ON POLYGONAL MEASURES WITH VANISHING HARMONIC MOMENTS

By

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TO THE MEMORY OF ANDREI ZELEVINSKY

Abstract. A polygonal measure is the sum of finitely many real constant density measures supported on triangles in $\mathbb{C}$. Given a finite set $S \subset \mathbb{C}$, we study the existence of polygonal measures spanned by triangles with vertices in S, all of whose harmonic moments vanish. We show that for generic S, the dimension of the linear space of such measures is $\binom{|S|-3}{2}$. We also investigate the situation in which the density for such measure takes on only values 0 or ± 1. This corresponds to pairs of polygons of unit density having the same logarithmic potential at ∞. We show that such (signed) measures do not exist for $|S| \leq 5$, but that for each $n \geq 6$ one can construct an S, with $|S| = n$, giving rise to such a measure.

1 Introduction and main results

Inverse problems in logarithmic potential theory have attracted substantial attention since the publication of the fundamental paper [15], where P. S. Novikov proved, in particular, that two convex (or, more generally, star-shaped) domains in $\mathbb{C}$ with unit density cannot have the same logarithmic potential near ∞. Observe that knowledge of the germ of a logarithmic potential of a finite compactly supported Borel measure μ at ∞ is equivalent to the knowledge of the sequence of its harmonic moments $m_j(\mu),\ j = 0, 1, \ldots$, where the j-th harmonic moment of μ is defined by $m_j(\mu) = \int_{\mathbb{C}} z^j d\mu(z)$.

More precisely, if

$$\mathfrak{u}_\mu(z) := \int_{\mathbb{C}} \ln |z - \xi| d\mu(\xi)$$

is the logarithmic potential of μ and

$$\mathfrak{C}_\mu(z) := \int_{\mathbb{C}} \frac{d\mu(\xi)}{z - \xi} = \frac{\partial \mathfrak{u}_\mu(z)}{\partial z}$$

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is its Cauchy transform, the Taylor expansion of $\mathfrak{C}_\mu(z)$ at ∞ has the form

$$\mathfrak{C}_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \cdots .$$

Thus Novikov's result can be reformulated as the statement that two convex domains in $\mathbb{C}$ with unit density cannot have coinciding sequences of harmonic moments. It is well known that for non-convex domains with unit density, the uniqueness in this problem no longer holds. For instance, examples of pairs of non-convex polygons with the same logarithmic potential near ∞ can be found in [6, p. 333]; see Figure 1 below.

The class of general polygons as well as domains bounded by lemniscates has attracted a substantial attention in this area. Several authors have also considered the case of polynomial densities instead of the unit density.

By a **convex polygon**, we mean the convex hull of a set of finite many points in the plane, at least 3 of which are non-colinear. A general **polygon** is the set-theoretic union of finitely many convex polygons. By a **vertex** of a polygon, we mean a point of the polygon's boundary, every sufficiently small ϵ-neighborhood of which in the polygon is different from a half-disk of radius ϵ.

Given an open set $D \subset \mathbb{C}$, we define its **standard measure** $\mu_D = \chi_D dxdy$, where χ_D is the characteristic function of D. The same measure is associated with the closure of D. We say that two polygons in $\mathbb{C}$ are **equipotential** if their standard measures create coinciding logarithmic potential outside their union. Here, we present one of the simplest examples of pairs of equipotential polygons given in [6, Example 1].

Example 1. Consider the 6-tuples

$$T = \{\pm\sqrt{3} \pm i, \pm 2i\} \quad \text{and} \quad T' = \left\{ \pm \frac{1 \pm \sqrt{3}i}{2}, \pm 1 \right\},$$

where $i = \sqrt{-1}$. Let $F \subset \mathbb{C}$ be the difference of the convex hull of T and the union of the set of 6 triangles obtained as the orbit of the triangle with nodes $(\sqrt{3} + i, \sqrt{3} - i, 1)$ under the rotation by $\pi/3$; see Figure 1. Let $F' \subset \mathbb{C}$ be the difference of the convex hulls of T and of T'. Then F and F' have the same logarithmic potential.

It is known that different polygons with constant (but not necessarily unit) density having the same logarithmic potential near ∞ must have the same set of vertices; see [6, Corollary 2 and Lemma 2]. The coincidence of the logarithmic potential near ∞ implies additional restrictions on the polygons as well beyond the coincidence of their sets of vertices; cf. [6].

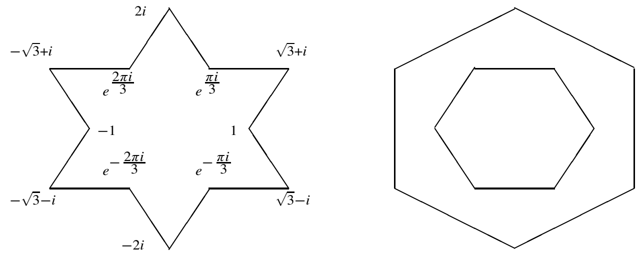


Figure 1. Two equipotential polygons: F on the left, F' on the right.

Taking this fact into account, we pose the following *classical inverse logarithmic potential problem for polygons in* $\mathbb{C}$.

Problem 1. Given a finite set $S \subset \mathbb{C}$, determine whether there exist two equipotential polygons whose sets of vertices coincide with S.

One can show that for generic S no pairs of equipotential polygons exist.

Definition 1. A complex (respectively, real) **polygonal measure** $\mu := \mu(\mathcal{D})$ is the sum

$$\mu := \sum_{\Delta \in \mathcal{D}} c_\Delta \mu_\Delta, \quad c_\Delta \in \mathbb{C} \ \text{(respectively, } c_\Delta \in \mathbb{R}), \tag{1.1}$$

where $\mathcal{D}$ is a finite set of closed triangles in the plane. The set of vertices of the triangles $\Delta \in \mathcal{D}$ with $c_\Delta \neq 0$ in (1.1) is called the set of **nodes** of this decomposition.

Observe that the decomposition (1.1) of a given μ need not be unique, and different choices of $\mathcal{D}$ can lead to different sets of nodes.

Besides the nodes of decompositions (1.1) of μ, it is natural to talk about the **vertices** of μ, defined as those $v \in \mathbb{C}$ such that for sufficiently small $\epsilon > 0$, the restriction of the density of μ to the ϵ-disk centered at v is neither constant, nor does there exists a line through v dividing the disk into two halves with different constant densities.

Obviously, the set of vertices of μ is a subset of the set of intersections of sides of the triangles in $\mathcal{D}$. There exists a finite collection $\tilde{\mathcal{D}}$ of triangles with pairwise empty intersections of interiors such that $\mu = \mu(\tilde{\mathcal{D}})$, and the nodes and vertices of μ coincide. However, such a representation of μ need not be the most economical one. For instance, in the notation of Example 1, consider $\tilde{\mu} := \mu_F - \mu_{F'}$. Observe that $\tilde{\mu}$ can be represented using only 6 nodes, although the polygons themselves have 12 vertices! This also illustrates the non-uniqueness of the representation of $\tilde{\mu}$ in the form (1.1). Indeed,

$$2\tilde{\mu} = \sum_{0\leq j\leq 5} \mu_{\exp(j\pi i/3)(\sqrt{3}+i,\sqrt{3}-i,-2i)} - \sum_{0\leq j\leq 1} \mu_{(\sqrt{3}+(-1)^j i,\sqrt{3}-(-1)^j i,-\sqrt{3}+(-1)^j i)}.$$

Let S admit a pair of equipotential polygons. Taking the difference of their standard measures, one obtains a polygonal measure supported on the convex hull conv(S) of S with density taking on only the values $0, \pm 1$ and with all harmonic moments vanishing. Conversely, if one can find a polygonal measure all of whose harmonic moments vanish and such that its density takes on only the values $0, \pm 1$, then one obtains a pair of equipotential polygons by taking the differences of conv(S) with the sets where the density attains value the 1, respectively -1.

If we weaken the condition that the density of a polygonal measure takes on only the values $0, \pm 1$, we arrive at the setup of the present paper. Given a **spanning** set S (i.e., S containing at least 3 non-colinear points), we introduce the linear spaces $\mathfrak{M}^{\mathbb{R}}(S) \subset \mathfrak{M}^{\mathbb{C}}(S)$ of real-valued, respectively, complex-valued polygonal measures obtained as real, respectively, complex linear spans of the standard measures of all triangles with vertices in S. Obviously, $\mathfrak{M}^{\mathbb{C}}(S) = \mathbb{C} \otimes \mathfrak{M}^{\mathbb{R}}(S)$.

We take a further step in the study of (non-)uniqueness in logarithmic potential theory by considering the following problem.

Problem 2. Given a finite $S \subset \mathbb{C}$, determine the linear subspace $\mathfrak{M}^{\mathbb{R}}_{null}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$ of real-valued polygonal measures (respectively, $\mathfrak{M}^{\mathbb{C}}_{null}(S) \subset \mathfrak{M}^{\mathbb{C}}(S)$ of complex-valued polygonal measures) with vanishing harmonic moments.

The main technical tool we use is the **normalized generating function $\Psi_\mu(u)$ for harmonic moments** of a measure μ, defined by

$$\Psi_\mu(u) = \sum_{j=0}^{\infty} \binom{j+2}{2} m_j(\mu) u^j. \tag{1.2}$$

Clearly, $\Psi_\mu(u)$ is closely related to the Cauchy transform $\mathfrak{C}_\mu(z)$ at ∞; indeed,

$$\Psi_\mu(u) = \frac{1}{2}\frac{d^2}{du^2}\left(\sum_{j=0}^{\infty} m_j(\mu) u^{j+2}\right).$$

Now, for a compactly supported measure μ and sufficiently large $|z|$, we have $z\mathfrak{C}_\mu(z) = \sum_{j=0}^{\infty} m_j(\mu)/z^j$. Thus for $|u|$ sufficiently small,

$$\Psi_\mu(u) = \frac{1}{2}\frac{d^2}{du^2}\left(u\mathfrak{C}_\mu\left(\frac{1}{u}\right)\right).$$

Similar multivariate generating functions were recently considered in [14]. Important in our consideration are the following observations.

Proposition 1. *For μ a measure with compact support,*

$$\Psi_{\mu}(u) = \sum_{j=0}^{\infty} \binom{j+2}{2} m_j(\mu) u^j = \int \frac{d\mu(z)}{(1-uz)^3}. \tag{1.3}$$

The normalized generating function $\Psi_{\Delta}(u)$ of (the standard measure of) the triangle $\Delta \subset \mathbb{C}$ whose vertices are located at a, b, c is given by

$$\Psi_{\Delta}(u) = \frac{\text{Area}\,\Delta}{(1-au)(1-bu)(1-cu)}.$$

Note that the integral transform in (1.3) appears to be a variant of the *Fantappiè transformation*; cf. [4].

Definition 2. A finite set $S = \{z_0, z_1, \ldots, z_n\}$ of points in $\mathbb{C}$ is **non-degenerate** if no three of its members are colinear.

Proposition 2. *For any non-degenerate set $S = \{z_0, z_1, \ldots, z_n\}$, $n \geq 2$, of points in $\mathbb{C}$ and a fixed non-negative integer $j \leq n$, the set of (standard measures of) triangles with a node at z_j is a basis of the spaces $\mathfrak{M}^{\mathbb{R}}(S)$ and $\mathfrak{M}^{\mathbb{C}}(S)$. In particular,*

$$\dim_{\mathbb{R}} \mathfrak{M}^{\mathbb{R}}(S) = \dim_{\mathbb{C}} \mathfrak{M}^{\mathbb{C}}(S) = \binom{n}{2}.$$

We are interested in the linear subspaces $\mathfrak{M}^{\mathbb{R}}_{null}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$ (respectively, $\mathfrak{M}^{\mathbb{C}}_{null}(S) \subset \mathfrak{M}^{\mathbb{C}}(S)$) of real-valued (respectively, complex-valued) measures all of whose harmonic moments vanish.

The main results of this paper are as follows.

Proposition 3. *For every non-degenerate set $S = \{z_0, z_1, \ldots, z_n\}$, $n \geq 2$, of points in $\mathbb{C}$,*

$$\dim_{\mathbb{C}} \mathfrak{M}^{\mathbb{C}}_{null}(S) = \binom{n-1}{2}.$$

Example 2. For $n = 3$, the space $\mathfrak{M}^{\mathbb{C}}_{null}(S)$ is spanned by the complex-valued measure $\tilde{\mu}$ whose densities with respect to the basis of triangles Δ_{012}, Δ_{013}, Δ_{023} are given by

$$\begin{aligned} d_{012} &= (z_1 - z_2)/|[012]|, \\ d_{013} &= (z_3 - z_1)/|[013]|, \\ d_{023} &= (z_2 - z_3)/|[023]|. \end{aligned}$$

Here,

$$[jkl] = \det \begin{pmatrix} 1 & 1 & 1 \\ x_j & x_k & x_l \\ y_j & y_k & y_l \end{pmatrix}$$

stands for twice the signed area of the triangle with nodes z_j, z_k, z_l, where $z_k = x_k + y_k i$.

Remark 1. For non-degenerate S, the space $\mathfrak{M}^{\mathbb{C}}_{null}(S)$ projects isomorphically onto the linear subspace of $\mathfrak{M}^{\mathbb{C}}(S)$ spanned by all triangles $\Delta_{0,j,k}$, $2 \le j < k \le n$. In other words, assigning arbitrarily complex-valued densities $d_{0,j,k}$, $2 \le j < k \le n$, we can determine uniquely the densities $d_{0,1,k}$, $k = 2, \dots, n$ to get a measure belonging to $\mathfrak{M}^{\mathbb{C}}_{null}(S)$.

Theorem 1. *For every non-degenerate set* $S = \{z_0, z_1, \dots, z_n\}$*,* $n \ge 2$*, of points in* $\mathbb{C}$*,*

$$\dim_{\mathbb{R}} \mathfrak{M}^{\mathbb{R}}_{null}(S) = \binom{n-2}{2}.$$

Remark 2. For non-degenerate S, the space $\mathfrak{M}^{\mathbb{R}}_{null}(S)$ projects isomorphically on the linear subspace of $\mathfrak{M}^{\mathbb{R}}(S)$ spanned by all triangles $\Delta_{0,i,j}$, $3 \le i < j \le n$. In other words, arbitrarily real-valued densities $d_{0,i,j}$, $3 \le i < j \le n$, determine uniquely the densities $d_{0,1,j}$, $j = 2, \dots, n$ and $d_{0,2,j}$, $j = 3, \dots, n$ of a measure belonging to $\mathfrak{M}^{\mathbb{R}}_{null}(S)$.

Theorem 2. *For every non-degenerate* 5*-tuple* $S = \{z_0, z_1, z_2, z_3, z_4\}$*, the space* $\mathfrak{M}^{\mathbb{R}}_{null}(S)$ *is spanned by the real measure* $\tilde{\mu}$ *with densities with respect to the basis of triangles* Δ_{012}, Δ_{013}, Δ_{014}, Δ_{023}, Δ_{024}, Δ_{034} *given by*

$$\begin{aligned} d_{012} &= |z_1 - z_2|^2 [134][234]/|[012]|, \\ d_{013} &= |z_1 - z_3|^2 [124][234]/|[013]|, \\ d_{014} &= |z_1 - z_4|^2 [123][234]/|[014]|, \\ d_{023} &= -|z_2 - z_3|^2 [124][134]/|[023]|, \\ d_{024} &= -|z_2 - z_4|^2 [134][123]/|[024]|, \\ d_{034} &= -|z_3 - z_4|^2 [123][124]/|[034]|. \end{aligned} \tag{1.4}$$

Example 3. The measure $3\tilde{\mu}$ for the 5-tuple $\{0, 2, 3+i, 1+3i, 2i\}$ is shown in Figure 2 . (In this case, $3\tilde{\mu}$ has integer densities which are easier to show TEXnically.)

Remark 3. Suppose that the densities of a polygonal measure $\mu \in \mathfrak{M}^{\mathbb{R}}_{null}(S)$ with respect to the basic triangles containing a fixed node (say z_0) are known. It

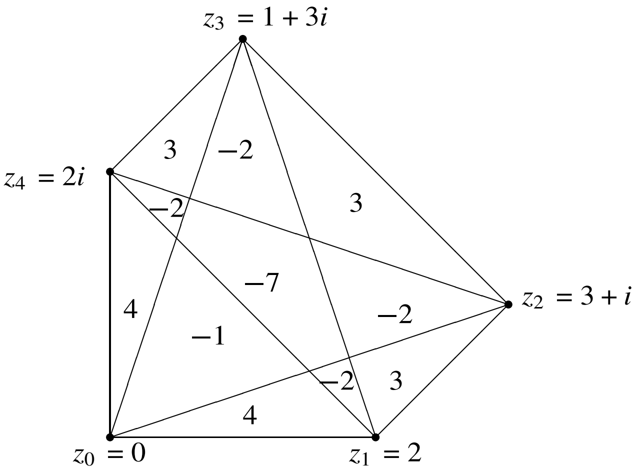


Figure 2. Measure $3\tilde{\mu}$ spanning $\mathfrak{M}^{\mathbb{R}}_{null}(0, 2, 3 + i, 1 + 3i, 2i)$.

is still desirable to find the densities in all its chambers, for instance, in view of the classical Problem 1. Here, by a **chamber**, we mean a connected component of $\mathrm{conv}(S) \setminus Arr(S)$, $Arr(S)$ being the union of all lines connecting pairs of points in S. (Integers in Figure 2 show the densities in the chambers they are placed in.) Each chamber is contained in a number of basic triangles, and the density of a given chamber equals the sum of the densities of all basic triangles containing it. Containment of chambers in triangles (and, more generally, in simplices in $\mathbb{R}^d$) can be coded by an appropriate incidence matrix whose rows correspond to simplices and columns correspond to chambers. If a simplex contains a chamber, the corresponding entry of the incidence matrix equals 1; otherwise, the entry equals 0. Examples of incidence matrices are given in the proof of Theorem 3 below.

This incidence matrix of chambers and simplices in $\mathbb{R}^d$ was studied for the first time in [3] and later in [1, 2]. It has rather delicate properties; already, the number of chambers is a complicated function of the initial non-degenerate set S. In particular, this number can change if we deform S within the class of non-degenerate sets. This observation partially explains why results of the present paper do not automatically solve Problem 1.

Remark 4. Notice that if $S = \{z_0, \dots, z_n\}$ consists of complex numbers having only rational real and imaginary parts, one can choose a basis of $\mathfrak{M}^{\mathbb{R}}_{null}(S)$ consisting of polygonal measures with integer densities.

Using Example 1 together with Theorem 2, we can prove the following result, which is related to the classical Problem 1.

Theorem 3. *For each* $n \geq 6$, *there exists* S, *with* $|S| = n$, *admitting a pair of equipotential polygons. No such* S *exists if* $|S| \leq 5$.

The essential part of the proof of Theorem 3 is dealing with the case $|S| = 5$.

Our final result concerns a natural cone spanned by the standard measures of triangles with nodes in S. Namely, for an arbitrary non-degenerate set $S = \{z_0, z_1, \dots, z_n\}$, denote by $\mathfrak{K}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$ the $\binom{n}{2}$-dimensional cone obtained by taking non-negative linear combinations of the standard measures of all triangles with nodes in S. (Recall that $\mathfrak{M}^{\mathbb{R}}(S)$ is the linear span of these measures.)

Theorem 4. *Extreme rays of* $\mathfrak{K}(S)$ *are spanned by (the standard measures) of triangles which do not contain any point of* S *different from the triangles' nodes. In particular, if* S *is a convex configuration, (i.e., each* z_j *belongs to the convex hull of* S*), every triangle with nodes in* S *spans an extreme ray of* $\mathfrak{K}(S)$.

We finish the introduction with a conjectural description of all faces of $\mathfrak{K}(S)$. We say that a pair of triangles with vertices in S **forms a flip** if its members have a common side and their convex hull is a 4-gon. With any pair of triangles forming a flip, we associate its **flipped pair**, which is obtained by removing the opposite diagonal from its member's convex hull; see Case a) Figure 3. (In this figure, the pairs of triangles $(\Delta_{013}, \Delta_{123})$ and $(\Delta_{012}, \Delta_{023})$ form a flip and each pair is the flipped one of the other pair.)

Conjecture 1. *A collection Col of triangles having no internal vertices spans a face of* $\mathfrak{K}(S)$ *if and only if, for each pair of triangles from Col forming a flip, its flipped pair of triangles is also contained in Col.*

The necessity of the stated condition is quite obvious; its sufficiency might follow from the results of [3].

2 Proofs

Proof of Proposition 1. First, we prove (1.3). Indeed,

$$\int \frac{d\mu(z)}{(1-uz)^3} = \sum_{k\geq 0} u^k \int \binom{k+2}{2} z^k \, d\mu(z) = \sum_{k\geq 0} u^k \binom{k+2}{2} m_k(\mu) = \Psi_\mu(u),$$

as required. By [7, (1)], for every f analytic in the closure of Δ,

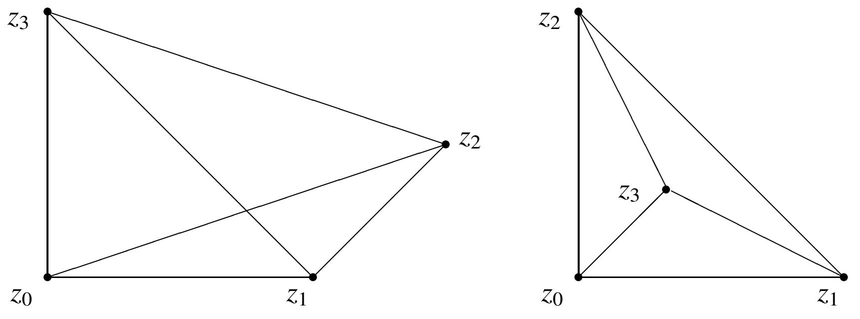
$$\frac{1}{2\text{Area}\,\Delta} \int_\Delta f''(z) dxdy = \sum_{k=1,\ j\neq i\in\{1,2,3\}\setminus\{k\}}^{k} \frac{f(z_k)}{(z_k - z_i)(z_k - z_j)}.$$

Applying the latter identity and (1.3) to $f(z) = (2u^2)^{-1}(1 - uz)^{-1}$, we obtain the claimed formula. □

To prove Proposition 2, we need to recall some basic notions. First we present a description of all linear dependences among the standard measures of all triangles with vertices in a non-degenerate set S. Namely, any 4-tuple of points (say, $S = \{z_0, z_1, z_2, z_3\}$) in S defines 4 triangles with vertices in S. To study linear dependences among these 4 triangles, one has to distinguish between two cases:

Case a) the convex hull of $\{z_0, z_1, z_2, z_3\}$, is a quadrangle,
Case b) the convex hull of $\{z_0, z_1, z_2, z_3\}$, is a triangle;

see Figure 3. Obviously, in Case a) we have (up to permutation of the vertices) the equality $\mu_{\Delta_{013}} + \mu_{\Delta_{123}} = \mu_{\Delta_{023}} + \mu_{\Delta_{012}}$. Analogously, in Case b), we have (up to permutation of the vertices) the relation $\mu_{\Delta_{012}} = \mu_{\Delta_{013}} + \mu_{\Delta_{123}} + \mu_{\Delta_{023}}$.



Case a) Case b)

Figure 3. Linear dependence of four triangles spanned by four points.

To complete the proof of Proposition 2, we need to show that if S is non-degenerate, the set of (the standard measures of) all triangles containing a given vertex $z_j \in S$ spans $\mathfrak{M}^{\mathbb{R}}(S)$ and that this set is linearly independent. The former claim follows immediately from the discussion preceding Figure 3. It remains to show the latter claim, for which we require additional notions.

Definition 3. By a 2-**chain** $\mathcal{C}^{(2)}$, we mean a formal linear combination

$$\mathcal{C}^{(2)} = \alpha_1\Delta_1 + \alpha_2\Delta_2 + \cdots + \alpha_s\Delta_s \tag{2.1}$$

of triangles $\Delta_1, \ldots \Delta_s$ in $\mathbb{C}$ with real or complex coefficients, where each triangle is equipped with the standard orientation induced from $\mathbb{C}$.

Using the standard pairing

$$\langle fdxdy, \mathcal{C}^{(2)}\rangle = \int_{\mathcal{C}^{(2)}} fdxdy = \sum_{j=1}^{s} \alpha_j \int_{\Delta_j} fdxdy,$$

one sees that a 2-chain (2.1) defines a linear functional on the space $\mathbf{\Omega}^{(2)}$ of smooth 2-forms on $\mathbb{C}$.

Definition 4. Analogously, by a 1-**chain** $\mathcal{C}^{(1)}$, we mean a formal linear combination

$$\mathcal{C}^{(1)} = \beta_1 I_1 + \beta_2 I_2 + \cdots + \beta_t I_t \tag{2.2}$$

of oriented finite intervals $I_1, \dots I_s$ in $\mathbb{C}$ with real or complex coefficients.

Again, using the standard pairing

$$\langle w, \mathcal{C}^{(1)}\rangle = \int_{\mathcal{C}^{(1)}} w = \sum_{j=1}^{t} \beta_j \int_{I_j} w,$$

where w is an arbitrary smooth 1-form, one sees that a 1-chain (2.2) defines a linear functional on the space $\mathbf{\Omega}^{(1)}$ of smooth 1-forms on $\mathbb{C}$.

Definition 5. We define the **boundary** $\partial\Delta$ of a given triangle Δ with vertices a, b, c, where the triple (a, b, c) is counterclockwise oriented, as the sum of the three oriented intervals $[ab]+[bc]+[ca]$. As usual, we extend the boundary operator ∂ to the linear space of all 2-chains by linearity.

Definition 6. A 2-chain (resp., 1-chain) is called **vanishing** if it defines the zero linear functional on $\mathbf{\Omega}^{(2)}$ (resp., $\mathbf{\Omega}^{(1)}$).

Lemma 1. *A* 2*-chain* $\mathcal{C}^{(2)}$ *is vanishing if and only if its boundary* $\partial\mathcal{C}^{(2)}$ *is a vanishing* 1*-chain.*

Proof. Stokes' Theorem says that $\int_{\partial\Delta} w = \int_{\Delta} dw$, where $w \in \mathbf{\Omega}^{(1)}$, Δ is an arbitrary triangle, $\partial\Delta$ is its boundary, and dw is the differential of w. (Recall that if $w = F(x, y)dx + G(x, y)dy$, then $dw = (G'_x - F'_y)dxdy$.) Observe that every 2-form $f(x, y)dxdy$ can be represented as dw_x, where $w_x = F(x, y)dx$ and $F(x, y)$ is the primitive function of $-f(x, y)$ along vertical lines. Analogously, $f(x, y)dxdy$ equals dw_y, where $w_y = G(x, y)dy$ and $G(x, y)$ is the primitive function of $f(x, y)$ along horizontal lines. Thus

$$\int_{\mathcal{C}^{(2)}} fdxdy = \int_{\partial\mathcal{C}^{(2)}} w_x = \int_{\partial\mathcal{C}^{(2)}} w_y.$$

The left-hand side vanishes for any measure $fdxdy$ if and only if $\partial\mathcal{C}^{(2)}$ vanishes. □

Proof of Proposition 2. We need to show that for every non-degenerate S, the standard measures of all triangles containing z_0 are linearly independent. Indeed, by Lemma 1, a 2-chain $\mathcal{C}^{(2)}$ of triangles vanishes if and only its boundary chain $\partial\mathcal{C}^{(2)}$ vanishes. But if S is non-degenerate, each triangle $\Delta_{0,i,j}$ has its unique edge (z_i, z_j) in the boundary, and no chain of the form $\beta_{i,j}(z_i, z_j)$ with non-trivial $\beta_{i,j}$ can be vanishing. Therefore, the standard measures of triangles $\Delta_{0,i,j}$ form a basis in $\mathfrak{M}^{\mathbb{C}}(S)$ and $\mathfrak{M}^{\mathbb{R}}(S)$. □

Proposition 2 has the following interesting and immediate consequence.

Corollary 1. *All linear dependences among the standard measures of all triangles with vertices in S are generated by the linear dependences shown in Figure 3 which come from all possible* 4*-tuples of vertices in S.*

Corollary 1 is a special case of [3, Theorem 1]. Unfortunately, it seems that a proof of this important statement is absent from the literature. However, J. A. De Loera has informed us that it can be derived from results in [8] or [9].

To prove Proposition 3 we require an additional statement. Given a non-degenerate $S = \{z_0, z_1, \dots, z_n\}$, consider the complex-valued measure μ obtained by assigning (complex) densities d_{0ij}, $1 \le i < j \le n$, to the triangles Δ_{0ij}. Set $m_{i,j} = d_{0ij}\,\text{Area}\,\Delta_{0ij}$ and arrange $\{m_{i,j}\}$ in the vector $\mathbf{m}_n = (m_{12}, m_{13}, \dots m_{n-1,n})^\top$ so that the $m_{i,j}$'s are ordered lexicographically. Introduce the $(n-1)\times\binom{n}{2}$-matrix $\mathcal{M}_n^{\mathbb{C}}$, whose columns correspond to $m_{i,j}$; namely, such a column contains consecutive elementary symmetric functions of the $(n-2)$-tuple

$$(-z_1, -z_2, \dots \widehat{-z_i}, \dots, \widehat{-z_j}, \dots, -z_n),$$

where $\widehat{-z_i}$ and $\widehat{-z_j}$ stand for the omission of these points.

Example 4. For $n = 4$ we get the following:

$$\mathbf{m}_4 = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})^\top \qquad \text{and}$$

$$\mathcal{M}_4^{\mathbb{C}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -z_3 - z_4 & -z_2 - z_4 & -z_2 - z_3 & -z_1 - z_4 & -z_1 - z_3 & -z_1 - z_2 \\ z_3z_4 & z_2z_4 & z_2z_3 & z_1z_4 & z_1z_3 & z_1z_2 \end{pmatrix}.$$

Consider the maximal minor $Min_n^{\mathbb{C}}$ of $\mathcal{M}_n^{\mathbb{C}}$ formed by the columns corresponding to $m_{12}, \dots, m_{1,n}$, i.e., the first $n-1$ columns of $\mathcal{M}_n^{\mathbb{C}}$.

Lemma 2. $det_n = (-1)^{n-1}\det(Min_n^{\mathbb{C}}) = (-1)^{n-1}\prod_{2\le i<j\le n}(z_i - z_j)$.

Proof. Indeed, the degree of $\det(Min_n^{\mathbb{C}})$ as a polynomial in $z_2, \dots, z_n$ equals $\binom{n-1}{2}$. We need to show that $\det(Min_n^{\mathbb{C}})$ vanishes if and only if $z_i = z_j$. The "if" part is obvious since the column corresponding to $m_{1,i}$ coincide with the column corresponding to $m_{1,j}$. To prove the "only if" part, we argue by contradiction and assume that $(\alpha_{12}, \dots, \alpha_{1n})$ is a nontrivial linear dependence among the columns of $Min_n^{\mathbb{C}}$. The $1k$-th column consists of the coefficients of the polynomial $g_{1k}(u) = \prod_{j=1}^{n}(1 - z_ju)/(1 - z_ku)$, and our linear dependence is a linear dependence among such polynomials. Evaluate these at $1/z_j$ and note that $g_{1k}(1/z_j)$ vanishes whenever $k \neq j$. Thus $\alpha_{1j} = 0$, which is a contradiction. Thus $\det(Min_n^{\mathbb{C}})$ is divisible by $\prod_{2\le i<j\le n}(z_i - z_j)$. Ater setting $z_2 = 0, z_3 = 1, \dots, z_n = n-2$, we can check that the normalizing factor equals $(-1)^{n-1}$.

Remark 5. Using Cramer's rule, we can easily give an explicit formula for the inverse $(Min_n^{\mathbb{C}})^{-1}$. From Lemma 2, we know that for any (not necessarily non-degenerate) $S = \{z_0, z_1, \dots, z_n\}$ with pairwise distinct points, the rank of $\mathcal{M}_n^{\mathbb{C}}$ equals $n-1$. Thus the kernel of $\mathcal{M}_n^{\mathbb{C}}$ has dimension

$$\binom{n}{2} - (n-1) = \binom{n-1}{2}.$$

Proof of Proposition 3. The case $n = 2$ is trivial, so we assume $n \geq 3$. In the above notation, the normalized generating function $\Psi_\mu(u)$ for harmonic moments of μ is given by

$$\begin{aligned}\Psi_\mu(u) = \sum_{1\le i<j\le n} d_{0ij}\Psi_{\Delta_{0ij}}(u) &= \sum_{1\le i<j\le n} \frac{m_{ij}}{(1-z_0u)(1-z_iu)(1-z_ju)} \\ &= \frac{1}{1-z_0u}\frac{P(u)}{\prod_{i=1}^{n}(1-z_iu)},\end{aligned} \tag{2.3}$$

where $P(u)$ is a polynomial of degree at most $n-2$. The coefficients of P at $1, u, u^2, \dots, u^{n-2}$ are the consecutive entries of the vector $\mathcal{M}_n^{\mathbb{C}} \cdot \mathbf{m}_n$, where $\mathbf{m}_n$ and $\mathcal{M}_n^{\mathbb{C}}$ are se introduced above. Since by Lemma 2 the rank of $\mathcal{M}_n^{\mathbb{C}}$ equals $n-1$ for any non-degenerate configuration S and, by definition, the kernel of $\mathcal{M}_n^{\mathbb{C}}$ coincides with $\mathfrak{M}_{null}^{\mathbb{C}}(S)$. Proposition 3 follows. ☐

Example 5. For $n = 4$, the coefficients at $(1, u, u^2)$ of the numerator $P(u)$ of (2.3) are the consecutive entries of the vector $\mathcal{M}_4^{\mathbb{C}} \cdot \mathbf{m}_4$ where $\mathbf{m}_n$ and $\mathcal{M}_n^{\mathbb{C}}$ are given in Example 4 above.

In other words,

$$P(u) = \sum_{\substack{1\le i<j\le 4\\ k<\ell,\ \{ij\}\cap\{k\ell\}=\varnothing}} (m_{ij} - (z_i + z_j)m_{k\ell}u + z_iz_jm_{k\ell}u^2).$$

In order to prove Theorem 1, observe that the space $\mathfrak{M}^{\mathbb{R}}_{null}(S) \subset \mathfrak{M}^{\mathbb{C}}_{null}(S)$ is the maximal (under the inclusion relation) real subspace of the complex kernel. It can be interpreted as the real kernel of the (real-valued) $(2n-3)\times\binom{n}{2}$-matrix $\mathcal{M}^{\mathbb{R}}_n$ obtained by taking the real and imaginary parts of all rows of $\mathcal{M}^{\mathbb{C}}_n$. In other words, we substitute each row of $\mathcal{M}^{\mathbb{C}}_n$ with two rows, the first being the row of real parts and the second being the row of imaginary parts of the corresponding row of $\mathcal{M}^{\mathbb{C}}_n$.

Example 6. For $S = \{0, z_1, z_2, z_3, z_4\}$, the space $\mathfrak{M}^{\mathbb{R}}_{null}(S)$ is given by the system

$$\mathcal{M}^{\mathbb{R}}_4 \cdot \mathbf{m}_4 = 0, \qquad \text{where } \mathbf{m}_4 = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})^\top \text{ and}$$

$$\mathcal{M}^{\mathbb{R}}_4 = s\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ -x_3-x_4 & -x_2-x_4 & -x_2-x_3 & -x_1-x_4 & -x_1-x_3 & -x_1-x_2\\ -y_3-y_4 & -y_2-y_4 & -y_2-y_3 & -y_1-y_4 & -y_1-y_3 & -y_1-y_2\\ x_3x_4-y_3y_4 & x_2x_4-y_2y_4 & x_2x_3-y_2y_3 & x_1x_4-y_1y_4 & x_1x_3-y_1y_3 & x_1x_2-y_1y_2\\ x_3y_4+x_4y_3 & x_2y_4+x_4y_2 & x_2y_3+x_3y_2 & x_1y_4+x_4y_1 & x_1y_3+x_3y_1 & x_1y_2+x_2y_1 \end{pmatrix}. \tag{2.4}$$

Since the first row of $\mathcal{M}^{\mathbb{C}}_n$ equals $(1, 1, \dots, 1)$, the matrix $\mathcal{M}^{\mathbb{R}}_n$ has $2n-3$ rows; see (2.4). Ordering the m_{ij}'s lexicographically, consider the maximal minor $Min^{\mathbb{R}}_n$ of $\mathcal{M}^{\mathbb{R}}_n$ formed by the columns corresponding to $(2n-3)$ variables

$$m_{12}, m_{13}, \dots, m_{1n}, m_{23}, m_{24}, \dots, m_{2n},$$

i.e. the first $(2n-3)$ columns of $\mathcal{M}^{\mathbb{R}}_n$.

Lemma 3. $\det Min^{\mathbb{R}}_n = C[123][124]\cdots[12n]\prod_{3\le i<j\le n}|z_i - z_j|^2$, $0 \ne C \in \mathbb{R}$.

Proof. We begin by showing that $\Theta := \det Min^{\mathbb{R}}_n$ is divisible by $[12k]$ for every $3 \le k \le n$. Since $[12k]$ is an irreducible quadratic polynomial in x_1, x_2, x_k and y_1, y_2, y_k, it suffices to show that the vanishing of $[12k]$ implies the vanishing of Θ. The vanishing of $[12k]$ is equivalent to the existence of $a \in \mathbb{R}$ satisfying $z_k = az_1 + (1-a)z_2$. The latter implies that $Min^{\mathbb{R}}_n$ has linearly dependent columns 12, $1k$, and $2k$. Indeed, they consist, respectively, of the coefficients of the linearly dependent polynomials

$$g_{12}(u) = (a-1)g_{1k}(u) - ag_{2k}(u); \tag{2.5}$$

$$g_{1k}(u) = (1-z_2u)(1-z_3u)\cdots(1-z_{k-1}u)(1-z_{k+1}u)\cdots(1-z_nu); \tag{2.6}$$

$$g_{2k}(u) = (1-z_1u)(1-z_3u)\cdots(1-z_{k-1}u)(1-z_{k+1}u)\cdots(1-z_nu). \tag{2.7}$$

To show that Θ is divisible by $|z_i - z_j|^2 = (z_i - z_j)(\overline{z}_i - \overline{z}_j)$ for all $3 \leq i < j \leq n$, observe that $z_i = z_j$ implies $g_{ki}(u) = g_{kj}(u)$ for $k = 1, 2$.

It remains to show that Θ is not identically 0. Arguing by contradiction, let $(\alpha_{12}, \alpha_{13}, \dots, \alpha_{1n}, \alpha_{23}, \dots, \alpha_{2n})$ be the coefficients of a nontrivial real linear dependence among the columns of $Min_n^{\mathbb{R}}$. The latter columns correspond to the coefficients of $g_{ij}(u)$. Evaluating these at $u = 1/z_k$, for $3 \leq k \leq n$, shows that all of them except for g_{1k} and g_{2k} vanish. This operation corresponds to adding the rows of $Min_n^{\mathbb{R}}$ multiplied by $1, \Re z_k^{-1}, \Im z_k^{-1}, \Re z_k^{-2}, \Im z_k^{-2}, \dots$. Thus we obtain the following single linear equation in two real variables α_{1k} and α_{2k}:

$$\alpha_{1k} g_{1k}(z_k^{-1}) + \alpha_{2k} g_{2k}(z_k^{-1}) = 0. \tag{2.8}$$

Using (2.6) and (2.7), we see that (2.8) implies

$$\alpha_{1k} = -\alpha_{2k} \frac{z_k - z_2}{z_k - z_1}, \quad \text{and so} \quad \frac{z_k - z_2}{z_k - z_1} \in \mathbb{R}.$$

A direct computation shows that the rightmost relation is equivalent to $[12k] = 0$, a contradiction, since $[12k] = 0$ means that z_1, z_2, and z_k are colinear. □

Proof of Theorem 1. The case $n = 2$ is trivial. The case $n = 3$ can be dealt with by explicitly computing the kernel of $\mathfrak{M}_3^{\mathbb{C}}$ and seeing that it does not contain real vectors if S is non-degenerate. Thus we assume $n \geq 4$. Lemma 3 implies that for every non-degenerate S, the matrix $\mathcal{M}_n^{\mathbb{R}}$ has rank equal to $2n - 3$. Therefore,

$$\dim \mathfrak{M}_{null}^{\mathbb{R}}(S) = \binom{n}{2} - (2n-3) = \binom{n-2}{2}.$$

□

Proof of Theorem 2. For $S = \{0, z_1, z_2, z_3, z_4\}$, the space $\mathfrak{M}_{null}^{\mathbb{R}}(S)$ is given by the system

$$\mathcal{M}_4^{\mathbb{R}} \cdot \mathbf{m}_4 = 0, \qquad \text{where } \mathbf{m}_4 = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})^{\top},$$

$$\mathcal{M}_4^{\mathbb{R}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -x_3-x_4 & -x_2-x_4 & -x_2-x_3 & -x_1-x_4 & -x_1-x_3 & -x_1-x_2 \\ -y_3-y_4 & -y_2-y_4 & -y_2-y_3 & -y_1-y_4 & -y_1-y_3 & -y_1-y_2 \\ x_3x_4-y_3y_4 & x_2x_4-y_2y_4 & x_2x_3-y_2y_3 & x_1x_4-y_1y_4 & x_1x_3-y_1y_3 & x_1x_2-y_1y_2 \\ x_3y_4+x_4y_3 & x_2y_4+x_4y_2 & x_2y_3+x_3y_2 & x_1y_4+x_4y_1 & x_1y_3+x_3y_1 & x_1y_2+x_2y_1 \end{pmatrix}. \tag{2.9}$$

Recall that the right kernel of a $k \times (k+1)$-matrix T of rank k is spanned by the vector $(T^{(1)}, \dots, T^{(k+1)})$, where $T^{(j)}$ is the minor of T with jth column removed multiplied by $(-1)^j$. Thus (2.9) has a unique (up to a scaling) solution of the form

$$\begin{aligned}
m_{12} &= \ \ |z_1 - z_2|^2[134][234],\\
m_{13} &= \ \ |z_1 - z_3|^2[124][234],\\
m_{14} &= \ \ |z_1 - z_4|^2[123][234],\\
m_{23} &= -|z_2 - z_3|^2[124][134],\\
m_{24} &= -|z_2 - z_4|^2[134][123],\\
m_{34} &= -|z_3 - z_4|^2[123][124].
\end{aligned}$$

It is easy to prove this. We give a sketch here for m_{12}. Note that m_{12} equals the determinant of the matrix $A^{(12)}$ obtained from $\mathcal{M}_4^{\mathbb{R}}$ by removing the 1st column. Then, $\det A^{(12)}$ is divisible by $|z_1 - z_2|^2$, as the rank of $A^{(12)}$ drops when $z_1 = z_2$ and $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$ is the product of two irreducible polynomials with complex coefficients.

Similarly, $\det A^{(12)}$ is divisible by [234] (and a very similar argument applies to [134]). To see this, note that, since [234] is irreducible, it suffices to show that its vanishing implies the vanishing of $\det A^{(12)}$. To see that $\det A^{(12)}$ vanishes, assume that $z_4 = az_2 + (1 - a)z_3$, with $a \in \mathcal{R}$, and make this substitution into $A^{(12)}$. The last three columns of $A^{(12)}$ become

$$\begin{pmatrix}
1 & 1 & 1\\
-ax_2 + ax_3 - x_1 - x_3 & -x_1 - x_3 & -x_1 - x_2\\
-ay_2 + ay_3 - y_1 - y_3 & -y_1 - y_3 & -y_1 - y_2\\
ax_1x_2 - ax_1x_3 - ay_1y_2 + ay_1y_3 + x_1x_3 - y_1y_3 & x_1x_3 - y_1y_3 & x_1x_2 - y_1y_2\\
ax_1y_2 - ax_1y_3 + ax_2y_1 - ax_3y_1 + x_1y_3 + x_3y_1 & x_1y_3 + x_3y_1 & x_1y_2 + x_2y_1
\end{pmatrix}.$$

They are linearly dependent with coefficients $(1, a - 1, -a)$. □

To prove Theorem 3, we need the following elementary statement. In what follows, $|[K]|$ denotes twice the area of the polygon K.

Lemma 4. *For an arbitrary triangle* $\Delta_{\alpha\beta\gamma}$ *and arbitrary secants* $\alpha\epsilon$, $\beta\delta$ *(see Figure 4),* $|[\Delta_{\alpha\beta\zeta}]| > |[\Delta_{\epsilon\delta\zeta}]|$.

Proof. Draw the line $\alpha\kappa$ through α and parallel to $\beta\gamma$, and extend $\beta\delta$ until it inersects $\alpha\kappa$. Denote the intersection point by η; see Figure 4. Now $\Delta_{\alpha\beta\eta}$ and $\Delta_{\alpha\epsilon\eta}$ have equal areas since they have the same base $\alpha\eta$ and equal altitudes. Hence $\Delta_{\alpha\beta\zeta}$ and $\Delta_{\eta\epsilon\zeta}$ have equal areas since they are obtained by removing $\Delta_{\alpha\zeta\eta}$ from $\Delta_{\alpha\beta\eta}$ and $\Delta_{\epsilon\delta\zeta}$, respectively. The lemma follows since $\Delta_{\eta\epsilon\zeta}$ contains $\Delta_{\epsilon\delta\zeta}$. □

Proof of Theorem 3. To prove the first part, we recall that Example 1 settles the case $|S| = 6$. To settle the case $|S| = 6 + q$, we modify Example 1

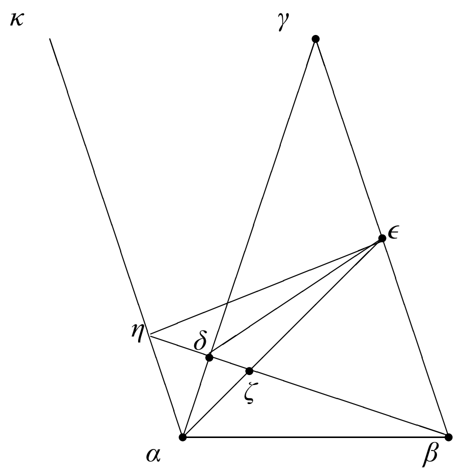


Figure 4. Illustration for Lemma 4.

accordingly, using the sets T, F, and F' introduced there. Pick q points $P_1, \ldots, P_q$ such that $Q := \operatorname{conv}(P_1, \ldots, P_q, \sqrt{3} + I, \sqrt{3} - I)$ is a convex $(q + 2)$-gon and $Q \cap \operatorname{conv}(T) = \operatorname{conv}(\sqrt{3} + I, \sqrt{3} - I)$. Then $F \cup Q$ and $F' \cup Q$ are equipotential $(q + 6)$-gons, by additivity of the measure.

To prove the second part, we have to consider the cases $|S| = 3, 4, 5$ separately. Cases $|S| = 3, 4$ follow from Theorem 1.

It remains to deal with the only non-trivial case $|S| = 5$. We have to consider the incidence matrices between the chambers and the basic simplices for all possible non-degenerate 5-tuples of points S. One can easily see that for non-degenerate 5-tuples, there are (up to permutation of the vertices) only three different cases to consider. They depend on the shape of conv(S), which can be a 5-gon, 4-gon, or triangle. The corresponding incidence matrices Inc_1, Inc_2, Inc_3 are given below using the labeling presented in Figures 5 and 6 for these cases. We show that in none of these cases does there exist a pair of equipotential polygons.

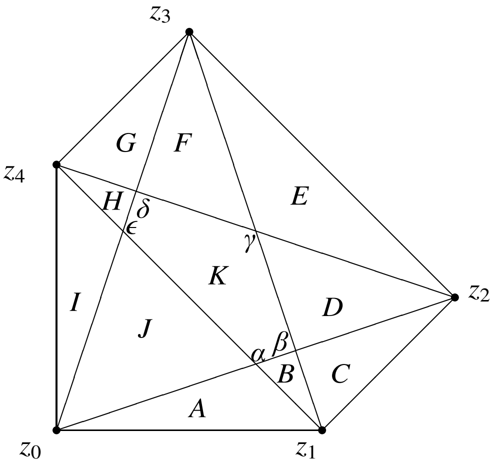
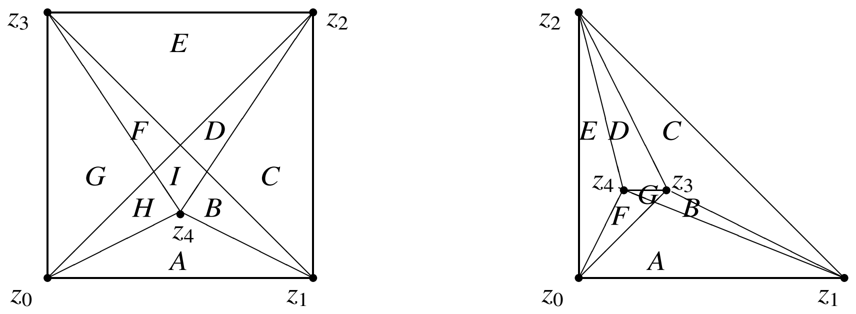


Figure 5. Chambers and their labeling for conv(S) a 5-gon. Greek letters denote the vertices of the inner 5-gon.

$$Inc_1 = \begin{array}{c|ccccccccccc} & A & B & C & D & E & F & G & H & I & J & K \\ \hline \Delta_{012} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_{013} & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ \Delta_{014} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \Delta_{023} & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ \Delta_{024} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \Delta_{034} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array},$$

$$Inc_2 = \begin{array}{c|ccccccccc} & A & B & C & D & E & F & G & H & I \\ \hline \Delta_{012} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ \Delta_{013} & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \Delta_{014} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_{023} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \Delta_{024} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ \Delta_{034} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array},$$

$$Inc_3 = \begin{array}{c|ccccccc} & A & B & C & D & E & F & G \\ \hline \Delta_{012} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \Delta_{013} & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \Delta_{014} & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \Delta_{023} & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \Delta_{024} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \Delta_{034} & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}.$$



4-gon triangle

Figure 6. Chambers and their labeling for conv(S) a 4-gon or a triangle.

Case 1: conv(S) is a 5-gon. Using the labeling in Figure 5 and (1.4), we see that densities d_{012}, d_{014}, d_{023}, and d_{034} are positive while d_{013} and d_{024} are negative. Assuming that the densities of all chambers attain only values 0, ±1 and looking at chambers C, E, G, we get that $d_{023} = d_{012} = d_{034} = 1$. Looking at chamber D, we conclude that $d_{024} = -1$. (d_{024} might be equal -2 as well, but then, looking at chamber K, we have to conclude that $d_{013} = 0$, which is impossible.) From chamber K, we get $d_{013} = -1$, and from chamber I, we get $d_{014} = 1$. Thus the density equals 1 in chambers A, C, E, G, I, equals -1 in chamber K, and vanishes in the remaining chambers. Notice that the total mass of the measure vanishes. To see that this cannot happen, we show that $|[K]| < |[A]|+|[C]|+|[E]|+|[G]|+|[I]|$. Using Lemma 4, we conclude that $|[A]| > |[\Delta_{\alpha\beta\epsilon}]|$, $|[C]| > |[\Delta_{\alpha\beta\gamma}]|$, $|[E]| > |[\Delta_{\beta\gamma\delta}]|$, $|[G]| > |[\Delta_{\delta\epsilon\gamma}]|$, and $|[I]| > |[\Delta_{\epsilon\alpha\delta}]|$; see Figure 5. Triangles $\Delta_{\alpha\beta\epsilon}$, $\Delta_{\alpha\beta\gamma}$, $\Delta_{\beta\gamma\delta}$, $\Delta_{\delta\epsilon\gamma}$, $\Delta_{\epsilon\alpha\delta}$ overlap pairwise. These overlaps consist of five smaller triangles inside K. The complement in K to the union of triangles $\Delta_{\alpha\beta\epsilon}$, $\Delta_{\alpha\beta\gamma}$, $\Delta_{\beta\gamma\delta}$, $\Delta_{\delta\epsilon\gamma}$, $\Delta_{\epsilon\alpha\delta}$ is a small 5-gon inside K. Now we can use these five small triangles to cover the small 5-gon inside K. We arrive at exactly the same situation as the original one, and we can apply the same argument we already used and cover a substantial part of the small 5-gon etc. Continuing this process, wecover the original 5-gon K in infinitely many steps. Thus the required measure does not exist.

Case 2: conv(S) is a 4-gon. Using labeling on the left part of Figure 6 and formulas (1.4), we conclude that densities d_{012}, d_{014}, d_{023}, and d_{034} are positive while densities d_{013} and d_{024} are negative. From chambers E and C, we conclude

$d_{023} = d_{012} = 1$. Then from chamber D we have that either $d_{024} = -1$ or -2. The second case leads to $d_{013} = 0$, a contradiction. Thus $d_{024} = -1$ which, from chamber I, gives $d_{013} = -1$. Finally, $d_{034} = d_{014} = 1$. Thus chambers A, C, E,and G have density 1, chamber I has density -1, and the remaining chambers have vanishing density. We need to show that $||[I]|| < ||[A]|| + ||[C]|| + ||[E]|| + ||[G]||$. We acutally show that $||[I]|| < ||[C]|| + ||[G]||$. Cut I into two triangles by drawing its diagonal connecting z_4 with non-neighboring vertex p of I (lying strictly above z_4 in the left part of Figure 6). Extending z_3p and z_0z_4, we get a triangle containing G and the left half of I, and we can apply Lemma 4. Analogously, extending z_2p and z_1z_4 we get a triangle containing C and the right half of I and we can apply Lemma 4. Thus the required measure does not exist.

Case 3: conv(S) is a triangle. Using labeling on the right part of Figure 6 and formulas (1.4). we again conclude that densities d_{012}, d_{014}, and d_{024} are negative, while d_{013}, d_{023}, and d_{034} are positive. Similar considerations as above give $d_{012} = d_{014} = d_{024} = -1$ and $d_{013} = d_{023} = d_{034} = 1$. Thus, the densities of A, C, and E are -1, and the density of G is 1. In fact, $||[G]|| < ||[A]||$ already. Indeed, extending the interval z_0z_4 and z_1z_3 to where they intersect, say at p, we obtain the triangle z_0z_1p to which we apply Lemma 4. Thus the required measure does not exist. □

To prove Theorem 4, we state the following observation, which is obvious from Figure 3.

Lemma 5. *The convex hull of the union of the supports of the standard measures of four triangles as in Figure 3, Case a) (i.e., two pairs forming a flip) is a plane quadrangle. The convex hull of the union of the supports of the standard measures of the standard measures of four triangles as in Figure 3, Case b) is a plane triangle.*

Proof of Theorem 4. If a triangle Δ contains an interior point other than its vertices, then μ_Δ is the sum of three triangles in which it is subdivided by an inner vertex; see Lemma 5. (Recall that S is non-degenerate, by assumption.) Thus μ_Δ is not an extreme ray. On the other hand, assume that no point in S other than its vertices is contained in Δ and that μ_Δ is a linear combination of the standard measures of some other triangles with vertices in S with positive coefficients. Since no such triangle can be contained strictly inside Δ, by assumption, and all coefficients are positive, any such linear combination necessarily has positive density somewhere outside Δ, a contradiction. □

3 Open problems

1. Theorem 1 gives the dimension of $\mathfrak{M}^{\mathbb{R}}_{null}(S)$ for non-degenerate S. Its dimension for arbitrary S is elusive. On one hand, if S is degenerate, then $\dim \mathfrak{M}^{\mathbb{R}}(S)$ decreases. On the other hand, the number of equations imposed on the densities might also decrease. It seems highly plausible that for an arbitrary S, $\dim \mathfrak{M}^{\mathbb{R}}_{null}(S)$ depends only on the non-oriented matroid associated to this set; see e.g., [10]. An algorithm for calculating this dimension is given in [1].

2. Besides the cone $\mathfrak{K}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$, one can introduce a more important, larger, cone $\mathfrak{K}_{pos}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$, where $\mathfrak{K}_{pos}(S) \supset \mathfrak{K}(S)$ consists of all non-negative measures from $\mathfrak{M}^{\mathbb{R}}(S)$.

Conjecture 2. *The combinatorial structure of $\mathfrak{K}_{pos}(S)$ depends only on the oriented matroid associated to S.*

Already, for generic configurations S with six points, the combinatorial structure of $\mathfrak{K}_{pos}(S)$ and, in particular, the set of its extreme rays seems to be quite complicated. We plan to study this fascinating subject in the future.

3. Ovserve that there exists a natural linear map $\Psi_\mu : \mathfrak{M}^{\mathbb{R}}(S) \to Rat_n$ obtained by associating to each measure $\mu \in \mathfrak{M}^{\mathbb{R}}(S)$ its normalized generating function (1.2). Here, Rat_n is the linear space of rational functions of the form

$$R(u) = \frac{P(u)}{\prod_{j=0}^{n}(1 - z_j u)}, \quad \deg P(u) \leq n-2,$$

having real constant term. Obviously, $\dim Rat_n = 2n - 3$; and, using Theorem 1, we see that $\mathfrak{M}^{\mathbb{R}}(S)$ is mapped onto Rat_n. The following question is very natural in connection with the inverse problem for the class of non-negative measures.

Problem 3. Describe the extreme rays/faces of the image cones $\Psi_\mu(\mathfrak{K}(S))$ and $\Psi_\mu(\mathfrak{K}_{pos}(S)$ in Rat_n.

4. We have given an example of a pair of equipotential polygons with $|S| = 6$; see Figure 1.

Problem 4. Describe all 6-tuples S which admit a pair of equipotential polygons.

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