{2}$.

**Proposition 1.** *Let $P, Q \in \mathcal{H}_n$ be two distinct strictly hyperbolic polynomials such that $P \preccurlyeq Q$. Then there exists a finite sequence $P_1, \ldots, P_m \in \mathcal{H}_n$ such that $P_1 = Q$, $P_m = P$, and $P_{i+1}$ is a simple non-degenerate contraction of $P_i$, $1 \leq i \leq m-1$.*

**Remark 2.** Proposition 1 is true even for polynomials with multiple zeros if the non-degeneracy condition is omitted.

**Proposition 2.** *Theorem 2 is true if $P$ and $Q$ are strictly hyperbolic polynomials and $P$ is a simple (non-degenerate) contraction of $Q$.*

From Propositions 1 and 2 we deduce that Theorem 2 is true in the generic case when $P$ and $Q$ have simple zeros. If this is not the case then we let $x_i$, $1 \leq i \leq n$, and $y_i$, $1 \leq i \leq n$, denote the zeros of $P$ and $Q$, respectively, and we choose an arbitrary positive number $\varepsilon$. Let $P_\varepsilon$ and $Q_\varepsilon$ be the polynomials with zeros $x_i - (n-i)\varepsilon$, $1 \leq i \leq n-1$, $x_n + \frac{n(n-1)}{2}\varepsilon$, and $y_i - (n-i)\varepsilon$, $1 \leq i \leq n-1$, $y_n + \frac{n(n-1)}{2}\varepsilon$, respectively. Note that $P_\varepsilon$ and $Q_\varepsilon$ are strictly hyperbolic and that $P_\varepsilon \preccurlyeq Q_\varepsilon$. The above arguments imply that $P_\varepsilon + \lambda P'_\varepsilon \preccurlyeq Q_\varepsilon + \lambda Q'_\varepsilon$ for any $\lambda \in \mathbb{R}$. Theorem 2 now follows by letting $\varepsilon \to 0$.

Let $P(x) = \prod_{i=1}^n (x - x_i) \in \mathcal{H}_n$, $n \geq 2$, and $\lambda \in \mathbb{R}$. Assume that $x_i < x_{i+1}$, $1 \leq i \leq n-1$, and let $x_i(\lambda)$, $1 \leq i \leq n$, denote the zeros of $D_\lambda P$. If these are labeled so that $x_i(0) = x_i$, $1 \leq i \leq n$, then one can show that $x_i(\lambda) < x_{i+1}(\lambda)$ and that by varying $x_n$ and keeping $\lambda$ fixed each $x_i(\lambda)$, $1 \leq i \leq n-1$, is an increasing function of $x_n$. This makes it possible to prove Theorem 3 by induction on $n$ in the generic case. If $P \in \mathcal{H}_n$ has multiple zeros we notice that $\left(1 - \varepsilon \frac{d}{dx}\right)^n P$ has simple zeros for any $\varepsilon \neq 0$. Since Theorem 3 is true for the latter polynomial, we get that Theorem 3 holds in the general case by letting $\varepsilon \to 0$.

To prove Theorem 4 we first use Theorem 7 in order to show that if $P$ is strictly hyperbolic and $Q$ satisfies the assumptions of Theorem 4 then $Q(x_i) = P'(x_i)$, where $x_i$, $1 \leq i \leq n$, are the zeros of $P$. Using again the fact that $\left(1-\varepsilon\frac{d}{dx}\right)^n P$ has simple zeros if $P \in \mathcal{H}_n$ and $\varepsilon \neq 0$ and also that the operator $\left(1-\varepsilon\frac{d}{dx}\right)^n$ preserves the ordering on $\mathcal{H}_n$ (cf. Theorem 2) we deduce that Theorem 4 holds for any $P \in \mathcal{H}_n$.

Let now $P(x) = \prod_{i=1}^n (x - x_i) \in \mathcal{H}_n$, $n \geq 2$, $\lambda \in \mathbb{R}$, and assume that $x_i < x_{i+1}$, $1 \leq i \leq n-1$. Denote the zeros of $D_\lambda P$ by $x_i(\lambda)$, $1 \leq i \leq n$, those of $P'$ by $w_j$, $1 \leq j \leq n-1$, and those of $D_\lambda P'$ by $w_j(\lambda)$, $1 \leq j \leq n-1$. If these are labeled so that $x_i(0) = x_i$, $1 \leq i \leq n$, and $w_j(0) = w_j$, $1 \leq j \leq n-1$, then one can show that $x_i(\lambda) < w_i(\lambda) < x_{i+1}(\lambda)$, $1 \leq i \leq n-1$, and that for any $\lambda > 0$ and $1 \leq m \leq n-1$ one has

$$\sum_{i=1}^m x_i'(\lambda) = \lambda \sum_{j=1}^{n-1} \frac{P''(w_j(\lambda)+\lambda)}{D_\lambda P''(w_j(\lambda))} \left( \sum_{i=1}^m \frac{1}{x_i(\lambda) - w_j(\lambda)} \right) < 0.$$

Thus $\sum_{i=1}^m x_i(\lambda)$, $1 \leq m \leq n-1$, are decreasing functions on $]0,\infty[$, which combined with Theorem 7 proves Theorem 5 for strictly hyperbolic polynomials (the case $\lambda < 0$ is similar). The same density arguments as those used for Theorem 4 show that Theorem 5 is true for all $P \in \mathcal{H}_n$.

Keeping the same notations as above, one can show that if $P \in \mathcal{H}_n$ is strictly hyperbolic then

$$x_n''(\lambda) = 2(x_n'(\lambda)+1)^2 \sum_{i=1}^{n-1} \frac{w_i - x_i(\lambda) - \lambda}{(x_n(\lambda) + \lambda - w_i)(x_n(\lambda) - x_i(\lambda))} > 0$$

for any $\lambda \in \mathbb{R}$, which proves Theorem 6 in the generic case. If $P$ has multiple zeros then one can use strictly hyperbolic polynomials of the form $\left(1 - \frac{1}{s}\frac{d}{dx}\right)^n P$, $s \in \mathbb{Z}_+$, in order to approximate the function $\lambda \mapsto \max Z(D_\lambda P)$ uniformly on compact intervals by convex $C^2$-functions. This proves Theorem 6.

## 2. Remarks and open questions

The manifold $\mathcal{C}_n$ of monic complex polynomials of degree $n$ is a natural context for discussing possible extensions of the above results to the complex case. By analogy with the hyperbolic case we may view $(\mathcal{C}_n, \preccurlyeq)$ as a partially ordered set, where the ordering relation $\preccurlyeq$ is now induced by the spectral order on $n$-tuples of vectors in $\mathbb{R}^2$ (cf. Theorem 1). This means that the zero sets of polynomials in $\mathcal{C}_n$ are viewed as subsets of $\mathbb{R}^2$ and that if $P, Q \in \mathcal{C}_n$ then $P \preccurlyeq Q$ if and only if $Z(P) \prec Z(Q)$.

The following example shows that if the partial ordering $\preccurlyeq$ on $\mathcal{C}_n$ is defined as above then one cannot expect a complex analogue of Theorem 3.

**Proposition 3.** *Let $P(z) = z^n - 1$ and $\lambda \in \mathbb{C}$. If $n \geq 3$ and $|\lambda|$ is small enough then $D_\lambda P$ and $P$ are incomparable with respect to the partial ordering $\preccurlyeq$ on $\mathcal{C}_n$.*

We also note that the results of the previous section (for the hyperbolic case) are valid only for real values of the parameter $\lambda$:

**Proposition 4.** *Let $P \in \mathcal{H}_n$ be a strictly hyperbolic polynomial and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If $n \geq 2$ and $|\lambda|$ is sufficiently small then $D_\lambda P$ and $P$ are incomparable with respect to the partial ordering $\preccurlyeq$ on $\mathcal{C}_n$.*

These examples suggest that complex generalizations of Theorem 3 - if any - should involve only classical majorization and real values of the parameter $\lambda$. Based on extensive numeric calculations, we make the following

**Conjecture 1.** *If $P \in \mathcal{C}_n$ then $\Re Z(P) \preccurlyeq \Re Z(D_\lambda P)$ for any $\lambda \in \mathbb{R}$.*

We end with a few questions related to the semigroups $\mathcal{S}$ and $\tilde{\mathcal{S}}$. Let $\mathcal{R}_n$ denote the set of all monic real polynomials of degree $n$. There is reason to believe that any two $\mathcal{S}$-orbits in $\mathcal{R}_n$ have a non-empty intersection:

**Conjecture 2.** *If $P_1, P_2 \in \mathcal{R}_n$ then there exist differential operators $\Psi_1, \Psi_2 \in \mathcal{S}$ such that $\Psi_1 P_1 = \Psi_2 P_2$.*

If true, Conjecture 2 would imply in particular that $\mathcal{S}P \cap \mathcal{H}_n \neq \emptyset$ for any $P \in \mathcal{R}_n$, which would answer in the affirmative a question of I. Krasikov.

Let finally $P \in \mathcal{H}_n$ and set $P_{\preccurlyeq} = \{Q \in \mathcal{H}_n \mid P \preccurlyeq Q\}$. One can easily check that if $P$ is strictly hyperbolic and $n \geq 3$ then $\tilde{\mathcal{S}}P \subsetneq P_{\preccurlyeq}$. It would be interesting to know whether there exists a (semi)group of differential operators $\mathcal{D} \supsetneq \tilde{\mathcal{S}}$ such that $P_{\preccurlyeq} = \mathcal{D}P$ for any $P \in \mathcal{H}_n$. This would give a completely new way of describing classical majorization.

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