THE TRANSLATION GEOMETRY OF PÓLYA'S SHIRES

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Abstract. In his shire theorem, G. Pólya proves that the zeros of iterated derivatives of a meromorphic function in the complex plane accumulate on the union of edges of the Voronoi diagram of the poles of this function. By recasting the local arguments of Pólya into the language of translation surfaces, we prove its generalization describing the asymptotic distribution of the zeros of a meromorphic function on a compact Riemann surface under the iterations of a linear differential operator $T_\omega : f \mapsto \frac{df}{\omega}$ where ω is a given meromorphic 1-form. The accumulation set of these zeros is the union of edges of a generalized Voronoi diagram defined by the initial function f together with the singular flat metric on the Riemann surface induced by ω. This result provides a novel approach to the problem of finding a flat geometric presentation of a translation surface initially defined in terms of algebraic or complex-analytic data.

Contents

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1. Introduction

1.1. Short historical account. The classical shire theorem of G. Pólya claims that for a meromorphic function f with the set S of its poles, the zeros of its iterated derivatives $f^{(n)}$ accumulate when $n \to +\infty$ along the edges of the Voronoi diagram associated with S, see [9]. An illustration of this famous result is shown in Figure 1.

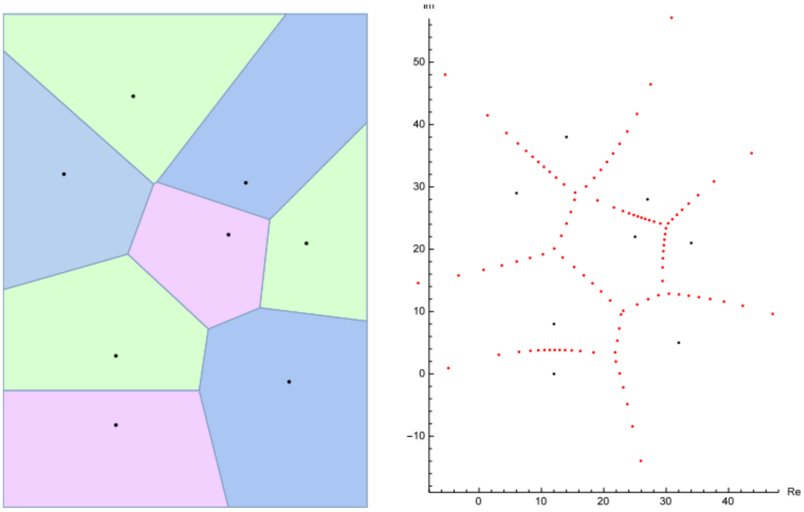

Figure 1. The Voronoi diagram of the roots of a polynomial f of degree 8 (left), and zeros of $(1/f)^{(15)}$ (right).

Several prominent mathematicians including N. Wiener, E. Hille, and R. P. Boas have continued Pólya's line of study soon after the publication of the latter theorem, see references in [17]. Over the years a number of articles extending and generalising the original result has appeared, see e.g. [5, 6, 7, 18, 19, 21].

More recent publications have concentrated on the weak limits of the root-counting measures for the zeros of $f^{(n)}$. In particular, Ch. Hägg and R. Bøgvad obtained a measure-theoretic refinement of Pólya's shire theorem for rational functions, see [2]. Using currents they also proved a similar result for Voronoi diagrams associated with generic hyperplane arrangements in $\mathbb{C}^m$.

Later Ch. Hägg extended the main result of [2] by considering meromorphic functions of the form $f = Re^U$ where R is a rational function with at least 2 distinct zeros and U is a non-constant polynomial, see [8]. The class of such functions coincides with the class of meromorphic functions that are quotients of two entire functions of finite order, each having a finite number of zeros, see [22].

In [11], V. Keo extended the results of [2] and [8] by studying a particular case of meromorphic functions $f(z) = 1/(1 - e^z)$ having an infinite number of poles and whose iterated derivatives are related to Eulerian polynomials. In addition, V. Keo considered iterations of rational functions under the action of differential operators of the form $\mathcal{D} = g(z)\dfrac{\partial}{\partial z}$ where $g(z)$ is a polynomial in z which is closely related to the topic of the present paper. He mainly studied a particular case of $z\dfrac{\partial}{\partial z}$ and formulated some conjectures.

Finally, in [25], M. Weiss provides a generalization of Pólya's classical theorem to automorphic functions in the half-plane. Observe that there are two natural ways to generalize the geometry of flat tori to surfaces of higher genus:

- hyperbolic surfaces (the metric still has constant curvature, but it is no longer flat);
- translation surfaces (the metric is still flat, but it has conical singularities).

1.2. **Our set-up.** Below we generalize Pólya's shire theorem to the case of meromorphic functions on compact Riemann surfaces and study a class of linear differential operators corresponding to the complex-analytic data defining a translation structure (see [28] for background on translation surfaces). Namely, let X be a compact Riemann surface with a fixed meromorphic 1-form ω. We associate to the pair (X, ω) the linear differential operator T_ω acting on meromorphic functions on X as

$$T_\omega : f \mapsto \frac{df}{\omega}. \tag{1.1}$$

Now given a meromorphic function f on X, we are interested in the asymptotic of zeros for the sequence $(f_n)_{n\in\mathbb{N}}$ of meromorphic functions $(f_n)_{n\in\mathbb{N}}$ defined inductively by

$$f_0 = f,\ f_{n+1} = T_\omega(f_n) = T_\omega^{n+1}(f),\ n \geq 1.$$

Definition 1.1. For a meromorphic function f on a Riemann surface X and any operator T acting on the space of meromorphic functions, define the *limit set* $\mathcal{L}(T, f) \subseteq X$ as the set of points $z \in X$ such that any open neighborhood of z in X contains a zero of $T^n(f)$ for infinitely many n.

In the classical shire theorem, the limit set coincides with the Voronoi diagram associated with the set of poles of the initial function. In our generalized settings, the Voronoi diagram is defined with respect to the so-called *principal polar locus* and the singular flat metric induced by the translation structure.

Definition 1.2. Given a point z_0 on a Riemann surface X, we say that a meromorphic function f is *locally factorized* by a primitive of meromorphic 1-form ω having no pole at z_0 if there exist:

- a neighborhood U of z_0 in X,
- a holomorphic function ϕ defined on U;
- a holomorphic function g defined on a neighborhood of $\phi(z_0)$ in $\mathbb{C}$

such that $\omega = d\phi$ and $f = g \circ \phi$ in U.

Below by a *non-zero meromorphic* 1*-form* we always mean a 1-form not vanishing identically on the underlying Riemann surface.

Definition 1.3. Consider a non-zero meromorphic 1-form ω and a fixed meromorphic function f on a Riemann surface X. The *principal polar locus* $\mathcal{PPL}(\omega, f)$ of pair (ω, f) is the subset of X containing:

- the poles of f that are not poles of ω;
- the zeros of ω where f is not locally factorized by a primitive of ω (see Definition 1.2).

Remark 1.4. In the original shire theorem, the Riemann surface is the extended complex plane $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, $\omega = dz$ and $\mathcal{PPL}(\omega, f)$ is the set of the affine poles of f (i.e. poles different from ∞).

The main result of our paper is as follows.

Theorem 1.5. *Consider a non-zero meromorphic* 1*-form* ω *on a compact Riemann surface* X*, its associated differential operator* T_ω *and any meromorphic function* f *on* X *such that* $\mathcal{PPL}(\omega, f) \neq \emptyset$*. Then the following two facts are valid:*

(i) the limit set $\mathcal{L}(T_\omega, f)$ is the union of:

- the Voronoi diagram $\mathcal{V}_{\omega,f}$ defined by $\mathcal{PPL}(\omega, f)$ (see Definition 2.10 below);
- the poles of ω of order at least two;
- the simple poles of ω that are not poles of f;

(ii) the asymptotic root-counting measure of the sequence $(T^n_\omega(f))_{n\in\mathbb{N}}$ is given by

$$\frac{\mu_{\omega,f}}{A\pi} + \frac{1}{A}\sum_{p\in\mathcal{P}}(d_p - 1)\delta_p,$$

where $\mu_{\omega,f}$ is the Cauchy measure of the Voronoi diagram $\mathcal{V}_{\omega,f}$ (see Definition 2.14), $\mathcal{P}$ is the set of poles of ω, and $A = \sum\limits_{z\in\mathcal{PPL}(\omega,f)} (a_z + 1)$ where z is a zero of ω of order a_z and $p \in \mathcal{P}$ is a pole of ω of degree d_p.

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2. Preliminary notions and results

2.1. **Growth of the order of poles under iterations of T_ω.** Notice that unless a function f has a very specific form, one expects it to develop poles at the zeros of ω under the iterations of the operator T_ω. The following lemma describes this phenomenon completely.

Lemma 2.1. *Consider a non-zero meromorphic 1-form ω and a meromorphic function f on a Riemann surface X. Then for any point $z_0 \in X$, the following statements are equivalent:*

- *no function in the sequence $(f_n)_{n\in\mathbb{N}}$ has a pole at z_0;*
- *f is locally factorized at z_0 by a primitive of ω (see Definition 1.2).*

Proof. Up to local biholomorphic change of z, we can assume that $z_0 = 0$ and that $\omega = z^m dz$ for some $m \in \mathbb{N}$. An arbitrary locally defined holomorphic function f can be written as a power series in z, i.e.,

$$f(z) = \sum_{k=0}^{+\infty} a_k z^k.$$

Then $T_\omega(f)$ is given by a Laurent series of the form:

$$T_\omega(f) = \sum_{k=0}^{+\infty} a_k k z^{k-m-1}.$$

It follows immediately that if no function in sequence $(f_n)_{n\in N}$ has a pole at 0, then $a_k = 0$ for all $k \notin (m+1)\mathbb{Z}$.
Introducing the local primitive $\phi = \frac{z^{m+1}}{m+1}$ of ω, we obtain

$$f(z) = \sum_{k=0}^{+\infty} a_{k(m+1)}(m+1)^k \phi^k.$$

Therefore, in a neighborhood of 0, f factorizes as $g \circ \phi$ where g is a holomorphic function given by the above series and well-defined near 0.

Conversely, for any meromorphic function f which locally factorizes as $g \circ \phi$, direct computation proves that for any k, we have:

$$T^k_\omega(f) = g^{(k)} \circ \phi.$$

Therefore no iterate $T_\omega^k(f)$ of f has a pole at z_0. □

The following statement establishes a dichotomy between the points belonging to the principal polar locus $\mathcal{PPL}(\omega, f)$ and points outside it (see Definition 1.3). In the principal polar locus $T_\omega^k(f)$ has poles whose orders grow linearly with n while the sum of orders of the poles outside $\mathcal{PPL}(\omega, f)$ remains bounded. In particular, we obtain that when the sum of orders of poles (and thus the sum of orders of zeros) grow infinitely, the principal polar locus must be non-empty. We prove that the total order of poles has linear growth and provide a formula for the coefficient of the latter linear dependence.

Proposition 2.2. *Consider a non-zero meromorphic* 1*-form* ω *and a meromorphic function* f *on a compact Riemann surface* X*. Then there is a number* $M > 0$ *such that:*

- *for any* $k > M$ *and any point* $p \in \mathcal{PPL}(\omega, f)$*,* p *is a pole of* $T_\omega^k(f)$ *of order* $\alpha_p + k(d_p + 1)$*, where* d_p *is the order of* ω *at* p *and* α_p *is some constant;*
- *for any* $k > M$*, the total order of the poles of* $T_\omega^k(f)$ *outside* $\mathcal{PPL}(\omega, f)$ *is constant. All these poles are simple poles of* ω*.*

Proof. Iterating T_ω we get that for any $k \in \mathbb{N}$, any pole of $T_\omega^k(f)$ is either a pole of f or a zero of ω. Since X is compact, we have only finitely many points to examine. Let us first consider the case of a point p which is a pole of f of order m and a pole of ω of order $-d_p$. Direct computation shows that p is a singularity of $T_\omega^k(f)$ of order $-m - k(d_p + 1)$. Therefore, if p is a simple pole of ω, it still remains a pole of $T_\omega^k(f)$ of order m for any $k \in \mathbb{N}$. In contrast, if p is a pole of ω of order at least two, then $T_\omega^k(f)$ is holomorphic at p provided that k is large enough. In both cases, p does not belong to $\mathcal{PPL}(\omega, f)$.

Now, let us consider the case of a point p which is a zero of ω and f is locally factorized by a primitive of ω. If p is not a pole of f, then Lemma 2.1 proves that p is not a pole of any function in the sequence $(f_n)_{n \in N}$. Thus, we have proved that after finitely many steps, the total order of the poles of $T_\omega^k(f)$ outside $\mathcal{PPL}(\omega, f)$ stabilizes to the total order of the poles of f at simple poles of ω.

Lemma 2.1 implies that for any point p of $\mathcal{PPL}(\omega, f)$, there exists $k_p \in \mathbb{N}$ such that $T_\omega^{k_p}(f)$ has a pole at p. Then direct computation proves that the order of the pole increases with each application of T_ω by $d_p + 1$ which finishes the proof. □

Corollary 2.3. *Consider a non-zero meromorphic* 1*-form* ω *and a meromorphic function* f *on a compact Riemann surface* X *such that* $\mathcal{PPL}(\omega, f) \neq \emptyset$*. For any* $z \in \mathcal{PPL}(\omega, f)$*, let* a_z *be the order of* ω *at* z*.*

Then the sum of orders $\mathcal{Z}_n$ *of the poles of meromorphic function* $T_\omega^n(f)$ *(and therefore the sum of orders of its zeros) has the asymptotics* $\mathcal{Z}_n \sim An$ *as* $n \to +\infty$ *where*

$$A = \sum_{z \in \mathcal{PPL}(\omega, f)} (a_z + 1).$$

2.2. **Translation structures.** A non-zero meromorphic 1-form ω on a (possibly open) Riemann surface X defines a geometric structure as follows. Denote by

- X^* the surface X punctured at the poles of ω;
- X^{**} the surface X punctured at the zeros and the poles of ω.

Local primitives of ω are locally injective on X^{**}. They form an atlas of local biholomorphisms of X to $\mathbb{C}$. Since two local primitives of the same differential are defined up to the addition of a constant, transition maps between two distinct charts of the atlas are *translations* of the complex plane. Therefore, ω endows the

Riemann surface X with a *translation structure*. The pair (X, ω) is called a *translation surface* (see [28] for general background on translation surfaces).

For example, the standard differential dz endows the complex plane $\mathbb{C}$ with the standard Euclidean metric $|dz|$. Any chart ϕ of the translation atlas (in other words a local primitive of ω) conjugates the standard differential dz in the complex plane with a differential ω defined on X: we have $\phi^* dz = \omega$. Consequently, we can think about a translation surface as formed by pieces of the standard flat plane glued together along translations.

The punctured surface X^{**} is thus endowed with a flat metric $|\omega|$. The latter naturally extends to the zeros of ω. A neighborhood of a zero of ω of order k is mapped (by any local primitive) to the complex plane as a ramified cyclic cover of degree $k+1$. It follows that the flat metric $|\omega|$ extends to such a zero as having a conical singularity of angle $2(k+1)\pi$. Therefore the punctured surface X^* is a Euclidean surface with conical singularities, see [23].

Remarkably, as soon as a meromorphic differential defined on a compact Riemann surface has poles, the metric structure it defines on the surface punctured at these poles is no longer compact.

Lemma 2.4. *Let X be a compact Riemann surface endowed with a non-zero meromorphic 1-form ω. Then the punctured surface X^* is a complete metric space for the singular flat metric $|\omega|$.*

Proof. We have to prove that any Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ of points in X^* converges to some limit point in X^*. Since X^* is locally compact and ω has finitely many poles, the question reduces to the only case when $(z_n)_{n\in\mathbb{N}}$ converges in X to a pole z_∞ of ω. We will prove that no such sequence is a Cauchy sequence in the flat metric.

Indeed, let $k \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}$ be the order and the residue of ω at the pole z_∞ respectively. Up to a biholomorphic change of z, we can assume that $z_\infty = 0$ and normalize ω as $(\frac{\lambda}{z} + \frac{1}{z^k})dz$ for $k \geq 2$ and as $\frac{\lambda dz}{z}$ for $k = 1$, (see [3] for the details on local models for poles in translation surfaces).

Up to a choice of a subsequence, we can assume that the sequence $(z_n)_{n\in\mathbb{N}}$ belongs to the domain of a unique chart $\phi : z \mapsto \lambda \ln(z) - \frac{1}{(k-1)z^{k-1}}$ if $k \geq 2$ and $\phi : z \mapsto \lambda \ln(z)$ if $k = 1$. The fact that this sequence $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric $|\omega|$ precisely means that the sequence $(\phi(z_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the complex plane and therefore it converges to some point. It is therefore disjoint from any small enough neighborhood of z_∞ when $n \to +\infty$ which finishes the proof. □

2.3. **Voronoi functions.** For a meromorphic function f in $\mathbb{C}$, the Voronoi diagram of its set of poles can be defined in terms of maximal embedded disks that are disjoint from these poles. *Voronoi functions* play a similar role on translation surfaces.

Definition 2.5. Consider a compact Riemann surface X endowed with a non-zero meromorphic 1-form ω and a meromorphic function f on X. Let $z \in X^* \setminus \mathcal{PPL}(\omega, f)$.

Denoting by ϕ a local primitive of ω such that $\phi(z) = 0$, the *Voronoi function* g_z is the holomorphic function defined in the neighborhood of the origin in the complex plane $\mathbb{C}$ and satisfying $g_z \circ \phi = f$ (see Definitions 1.2 and 1.3 for the existence and uniqueness of g_z if z is a zero of ω).

The *critical radius* $\rho(z)$ is the radius of convergence of g_z in 0. Immediate computation shows that for any $k \geq 1$, $g_z^{(k)} \circ \phi = T_\omega^k(f)$.

The behaviour of g_z on the boundary of its disk of convergence at 0 reflects the geometry of the translation surface (X, ω).

Lemma 2.6. *Consider a compact Riemann surface X endowed with a non-zero meromorphic 1-form ω and a meromorphic function f. Let $z \in X^* \setminus \mathcal{PPL}(\omega, f)$. Denote by $\tilde{z}$ some preimage of z in the universal cover $\pi : \tilde{X^*} \to X^*$. Introduce the primitive $\tilde{\phi}$ of $\pi^*\omega$ in $\tilde{X^*}$ such that $\tilde{\phi} = 0$.*

Then for any path $\sigma : [0,1] \to \mathbb{C}$ such that

- *$\sigma(0) = 0$;*
- *$|\sigma(s)| < \rho(z)$ for any $s \in [0,1[$;*
- *$|\sigma(1)| \leq \rho(z)$;*

there exists a path $\tilde{\gamma} : [0,1] \to \tilde{X^*}$ such that $\tilde{\phi} \circ \tilde{\gamma} = \sigma$. Besides, one of the following two statements holds:

(1) $\pi \circ \tilde{\gamma}(1) \notin \mathcal{PPL}(\omega, f)$ and g_z extends to $\sigma(1)$ as a holomorphic function;
(2) $\pi \circ \tilde{\gamma}(1) \in \mathcal{PPL}(\omega, f)$ and g_z does not extends to $\sigma(1)$ as a holomorphic function. Nevertheless, it extends in a neighborhood of $\sigma(1)$ as a convergent Puiseux series.

Proof. The existence of a path $\tilde{\gamma} : [0,1] \to \tilde{X^*}$ satisfying $\tilde{\phi} \circ \tilde{\gamma} = \sigma$ follows from the metric completeness of $(X^*, |\omega|)$ (see Lemma 2.4).

Analytic continuation along the path $\tilde{\gamma}$ proves that the function $g_z \circ \tilde{\phi}$ coincides with $f \circ \pi$. (Recall that π is the projection of $\tilde{X^*}$ to X^*).)

We first consider the case when $\pi \circ \tilde{\gamma}(1) \notin \mathcal{PPL}(\omega, f)$. Following Definitions 1.2 and 1.3, there is a neighborhood of $\pi \circ \tilde{\gamma}(1)$ in X^* in which f is factorized by ϕ. Thus g_z extends to $\sigma(1)$ as a holomorphic function.

If $\pi \circ \tilde{\gamma}(1) \in \mathcal{PPL}(\omega, f)$ but is not a zero of ω, then ϕ is locally injective in a neighborhood of $\pi \circ \tilde{\gamma}(1)$ and the meromorphic function f can be factorized by ϕ. Therefore g_z extends to $\sigma(1)$ as a pole of a meromorphic function.

In the latter case, $\pi \circ \tilde{\gamma}(1)$ is a zero of ω of order k. Equivalently, $\tilde{\gamma}(1)$ is a critical point of $\tilde{\phi}$ of order k. The equation $f \circ \pi = g \circ \tilde{\phi}$ can be locally solved at $\tilde{\gamma}(1)$ if we consider g as a convergent Puiseux series with exponents belonging to $\frac{1}{k+1}\mathbb{Z}$. This provides the correct extension of g_z to $\sigma(1)$. The function g_z can not extend to $\sigma(1)$ holomorphically because in this case, f would be locally factorized (see Definition 1.2) by a primitive of ω and $\tilde{\gamma(1)}$ would not belong to the principal polar locus. □

Corollary 2.7. *Consider a compact connected Riemann surface X endowed with a non-zero meromorphic 1-form ω and a meromorphic function f on X such that $\mathcal{PPL}(\omega, f) \neq \emptyset$. Then any point $z \in X^* \setminus \mathcal{PPL}(\omega, f)$ has a finite critical radius $\rho(z)$.*

Proof. We assume by contradiction that $\rho(z) = +\infty$, i.e. that g_z is an entire function. Since X is connected and $\mathcal{PPL}(\omega, f) \neq \emptyset$, there is a real-analytic path $\gamma : [0,1] \to X^*$ such that $\gamma(0) = 0$ and $\gamma(1) = z_0$ where $z_0 \in \mathcal{PPL}(\omega, f)$. The path γ lifts to a path $\tilde{\gamma}$ on the universal cover $\tilde{X^*}$. The path $\tilde{\gamma}$ satisfies the assumptions of Lemma 2.6 and we conclude that g_z is not holomorphic at $\tilde{\phi} \circ \tilde{\gamma}(1)$ which is a contradiction. □

2.4. **Voronoi diagrams.** Next we stratify the translation surface (X, ω) according to the values of the *Voronoi index* defined below.

Definition 2.8. Let X be a compact connected Riemann surface X endowed with a non-zero meromorphic 1-form ω and a meromorphic function f such that $\mathcal{PPL}(\omega, f) \neq \emptyset$. For any point $z \in X^* \setminus \mathcal{PPL}(\omega, f)$, let g_z be the Voronoi function of z. Let $\rho(z)$ be the critical radius at z.

Then the *Voronoi index* ν_z is the number of points on the circle of radius $\rho(z)$ centered at 0 at which g_z does not extend as a holomorphic function.

Lemma 2.9. *For any point $z \in X^* \setminus \mathcal{PPL}(\omega, f)$, ν_z is a positive integer.*

Proof. Following Corollary 2.7, the radius $\rho(z)$ of convergence of g_z at 0 is finite and g_z fails to extend to a holomorphic function at one point of the circle of radius $\rho(z)$ at least. At each of these singular points, g_z extends as a Puiseux series converging in some neighborhood of such a point (see Lemma 2.6). Compactness argument then proves that there can be at most finitely many such points inside the considered circle. □

Assuming that the principal polar locus is nonempty, the values of the Voronoi index decompose the underlying surface X into:

- the Voronoi cells (for which $\nu = 1$);
- the Voronoi edges (for which $\nu = 2$);
- the Voronoi vertices (for which $\nu \geq 3$).

Definition 2.10. Consider a compact Riemann surface X endowed with a non-zero meromorphic 1-form ω and a meromorphic function f on X such that $\mathcal{PPL}(\omega, f) \neq \emptyset$. The *Voronoi diagram* $\mathcal{V}_{\omega,f}$ of pair (ω, f) is the union of all points $z \in X$ such that $\nu_z \geq 2$.

Proposition 2.11. *Consider a compact Riemann surface X endowed with a non-zero meromorphic* 1*-form ω and a meromorphic function f on X such that $\mathcal{PPL}(\omega, f) \neq \emptyset$. Then $\mathcal{V}_{\omega,f}$ is the union of geodesic segments in the (singular) flat metric $|\omega|$.*

Proof. For any point $z_0 \in \mathcal{V}_{\omega,f}$ of critical radius r and Voronoi index ν, we denote by $\alpha_1, \dots, \alpha_\nu$ the points of $\{t \in \mathbb{C} \mid |t| = r\}$ where the Voronoi function g_{z_0} does not extend as a holomorphic function.

Actually, there exists $\epsilon > 0$ such that g_{z_0} extends as a holomorphic function to a domain $\mathcal{D}$ formed by the points of the open disk $\{x \in \mathbb{C} \mid |x| < r + \epsilon\}$ which do not belong to the cuts $[1, \frac{r+\epsilon}{r}[\alpha_i$ for $1 \leq i \leq \nu$.

For any two distinct $i, j \in \{1, \dots, \nu\}$, let the line L_{ij} be the bisector of the segment $[\alpha_i, \alpha_j]$. Denote by $\mathcal{L}$ the union of all these lines L_{ij} for $1 \leq i < j \leq \nu$.

Consider a small neighborhood U of z_0 in X in which ω has a well-defined primitive ϕ satisfying the condition $\phi(z_0) = 0$. We can assume that for any $z \in U$, $|\phi(z)| < \delta$ where $\delta < \frac{\epsilon}{2}$. It follows that for any $z \in U$, the disk of convergence of g_z in $\phi(z)$ is contained in $\mathcal{D}$, see Figure 2. It follows that unless $\phi(z) \in \mathcal{L}$, we have $\nu(z) = 1$.

In the open set U, $\phi^{-1}(\mathcal{L})$ is the union of straight segments in the flat coordinate ϕ which finishes the proof. □

Remark 2.12. Voronoi diagrams cannot be given a purely metric definition, as demonstrated in the following example. Consider $\omega = zdz$ and $f(z) = \frac{1}{z^2-1}$. Since f is factorized by a primitive $\phi : z \mapsto \frac{1}{2}z^2$ of ω, we have $\mathcal{PPL}(\omega, f) = \{\pm 1\}$.

The points equidistant from 1 and -1 (with respect to $|zdz|$) are those satisfying $\mathrm{Re}(z) = 0$. However, since 1 and -1 have the same image under ϕ, there is no point z in $\mathbb{C} \setminus \{\pm 1\}$ such that the corresponding Voronoi function g_z has more than one singular point in its critical circle. The Voronoi diagram is empty and, in particular, it does not coincide with the imaginary axis as a purely metric definition would suggest.

2.5. **Cauchy measure of a Voronoi diagram.** Using central angles at points of $\mathcal{PPL}(\omega, f)$, let us define the *Cauchy measure* $\mu_{\omega,f}$ of the Voronoi diagram $\mathcal{V}_{\omega,f}$. It is obtained by the pullback of a certain differential form by a local primitive of ω.

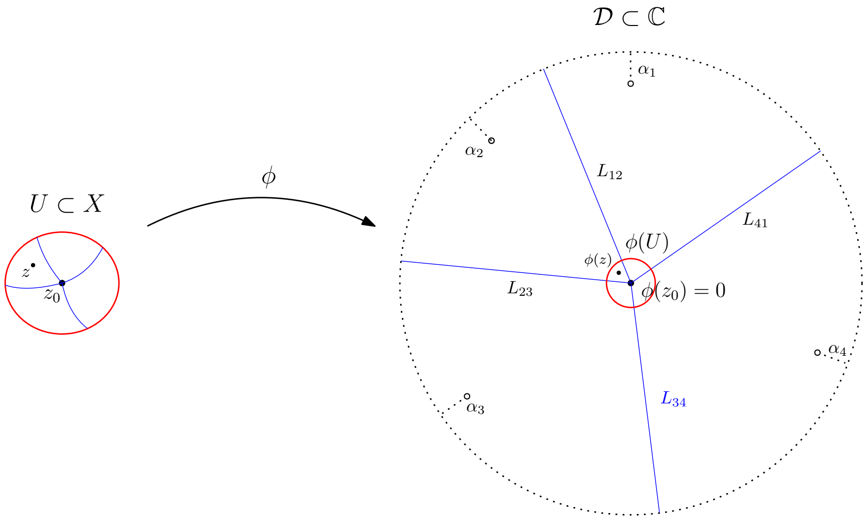

FIGURE 2. The Voronoi function g_{z_0} extends to a holomorphic function in the domain $\mathcal{D}$. The blue lines form the union $\mathcal{L}$ of the bisector rays contained in $\mathcal{D}$. The preimage $\phi^{-1}(\mathcal{L})$ is the union of straight segments with respect to the coordinate ϕ in U.

Definition 2.13. Let A, B two distinct points in the complex plane. We define the *unnormalized Voronoi angular measure* μ on the bisector L_{AB} as follows. For any segment $[M, N]$ contained in L_{AB}, let $\mu([M, N])$ be the angle at A of triangle AMN (see Figure 3). Obviously, we have $\mu(L_{AB}) = \pi$, and we define the (normalized) *Voronoi measure* as $\frac{1}{\pi}\mu$. This measure is then a probability measure and we note for later use that it can also be described as obtained by integration of the real differential form

$$d\theta_{AB} := \frac{|A - B|}{2\pi|P - A||P - B|} d\lambda(P)$$

on L_{AB}, where $d\lambda$ is Euclidean length measure on L_{AB}.

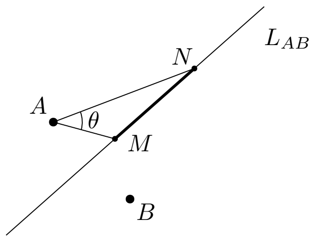

FIGURE 3. Illustration of the angular measure μ on the bisector L_{AB} of two points A and B in the plane. Here, $\mu([M, N]) = \theta$.

Definition 2.14. Consider a compact Riemann surface X endowed with a non-zero meromorphic 1-form ω and a meromorphic function f such that $\mathcal{PPL}(\omega, f) \neq \emptyset$.

Then the *Cauchy measure* of the Voronoi diagram $\mathcal{V}_{\omega,f}$ is obtained by the integration of the differential form obtained as the pullback of the differential form $d\theta_{AB}$ at each point z such that $\nu_z = 2$, (see Definition 2.13) by a local primitive ϕ of ω satisfying $\phi(z) = 0$.

Remark 2.15. Since in the principal polar locus there are finitely many points with a finite conical angle at each of them, the Cauchy measure $\mu_{\omega,f}$ is finite. Besides, it is invariant under rescaling $\omega \mapsto \lambda\omega$ (with $\lambda \in \mathbb{C}^*$).

3. Zero-free regions

In this subsection, we prove that zeros of iterates of the operator T_ω cannot accumulate inside Voronoi cells.

Proposition 3.1. *Let X be a compact connected Riemann surface X endowed with a non-zero meromorphic 1-form ω and a meromorphic function f such that $\mathcal{PPL}(\omega, f) \neq \emptyset$.*

Then, in the open subset $U \subset X^* \setminus \mathcal{PPL}(\omega, f)$ formed by points z satisfying $\nu_z = 1$ (see Definition 2.8),

$$\frac{1}{n}\log\left|\frac{T^n_\omega f(z)}{n!}\right| \to \log\frac{1}{\rho(z)}$$

on every compact subset as $n \to +\infty$, where $\rho(z)$ is the critical radius of the Voronoi function g_z, see Definition 2.5.

In particular, for any compact subset K of U, there is a bound $N > 0$ such that for any $n > N$, $T^n_\omega(f)$ has no zeros in K.

Proposition 3.1 will be deduced from a purely analytic lemma about the asymptotics of the coefficients of the Taylor series of a Puiseux series. The lemma applies a uniform version of a theorem proved by Orlov in [15], see our Corollary A.2.

Lemma 3.2. *Let g be a holomorphic function on the open centered disk $D(R) \subset \mathbb{C}$ of radius $R > 0$. Suppose that g can be extended holomorphically to $\partial D(R)$, except at only one point d on the boundary circle $\partial D(R)$ where it extends as a Puiseux series. Then, there exists $\delta > 0$ such that*

$$\frac{1}{n}\log\left|\frac{g^{(n)}(x)}{n!}\right| \to \log\frac{1}{|x-d|}$$

uniformly on the closed centered disk $\overline{D}(\delta)$ of radius δ.

Proof. Without loss of generality, we may assume that $D(R)$ is centered at 0. Suppose that the Puiseux series at d has a leading exponent m_0, meaning that $B_{m_0}(d-x)^{m_0}$ is the term with the smallest non-integer exponent, or smallest negative exponent.

By Corollary A.2, there exist $\epsilon, \delta > 0$ such that the asymptotic behavior

$$\frac{g^{(n)}(x)}{n!} = b(n,x)(d-x)^{-n} + O\left((|d-x|(1+\epsilon))^{-n}\right)$$

holds on $\overline{D}(\delta)$, where

$$b(n,x) = K(x-d)^{m_0} n^{-(m_0+1)}(M + O(n^{-\frac{1}{N}}|x-d|^{\frac{1}{N}})),$$

with all constants depending only on g and R. Then,

$$\left|\frac{g^{(n)}(x)}{n!}\right| = |d-x|^{-n}\left|b(n,x) + O((1+\epsilon)^{-n})\right|. \tag{3.1}$$

We claim that $\frac{1}{n}\log|b(n,x) + O((1+\epsilon)^{-n})|$ converges to 0 uniformly on $\overline{D}_\delta$ as $n \to +\infty$. For any given $\epsilon' > 0$, there is some $N' \in \mathbb{N}$ such that $(1+\epsilon)^{-n} < \frac{\epsilon'}{2}$ and

$b(n,x) < \frac{\epsilon'}{2}$ for all $n \geq N'$ and $x \in \overline{D}(\delta)$. Thus, $|b(n,x) + O((1+\epsilon)^{-n})|$ converges to 0 uniformly on $\overline{D}(\delta)$ as $n \to +\infty$. Taking logarithm and dividing by n, we obtain

$$\frac{1}{n}\log\left|\frac{g^{(n)}(x)}{n!}\right| = -\log|x-d| + \log|o(1)|.$$

Therefore,

$$\lim_{n\to+\infty}\frac{1}{n}\log\left|\frac{g^{(n)}(x)}{n!}\right| = -\log|x-d|$$

where the convergence is uniform in $\overline{D}(\delta)$ as desired. □

We now prove Proposition 3.1 by applying Lemma 3.2 to the Voronoi function of a point contained in a Voronoi cell.

Proof of Proposition 3.1. For any $z_0 \in U$, the Voronoi function g_{z_0} converges in the open disk of radius $\rho(z_0)$ centered at 0 and extends as a holomorphic function to every point of the boundary circle, except at one point d, to a neighborhood of which g_{z_0} extends as a Puiseux series (see Lemma 2.6 and Definition 2.8). We then deduce from Lemma 3.2 the existence of a neighborhood W of 0 where $\frac{1}{n}\log\left|\frac{g_{z_0}^{(n)}(x)}{n!}\right|$ converges to $-\log|x-d|$ uniformly. Therefore, there is a neighborhood of z_0 in X where

$$\frac{1}{n}\log\left|\frac{T_\omega^n(f)(z)}{n!}\right| = \frac{1}{n}\log\left|\frac{g_{z_0}^{(n)}(\phi(z))}{n!}\right|$$

converges uniformly to $-\log\rho(z)$ where $\rho(z)$ is the radius of convergence of g_z. The last claim follows immediately. □

4. Edges of the limit set

The remainder of the proof is an application of potential theory. In Proposition 4.1, we will show that

$$\frac{1}{n}\log\left|\frac{T_\omega^n f(z)}{n!}\right| \longrightarrow \log\frac{1}{\rho(z)}$$

as $n \to +\infty$ in the space $L^1_{loc}(X^*)$ of locally integrable functions on X^* (with respect to the measure $\omega \wedge \overline{\omega}$). Then, using the Laplacian operator Δ_ω induced by the flat metric $|\omega|$ on X^*, we prove that the Laplacian of $\frac{1}{n}\log\left|\frac{T_\omega^n f(z)}{n!}\right|$ coincides (up to normalization) with the root-counting measure of $T_\omega^n f(z)$ and converges as a distribution (positive measure) to the limit $\frac{\mu_{\omega,f}}{A\pi} + \frac{1}{A}\sum_{p\in\mathcal{P}}(d_p-1)\delta_p$ stated in the second part of Theorem 1.5. Finally, we will deduce the first part of the theorem from this convergence.

4.1. **Convergence in $L^1_{loc}(X^*)$.** Proposition 3.1 shows that $\frac{1}{n}\log\left|\frac{T_\omega^n f(z)}{n!}\right|$ converges uniformly to $\log\frac{1}{\rho(z)}$ on every compact subset inside Voronoi cells of X. Oscillations of $T_\omega^n f(z)$ near the edges of the Voronoi diagram lead us to consider the weaker L^1_{loc} convergence as follows.

Proposition 4.1. *Assume $\mathcal{PPL}(\omega,f) \neq \emptyset$. For any $z \in X^* \setminus \mathcal{PPL}(\omega,f)$, there exists a neighborhood U of z such that on any compact subset $K \subset U$, we have:*

$$\int_K \left|\frac{1}{n}\log\left|\frac{T_\omega^n f(z)}{n!}\right| - \log\frac{1}{\rho(z)}\right| \omega\wedge\overline{\omega} \longrightarrow 0$$

where $\rho(z)$ is the critical radius of the Voronoi function g_z.

Of course, we first need to check that $\log \frac{1}{\rho(z)}$ is a locally integrable function.

Lemma 4.2. *Assuming that $\mathcal{PPL}(\omega, f) \neq \emptyset$, the function $z \mapsto \log \frac{1}{\rho(z)}$ belongs to $L^1_{loc}(X^*)$.*

Proof. For any $z_0 \in X^* \setminus \mathcal{PPL}(\omega, f)$, we consider its Voronoi function g_{z_0} and a local primitive ϕ of ω such that $\phi(z_0) = 0$. Let $d_1, \dots, d_k$ be the points of the critical circle where g_{z_0} does not extend holomorphically, with k being the Voronoi index of z_0, see Lemma 2.9. Then, there is a neighborhood U of z_0 such that for any $z \in U$, $\rho(z) = \min\limits_{1 \leq i \leq k} |\phi(z) - d_i|$.

A simple integral computation shows that the function $w \mapsto \log|w - b|$ for some constant $b \in \mathbb{C}$ is locally integrable away from b. It follows that $x \mapsto -\min\limits_{1 \leq i \leq k} \log|x - d_i|$ is locally integrable in $\mathbb{C} \setminus \{d_1, \dots, d_k\}$. We deduce that $z \mapsto \log \frac{1}{\rho(z)}$ is integrable in a small enough neighborhood of z_0 in $X^* \setminus \mathcal{PPL}(\omega, f)$. The claim follows. □

Proposition 4.1 follows from the local lemma below, which we will prove in Section 4.2.

Lemma 4.3. *Let g be a holomorphic function on the open centered disk $D(\rho) \subset \mathbb{C}$ of radius $\rho > 0$. Suppose that g can be extended holomorphically to $\partial D(\rho)$, except at only finitely many points $d_1, \dots, d_k$ on the boundary circle $\partial D(\rho)$ where it extends as Puiseux series. Then, setting*

$$M(x) = -\min_{1 \leq i \leq k} \log|x - d_i|,$$

there exists $\delta > 0$ such that

$$\int_{\overline{D}(\delta)} \left| \frac{1}{n} \log \left| \frac{g^{(n)}(x)}{n!} \right| - M(x) \right| d\lambda \longrightarrow 0$$

where $\overline{D}(\delta)$ is the closed centered disk of radius δ and $d\lambda$ is the standard Lebesgue measure.

Proof of Proposition 4.1. For any $z_0 \in X^* \setminus \mathcal{PPL}(\omega, f)$, we apply Lemma 4.3 to the Voronoi function g_{z_0}. Notice that $\omega \wedge \overline{\omega} = \phi^* d\lambda$ and $T^k_\omega(f) = g^{(k)} \circ \phi$. □

4.2. **Minimum modulus principle.** This section is devoted to proving Lemma 4.3. We recall that the points $d_1, \dots, d_k$ are the points on the critical circle where g does not extend holomorphically. We assume that $k \geq 2$. Indeed, if $k = 1$, Lemma 4.3 immediately follows from Lemma 3.2.

Assuming that $d_1, \dots, d_k$ are cyclically ordered according to their argument, we introduce k points $c_1, \dots, c_k$ on the critical circle of radius ρ in such a way that c_i is the midpoint of the circular arc $(d_i d_{i+1})$. We define a subset $A \subset \mathbb{C}$ formed by the k segments $[0, c_1], \dots, [0, c_k]$. We denote by A_ϵ the open tubular ϵ-neighborhood of A in the centered disk $D(\rho)$, see Figure 4.2.

In order to prove Lemma 4.3, we need to prove in particular that the contribution of a neighborhood of A the integral of $\left| \frac{1}{n} \log \left| \frac{g^{(n)}(x)}{n!} \right| \right|$ can be made arbitrarily small. The remainder of the integral will then be computed using Lemma 3.2.

Lemma 4.4. *In the settings of Lemma 4.3, there exists a constant C depending only on g such that for any small enough $\epsilon > 0$ and $\delta > 0$, we have*

$$\frac{1}{n} \int_{A_\epsilon \cap D(\delta)} \left| \log \left| \frac{g^{(n)}(x)}{n!} \right| \right| d\lambda < C\epsilon$$

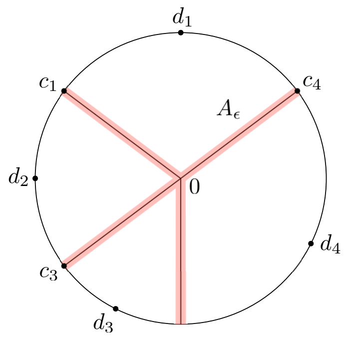

FIGURE 4. The open tubular neighborhood A_ϵ of A.

where $d\lambda$ is the standard Lebesgue measure on $\mathbb{C}$ and $\overline{A_\epsilon \cap D(\delta)}$ is the closure of the intersection of A_ϵ with the centered disk of radius δ.

The key ingredient is the classical Cartan-Boutroux lemma (the Minimum Modulus Principle) which provides an estimate of the growth of $\log |g(x)|$ near its singularities, in a form that is suitable for integrating. For the sake of completeness, we quote the following version from [13] (see also [27]). In the following, e is the usual Euler's number.

Lemma 4.5. *(*[13, Lecture 11, Theorem 4]*) Consider a holomorphic function g on the closed disk $\overline{D(0, 2er)}$ with $|g(0)| = 1$ and $r > 0$. Let η be an arbitrary number satisfying $0 < \eta < 3e/2$. Denote by M the maximal value of $\big|g(x)\big|$ on the circle $|x| = 2er$.*

Then, inside $\overline{D(0, 2er)}$, there is a family of excluded disks, the sum of whose radii is not greater than $4\eta r$, such that outside these disks we have

$$\log |g(x)| > -H(\eta) \log M, \tag{4.1}$$

for $H(\eta) = 2 + \log \frac{3e}{2\eta}$.

Remark 4.6. We want to apply this result to a small η. Note that the condition on η implies that:

(1) $H(\eta) \geq 2 > 0$;
(2) $H(\eta) \sim -\log \eta$ as $\eta \to 0$;
(3) the total area of the excluded discs in the lemma is less than $\pi(4\eta R)^2$ (since the maximum occurs in the case of one disk).

Remark 4.7. If $g(0) \neq 0$, we apply the lemma to $g(z)/g(0)$ and modify (4.1) to

$$\begin{aligned} &\log |g(x)| - \log |g(0)| > -H(\eta)(\log M - \log |g(0)|) \\ &\iff \log |g(x)| > -H(\eta) \log M + (1 + H(\eta)) \log |g(0)|. \end{aligned} \tag{4.2}$$

We will also make use of a specific covering of A_ϵ by a family of disks.

Lemma 4.8. *For every sufficiently small $\epsilon > 0$, there exist $r > 0$, $h > 0$, and s_ϵ points $x_1, \dots, x_{s_\epsilon} \in A_\epsilon \setminus A$ such that the following conditions are satisfied:*

(1) A_ϵ is covered by the union of disks $\bigcup_{i=1}^{s_\epsilon} D(x_i, r)$ of radius r centered at x_i;
(2) the area of the union $\bigcup_{i=1}^{s_\epsilon} D(x_i, r)$ is smaller than $C_A \cdot \epsilon$ where C_A is a constant depending only on A;
(3) the union $\bigcup_{i=1}^{s_\epsilon} D(x_i, 2er + h)$ has distance at least $d/2$ to $\{d_1, \dots, d_k\}$ where d is the distance between A and $\{d_1, \dots, d_k\}$.

Proof. Observe that it suffices that r and ϵ satisfy $2er + h < d/2 - \epsilon$ for (3) to be true. Given an $\epsilon < d/2$, this is clearly a feasible constraint on r and h. The rest is intuitively clear and can be made precise by Euclidean geometry. Domain A_ϵ decomposes into a central k-gon (contained in a disk of radius ϵ) and k strips (each contained in a rectangle of length ρ and width 2ϵ), see Figure 4.2. Each of these polygonal pieces can be covered by a family of equilateral triangles of size ϵ, with a number of equilateral triangles growing as $\frac{\rho}{\epsilon}$. We construct a family of disks, each circumscribing an equilateral triangle of the family. Then, the linear bound in ϵ on the area of the covering family of disks is easily realized. An arbitrarily small deformation of the covering ensures that the centers of the disks do not belong to A. Finally, we remark that since $r = r(\epsilon) \to 0$ as ϵ, we have $\lim_{\epsilon \to 0} d/2 - \epsilon - 2er = d/2$, and so we may choose the constant h so that it works for all (small enough) ϵ. $\square$

Now, we are ready to prove Lemma 4.4.

Proof of Lemma 4.4. The proof proceeds in four steps.

4.2.1. *First step.* Choose $0 < \epsilon < \delta < \rho$ small enough that g is holomorphic on $A_\epsilon \cap D(\delta)$. Let $r > 0$ be such that $\overline{A_\epsilon \cap D(\delta)}$ is covered by disks $D(x_i, r)$, $i = 1, \dots, s_\epsilon$ as in Lemma 4.8. Shrinking δ and ϵ if necessary, we require that each $D(x_i, 2er + h)$ is contained in $D(\rho)$, where h is the constant independent of ϵ, see the end of the proof of Lemma 4.8. Then, since x_i lies outside of A, Lemma 3.2 implies that

$$\lim_{n \to +\infty} \frac{1}{n} \log \left| \frac{g^{(n)}(x_i)}{n!} \right| = -\log |x_i - d_i| = M(x_i) \tag{4.3}$$

where d_i is the nearest point in $\{d_1, \dots, d_k\}$ to x_i. In particular, $\frac{g^{(n)}(x_i)}{n!} \neq 0$ for all sufficiently large n. Let $n_0 \in \mathbb{N}$ be such that

$$\left| M(x_i) - \frac{1}{n} \log \left| \frac{g^{(n)}(x_i)}{n!} \right| \right| \leq |M(x_i)| \text{ if } n > n_0, \tag{4.4}$$

for all $i = 1, ..., s_\epsilon$. Consequently,

$$\left| \log \left| \frac{g^{(n)}(x_i)}{n!} \right| \right| \leq C \cdot n, \text{ if } n > n_0, \tag{4.5}$$

where C is the maximal value of $2|M(x)|$ in a region containing $\overline{A_\epsilon \cap D(\delta)}$.

4.2.2. *Second step.* Condition (3) in Lemma 4.8 and the choice of a small $\delta > 0$ ensures that g is holomorphic on the union of the disks $\bigcup_{i=1}^{s_\epsilon} D(x_i, 2er + h)$. Let B be the maximal value of $|g(x)|$ in the union. Since circles of radius h around a point x on the circle $|x - x_i| = 2er$ are contained in $D(x_i, 2er + h)$, we can apply Cauchy's estimate of the derivatives of g to a point x with $|x - x_i| \leq 2er$ and obtain an upper bound

$$\left| \frac{g^{(n)}(x)}{n!} \right| \leq h^{-n} B. \tag{4.6}$$

This inequality is hence true for any $x \in \bigcup_{i=1}^{s_\epsilon} D(x_i, 2er)$.

4.2.3. *Third step.* We now apply Lemma 4.5 to $\frac{g^{(n)}(x)}{n!}$ on each $D_i = D(x_i, r)$. Let $M_n \leq h^{-n}B$ be the maximum value of $\left|\frac{g^{(n)}(x)}{n!}\right|$ on the circle $|x - x_i| = 2er$. Here, the inequality comes from (4.6).

For small $\eta > 0$, we introduce the subset $O_{\eta,i} \subset D_i$ such that for $x \in O_{\eta,i}$ we have

$$\log\left|\frac{g^{(n)}(x)}{n!}\right| > -H(\eta)\left(\log M_n - \log\left|\frac{g^{(n)}(x_i)}{n!}\right|\right) + \log\left|\frac{g^{(n)}(x_i)}{n!}\right|. \tag{4.7}$$

Following Lemma 4.5 and Remark 4.7, $O_{\eta,i}$ satisfies

$$\lambda(D_i \setminus O_{\eta,i}) \leq \pi(4\eta r)^2. \tag{4.8}$$

This will give us a rough upper estimate of $\left|\log|\frac{g^{(n)}(x)}{n!}|\right|$ as follows. First, by (4.6),

$$\log(h^{-n}B) \geq \log|\frac{g^{(n)}(x)}{n!}|. \tag{4.9}$$

Then, by the maximum modulus principle,

$$\log(h^{-n}B) - \log\left|\frac{g^{(n)}(x_i)}{n!}\right| \geq \log M_n - \log\left|\frac{g^{(n)}(x_i)}{n!}\right| \geq 0.$$

Using this and the fact that $-H(\eta) < 0$, we have that (4.7) implies

$$\log\left|\frac{g^{(n)}(x)}{n!}\right| \geq -H(\eta)\left(\log(h^{-n}B) - \log\left|\frac{g^{(n)}(x_i)}{n!}\right|\right) + \log\left|\frac{g^{(n)}(x_i)}{n!}\right| \tag{4.10}$$

For real numbers $c \geq b \geq a$, we have $|a| + |c| \geq |b|$. So, (4.9) and (4.10), together with (4.5), imply

$$(|H(\eta)| + 1)(|\log(h^{-n}B)| + Cn) \geq \left|\log|\frac{g^{(n)}(x)}{n!}|\right|. \tag{4.11}$$

Since $\bigcup_{i=1}^{s_\epsilon} D(x_i, r) = \bigcup_{i=1}^{s_\epsilon} D_i$ covers $A_\epsilon \cap D(\delta)$, we finally deduce from (4.8) that the subset $O_\eta = \bigcup_{i=1}^{s_\epsilon} O_{\eta,i} \subset \bigcup_{i=1}^{s_\epsilon} D_i$, where inequality (4.11) holds, satisfies

$$\lambda(\overline{A_\epsilon \cap D(\delta)} \setminus O_\eta) \leq s_\epsilon \pi(4\eta r)^2 = L\eta^2\epsilon, \tag{4.12}$$

for some constant L.

4.2.4. *Fourth step.* Fix some $\epsilon > 0$. The inequality (4.12) implies that the sets O_η are an exhausting sequence of open subsets of $\overline{A_\epsilon \cap D(\delta)}$ as $\eta \longrightarrow 0$.

Now, we can estimate

$$\frac{1}{n}\int_{\overline{A_\epsilon \cap D(\delta)}} \left|\log|\frac{g^{(n)}(x)}{n!}|\right| d\lambda.$$

Let $\eta_j = 1/j$ and $O_0 := \emptyset$. Subdivide the integral as follows:

$$\frac{1}{n}\int_{\overline{A_\epsilon \cap D(\delta)}} \left|\log|\frac{g^{(n)}(x)}{n!}|\right| d\lambda = \frac{1}{n}\sum_{j=0}^{+\infty}\int_{O_{\eta_{j+1}} \setminus O_{\eta_j}} \left|\log|\frac{g^{(n)}(x)}{n!}|\right| d\lambda. \tag{4.13}$$

By (4.11) and (4.12), the last sum is dominated by

$$\frac{1}{n}\sum_{j=1}^{+\infty}\frac{L\epsilon}{j^2}|(|H(1/j)| + 1)(|\log(h^{-n}B)| + Cn). \tag{4.14}$$

Now, observe that $|H(1/j)| + 1 \sim \log j$ by Remark 4.6. Second, $\frac{|\log(h^{-n}B)| + Cn}{n}$ is globally bounded (recall that h is a constant, independent from ϵ). Hence, (4.14) converges to $S_n\epsilon$, for some uniformly bounded S_n. This is the desired bound, and Lemma 4.4 is proved.

4.2.5. *Proof of Lemma 4.3.* By choosing a small enough ϵ and δ, Lemma 3.2 proves that the integral of $\left|\frac{1}{n}\log\left|\frac{g^{(n)}(x)}{n!}\right| - M(x)\right| d\lambda$ on $\overline{D}(\delta)\setminus A_\epsilon$ converges uniformly to zero. It remains to prove that the integral of the same measure over $\overline{A_\epsilon \cap D(\delta)}$ can be made arbitrarily small as $\epsilon \to 0$.

By choosing a small enough δ, $|M(x)|$ can be globally bounded on $\overline{D}(\delta)$ and the area of $A_\epsilon \cap D(\delta)$ can be bounded by a linear expression in ϵ. We obtain that

$$\int_{\overline{A_\epsilon \cap D(\delta)}} |M(x)|\, d\lambda < B \cdot \epsilon$$

for some constant $B > 0$.

Using the estimate of Lemma 4.4 (proved in Sections 4.2.1 to 4.2.4), we obtain that

$$\int_{\overline{A_\epsilon \cap D(\delta)}} \left|\frac{1}{n}\log\left|\frac{g^{(n)}(x)}{n!}\right|\right| d\lambda < C \cdot \epsilon$$

for some constant C that does not depend on n.

By the triangle inequality, we deduce that $\int_{\overline{A_\epsilon \cap D(\delta)}} \left|\frac{1}{n}\log\left|\frac{g^{(n)}(x)}{n!}\right| - M(x)\right| d\lambda$ can be made arbitrarily small as $\epsilon \to 0$. This proves the convergence to zero of the integral over $\overline{D}(\delta)$. $\square$

4.3. **A Laplace operator on $L^1_{loc}(X^*)$.** In the surface X^* punctured at the poles of ω, there is a singular flat metric $|\omega|$ with conical singularities at the zeros of ω. We denote by Δ_ω the corresponding Laplace operator defined on $L^1_{loc}(X^*)$ (see [10, 12] for a reference on Laplace operators of singular flat metrics). It is an operator on $L^2_{loc}(X^*)$, and acts in particular on test functions in a local parametrization as the ordinary Laplacian with respect to the parametrizing variable, even at the singular points(see [10, Prop.3.3 and (4.3)]). This allows us to first compute the Laplacian of $T^n_\omega f$, as a distribution.

Lemma 4.9. *For any $n \geq N$ for some $N \in \mathbb{N}$, we have:*

$$\frac{1}{2\pi}\Delta_\omega \log|T^n_\omega f(z)| = \sum_{z\in Z(T^n_\omega f)} \delta_z - n\left(\sum_{p\in\mathcal{PPL}(\omega,f)} \delta_p\right) - \nu$$

where $Z(T^n_\omega f)$ is the set of the zeroes of $T^n_\omega f$ in X^* (counted with multiplicities) while ν is a fixed distribution of finite weight.

Proof. Let z_0 be a point of the surface X^{**} punctured at the zeroes and poles of ω. For any small enough neighborhood U of z_0, a local primitive ϕ of ω such that $\phi(z_0)$ defines a local flat chart where Δ_ω is conjugated to the standard Laplace operator Δ. Since we have $T^k_\omega(f) = g^{(k)} \circ \phi$, on the open set U, we have

$$\Delta_\omega \log|T^n_\omega f(z)| = (\Delta \log\left|g^{(n)}\right|) \circ \phi(z). \tag{4.15}$$

It is well-known that the standard Laplacian of the logarithmic absolute value of a meromorphic function f, thought of as a L^1_{loc}-function and hence a distribution, is the divisor of the function expressed in terms of Dirac measures. We deduce that on any open set of X^{**}, $\frac{1}{2\pi}\Delta_\omega \log|T^n_\omega f(z)|$ coincides with the divisor of the zeroes and poles of $f^{(n)}$ (taken as a distribution). Following the results of Section 2.1, the poles of $T^n_\omega f$ that belong to X^{**} are located at the poles of f. They belong to

$\mathcal{PPL}(\omega, f)$ and the order of each of them increases by one after each iteration of T_ω. This proves the claim for any open subset of X^* that does not contain a zero of ω.

Now we consider an arbitrarily small open neighborhood U of a zero z_0 of ω. There is locally a primitive $\zeta = \phi$ of ω that maps the open set U to an open set V in the ζ parameterplane as a branched cover.

In the first case, $z_0 \notin \mathcal{PPL}(\omega, f)$ and $f = g \circ \phi$ is, as above, locally factorized by a primitive $\zeta = \phi$ of ω(by Definition 1.2). By the remark on the Laplacian before the proof (4.15) still holds, and the same argument as above gives that as a distribution on X^*(where we use the local coordinates of the complex structure on the Riemann surface)

$$\Delta_\omega \log |T^n_\omega f(z)| = \sum_{z \in Z(T^n_\omega f) \cap U} \delta_z.$$

In the second case, $z_0 \in \mathcal{PPL}(\omega, f)$ and provided n is large enough, $T^n_\omega f$ has a pole in z_0 (see Proposition 2.2). Moreover, if U is small enough, Proposition 3.1 shows that $T^n_\omega f$ has no zero in U. In terms of the local variable $\zeta = \phi(z)$, we have that $f(z) = g(\zeta^{1/k})$ may be developed in a Puiseaux series, and then it follows from Lemma A.3 that locally in U we have

$$\frac{1}{2\pi} \Delta_\omega \log |T^n_\omega f(z)| = -(n + \alpha(n))\delta_p,$$

where $\alpha(n)$ is bounded. Finally we let ν be the measure that is the sum of all constant parts $\alpha(n)\delta_p$, together with a similar sum coming from the poles of f at regular points . □

We proved in Proposition 4.1 that $\frac{1}{n} \log \left| \frac{T^n_\omega f(z)}{n!} \right|$ converges in $L^1_{loc}(X^* \setminus \mathcal{PPL}(\omega, f))$ to $\log \frac{1}{\rho(z)}$. We deduce that $\Delta_\omega \frac{1}{n} \log \left| \frac{T^n_\omega f(z)}{n!} \right|$ converges to $\Delta_\omega \log \frac{1}{\rho(z)}$ as a distribution. The Laplacian of $\log \frac{1}{\rho(z)}$ can be computed in terms of the Cauchy measure of the Voronoi diagram defined by f and ω.

Lemma 4.10. *In the space of distributions on X^*, we have:*

$$\frac{1}{2\pi} \Delta_\omega \log \frac{1}{\rho(z)} = \mu_{\omega,f} - \sum_{p \in \mathcal{PPL}(\omega, f)} \delta_p.$$

where $\rho(z)$ is the critical radius of the Voronoi function g_z (see Definition 2.5) and $\mu_{\omega,f}$ is the Cauchy measure of $\mathcal{V}_{\omega,f}$ (see Section 2.5).

Proof. Note first that for a chart $\zeta = \phi(z)$ and acting on test functions or real analytic functions such as as $\log(\frac{1}{\rho(z)})$,

$$\frac{1}{2\pi} \Delta_\omega = \frac{2}{\pi} \frac{\partial^2}{\bar{\partial}\zeta \partial\zeta}.$$

Thus it suffices to prove the lemma for a distinguished parameter ζ and a corresponding open set V. In a small enough neighbourhood of a $p \in \mathcal{PPL}(\omega, f)$, we have that $\rho(\zeta) = |\zeta - p|$, and $\frac{1}{2\pi} \Delta_\omega \log(\frac{1}{\rho(\zeta)}) = -\delta_p$. This gives the Dirac delta function part of the lemma. Besides these terms the support of the Laplacian will be contained in the Voronoi skeleton, as ρ otherwise is harmonic in an open Voronoi cell. Next, take a small neighbourhood N that intersects the Voronoi skeleton and which contains no $p \in \mathcal{PPL}(\omega, f)$. Let p_i $i = 1, \ldots$ be the elements of $\mathcal{PPL}(\omega, f)$ such that $\rho(\zeta) = |\zeta - p_i|$ is valid in a non-zero open subset O_i of N. Define $H_i(\zeta) := -\log |\zeta - p_i|$ which is a harmonic function in the complex plane except for $\zeta = p_i$. Let χ_i be the characteristic function of O_i. As L^1_{loc}-functions,

$M(z) := \log(\frac{1}{\rho(\zeta)}) = \sum H_i(\zeta)\chi_i$, and to calculate the derivatives, we need to recall how to calculate the derivatives of a characteristic function, the Sokhotski-Plemelj formulae. Let g be a test function (with compact support) which we may assume to vanish on the part of each ∂O_i which is not contained in the skeleton. Let finally ∂_v denote the directional derivative associated to a vector $v = (\alpha, \beta)$ in $\mathbb{R}^2 = \mathbb{C}$.

Green's theorem (in normal form) implies that

$$-\langle \partial_v \chi_i, g\rangle = \int\int_{O_i} \partial_v g dx dy = \int_{\partial O_i} (\alpha, \beta) \cdot N g ds, \tag{4.16}$$

where the boundary ∂O_i is oriented so that O_i is to the left, and N_i is the unit normal away from O_i. Note that, by the given orientation, $Nds = (dy, -dx)$. In terms of distributions, (and with a slight abuse of notation) this may be formulated as

$$\partial_v \chi_i = -(\alpha, \beta) \cdot Nds) = -\alpha dy + \beta dx, \tag{4.17}$$

and remains true if v is a complex vector. As a first corollary, Leibniz' rule implies that the distributional derivative

$$\frac{\partial M(\zeta)}{\partial \zeta} = \sum \frac{\partial H_i(\zeta)}{\partial \zeta}\chi_i, \tag{4.18}$$

since each part of the boundary V_{ij} between the cells O_i and O_j occurs twice with opposite normals in the second part of the sum

$$\frac{\partial M(\zeta)}{\partial \zeta} = \sum_i \frac{\partial H_i(\zeta)}{\partial \zeta}\chi_i - \sum_i H_i(\zeta)(\frac{1}{2})(1, -i) \cdot N_i ds,$$

and $M(\zeta)$ is continuous on V_{ij} (Clearly the endpoints of V_{ij} where the normal is undefined do not matter for the integral (4.16)). Thus the distributional derivative coincides with the a.e. pointwise derivative(and this is also a bona fide L^1_{loc}-function). Apply (4.17) and Leibniz rule again, to (4.18):

$$\frac{\partial^2 M(\zeta)}{\partial \bar{\zeta}\partial \zeta} = \sum \frac{\partial^2 H_i(\zeta)}{\partial \bar{\zeta}\partial \zeta}\chi_i - \sum \frac{1}{2}\frac{\partial H_i(\zeta)}{\partial \zeta}(1, i) \cdot Nds, \tag{4.19}$$

since $\frac{\partial}{\partial \zeta} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. The first sum is 0 by our assumptions on N. For the second sum, first note that $\frac{\partial H_i(\zeta)}{\partial \zeta} = 1/(2(\zeta - a_i))$, and second that since $Nds = (dy, -dx)$ the term $(1, i) \cdot N\,\mathrm{d}s = \mathrm{d}y - i\,\mathrm{d}x = -i\,\mathrm{d}\zeta$. Each piece of the segment V_{ij} that separates the Voronoi cell O_i(to the left) and O_j, will occur twice in the sum, and then contribute

$$(-i/4)(\frac{1}{z - a_i} - \frac{1}{z - a_j})\,\mathrm{d}z = (i/4)\frac{a_j - a_i}{(z - a_i)(z - a_j)}\,\mathrm{d}z. \tag{4.20}$$

V_{ij} is a segment of the orthogonal bisector of the line segment between a_i and a_j, which is given by the equation

$$\zeta = (1/2)(a_i + a_j) + ti(a_j - a_i) = (1/2 - ti)a_i + (1/2 + ti)a_j,\ t \in \mathbb{R}.$$

Then, if $\zeta \in V_{ij}$, $\zeta - a_i = (a_j - a_i)(1/2 + ti)$ and $\zeta - a_j = -(a_j - a_i)(1/2 - ti)$. Since (with the orientation that O_i is to the left of V_{ij}), $\mathrm{d}\zeta = i(a_j - a_i)\,\mathrm{d}t$, the measure in (4.20) is

$$\frac{\mathrm{d}t}{4(1/4 + t^2)} = \frac{|a_i - a_j|^2 dt}{4|\zeta - a_i||\zeta - a_j|},$$

using $|\zeta - a_i||\zeta - a_j| = |\zeta - a_i|^2 = |a_j - a_i|^2((1/4) + t^2)$. Length measure along V_{ij} is given by $\mathrm{d}s = |a_j - a_i|\,\mathrm{d}t$. To finish the proof of the lemma, it just remains to note that this, multiplied by $2/\pi$ is exactly the Cauchy measure (see Definitions 2.13 and 2.14.

□

4.4. **Proof of Theorem 1.5.** We are now ready to prove our main result.

Proof of Theorem 1.5. We proved in Proposition 4.1 that $\frac{1}{n}\log\left|\frac{T_\omega^n f(z)}{n!}\right|$ converges to $\log\frac{1}{\rho(z)}$ in the space of locally integrable functions on $X^* \setminus \mathcal{PPL}(\omega, f)$. Introducing the Laplace operator Δ_ω corresponding to the flat metric $|\omega|$ on X^* (see Section 4.3), we obtain that $\Delta_\omega \log|T_\omega^n f(z)|^{1/n}$ converges to $\Delta_\omega \log\frac{1}{\rho(z)}$ as distributions on X^*.

These distributions have been computed in Lemma 4.9 and Lemma 4.10. We deduce that $\frac{1}{n}\sum_{z\in Z(T_\omega^n f)} \delta_z$ converges to $\mu_{\omega,f}$ as positive distributions of finite weight on X^*. In other words, the zeros of $T_\omega^n f$ in the punctured surface X^* asymptotically distribute according to the Cauchy measure of the Voronoi diagram $\mathcal{V}_{\omega,f}$ (see Section 2.5). In particular, the Voronoi diagram is contained in the limit set $\mathcal{L}(T_\omega, f)$.

An immediate computation proves that at a pole p of order d of ω, for any large enough n, $T_\omega^n(f)$ has a zero at p of order $(d-1)n + b$ where b is some constant determined by the order of f at p. From this fact we conclude the following:

- if $d \geq 2$, then $p \in \mathcal{L}(T_\omega, f)$;
- if $d = 1$, then $p \in \mathcal{L}(T_\omega, f)$ if and only if p is not a pole of f.

Combining what precedes with the fact proved in Proposition 3.1 that any relatively compact open subset in a Voronoi cell is zero-free provided n is large enough, we obtain the complete description of $\mathcal{L}(T_\omega, f)$ as a subset of X, proving claim (i).

It remains to compute the asymptotic root-counting measure of the sequence $(T_\omega^n(f))_{n\in\mathbb{N}}$. It has been proved in Corollary 2.3 that the sum of orders $\mathcal{Z}_n$ of the poles of meromorphic function $T_\omega^n(f)$ (and therefore the sum of orders of its zeros) has the asymptotics $\mathcal{Z}_n \sim An$ as $n \to +\infty$ where

$$A = \sum_{z\in\mathcal{PPL}(\omega,f)} (a_z + 1).$$

The asymptotic root-counting measure of the sequence $(T_\omega^n(f))_{n\in\mathbb{N}}$ is then the sum of $\frac{1}{A}\mu_{\omega,f}$ (corresponding to the contribution of the punctured surface X^*) and a term $\frac{1}{A}(d_p - 1)\delta_p$ for each pole of ω. □

Remark 4.11. The computation of the asymptotic root-counting measure proves in particular that the Cauchy measure $\mu_{\omega.f}$ of the Voronoi diagram $\mathcal{V}_{\omega,f}$ is an integer multiple of π. It is a topological invariant of pair (ω, f) in the spirit of Gauss-Bonnet theorem.

5. Application to derivatives of algebraic functions

Our approach strengthens and generalizes the earlier results of Prather-Shaw in [18, 19] on the derivatives of (a restricted class of) algebraic functions.

Notation 5.1. Consider a plane algebraic curve $\Gamma \subset \mathbb{C}_z \times \mathbb{C}_w$ with coordinates (z, w) given as the zero locus of a bivariate polynomial $\Phi(z, w) = 0$. Take the projectivization $\widetilde{\Gamma} \subset \mathbb{C}P_z^1 \times \mathbb{C}P_w^1$ of Γ in $\mathbb{C}P_z^1 \times \mathbb{C}P_w^1$ with homogeneous coordinates $(Z_0 : Z_1), (W_0 : W_1)$. Denote the bidegree of $\widetilde{\Gamma}$ by (k, ℓ). To avoid trivialities we assume that both k and ℓ are positive. The case of rational functions considered by Pólya corresponds to the situation $\ell = 1,\ k \geq 2$.

Notation 5.2. In the above notation, consider the projection $\pi_z : \widetilde{\Gamma} \to \mathbb{C}P_z^1$. Under the assumption that Γ is reduced the preimage $\pi_z^{-1}(\hat{z}) \subset \widetilde{\Gamma}$ of a generic point $\hat{z} \in \mathbb{C}P^1$ consists of ℓ distinct points. By the *branching locus* $\mathcal{B}(\Gamma) \subset \mathbb{C}P_z^1$ we mean the set of points $\hat{z} \in \mathbb{C}P_z^1$ for which $\pi_z^{-1}(\hat{z}) \subset \mathbb{C}P_w^1$ is a positive divisor of degree ℓ with less than ℓ distinct points, i.e. some point has multiplicity exceeding 1. Points of multiplicity more than 1 in the preimage $\pi_z^{-1}(\hat{z})$ of a branching point $\hat{z}$ will be called the *critical points* of (the meromorphic function) $\pi_z : \widetilde{\Gamma} \to \mathbb{C}P^1_z$.

The *affine branching locus* $\mathcal{B}_a(\Gamma)$ is not just the restriction of $\mathcal{B}(\Gamma)$ to $\mathbb{C}_z$, but is its subset for which we additionally require that for $\hat{z} \in \mathcal{B}_a(\Gamma)$ at least one critical point of the divisor $\pi_z^{-1}(\hat{z})$ is affine, i.e. lies in $\mathbb{C}_w$ and not at $\infty_w \in \mathbb{C}P^1_w$.

We call a critical point $\hat{w} \in \pi_z^{-1}(\hat{z})|_{\mathbb{C}_w}$ *essential* if at least one of the local branches of Γ near $\hat{w}$ has a critical point at $\hat{w}$ as well, i.e. its projection on a small neighborhood of $\hat{z}$ is not a local biholomorphism. (In other words, this branch needs fractional powers for its presentation as a Puiseux series at $\hat{z}$ or, alternatively, $\hat{z}$ is a branch point of the lift $n\pi_z : N\widetilde{\Gamma} \to \mathbb{C}P^1_z$ where $N\widetilde{\Gamma}$ is the normalization of $\widetilde{\Gamma}$.) The branching point $\hat{z} \in \mathcal{B}_a(\Gamma)$ obtained as projection of an essential critical point $\hat{w}$ to $\mathbb{C}P^1_z$ is called *essential* as well. Let $\mathcal{B}_a^{ess}(\Gamma) \subset \mathcal{B}_a(\Gamma)$ denote the set of all essential affine branching points.

Finally, by the *locus of poles* $\mathcal{P}(\Gamma) \subset \mathbb{C}P^1_z$ we mean the set of points $\hat{z} \in \mathbb{C}P^1_z$ for which $\pi_z^{-1}(\hat{z}) \subset \mathbb{C}P^1_w$ contains $\infty_w \in \mathbb{C}P^1_w$ where $\mathbb{C}P^1_w = \mathbb{C}_w \cup \infty$. The *affine locus of poles* $\mathcal{P}_a(\Gamma) \subset \mathcal{P}(\Gamma)$ is the restriction of $\mathcal{P}(\Gamma)$ to $\mathbb{C}_z$.

Definition 5.3. Given Γ as above and a positive integer n, define the algebraic curve $\Gamma^{(n)} \subset \mathbb{C}_z \times \mathbb{C}_w$, called the *affine n-th derivative curve*, obtained by taking the n-th derivatives of all branches of Γ considered as (local) algebraic function $w(z)$ with respect to the variable z. The *n-th derivative curve* of Γ is the projectivization $\widetilde{\Gamma}^{(n)} \subset \mathbb{C}P^1_z \times \mathbb{C}P^1_w$ of $\Gamma^{(n)}$.

Definition 5.4. We define the *(projective) zero locus* $\widetilde{\mathcal{Z}}^{(n)}(\Gamma)$ *of the n-th derivative* of the algebraic function given by the algebraic curve Γ as the intersection locus of $\widetilde{\Gamma}^{(n)}$ with $(\mathbb{C}P^1_z, 0) \subset \mathbb{C}P^1_z \times \mathbb{C}P^1_w$. The *affine zero locus* $\mathcal{Z}_a^{(n)}(\Gamma)$ is the restriction of $\widetilde{\mathcal{Z}}^{(n)}(\Gamma)$ to $\mathbb{C}_z \subset \mathbb{C}P^1_z$. The *(projective) limit set* $\mathcal{F}(\Gamma) \subset \mathbb{C}P^1_z$ is the limiting set of the supports of the sequence $\{\widetilde{\mathcal{Z}}^{(n)}(\Gamma)\}$ and the *affine limit set* $\mathcal{F}_a(\Gamma) \subset \mathbb{C}_z$ is the limit set of the supports of the sequence $\{\widetilde{\mathcal{Z}}_a^{(n)}(\Gamma)\} \subset \mathbb{C}_z$.

Remark 5.5. In Pólya's original definition, see [16], the limit set for the usual derivative of a rational function consists of all points every neighborhood of which contains points from infinitely many $\widetilde{\mathcal{Z}}_a^{(n)}(\Gamma)$.

Problem 5.6. Given an affine (resp. projective) reduced algebraic curve Γ as above, describe its affine (resp. projective) limit set.

Remark 5.7. Obviously, if the bidegree of Γ is $(k, 1)$ we recover the classical question of Pólya about the accumulation of zeros of consecutive derivatives of a rational function. Moreover for algebraic functions given by a certain class of algebraic curves, their limit sets have been described in [19]. However, to the best of our knowledge, the description of the limit set for general reduced algebraic curves is new.

To derive results about the limit sets of general algebraic curves, i.e. about the accumulation of the zeros of consecutive derivatives of algebraic functions from the material of the previous sections we proceed as follows. Our main objects are as follows.

Definition 5.8. Given a reduced affine curve $\Gamma \subset \mathbb{C}_z \times \mathbb{C}_w$, take the normalization $N\widetilde{\Gamma}$ of the projectivization $\widetilde{\Gamma} \subset \mathbb{C}P^1_z \times \mathbb{C}P^1_w$ and define the meromorphic 1-form $\tilde{\omega}$ on $\widetilde{\Gamma}$ as the pullback of the standard meromorphic 1-form dz on $\mathbb{C}P^1_z$ under the projection $\pi_z : \widetilde{\Gamma} \to \mathbb{C}P^1_z$. Then take the pullback ω of $\tilde{\omega}$ from $\widetilde{\Gamma}$ to $N\widetilde{\Gamma}$. The meromorphic function $f : N\widetilde{\Gamma} \to \mathbb{C}P^1_w$ is obtained as composition of the normalization map $n : N\widetilde{\Gamma} \to \widetilde{\Gamma}$ with the projection $\pi_w : \widetilde{\Gamma} \to \mathbb{C}P^1_w$.

Lemma 5.9. *In the above notation, the principal polar locus (see Definition 1.3) $\mathcal{PPL}(\omega, f)$ of the pair (ω, f) on Riemann surface $N\widetilde{\Gamma}$ is given by:*

- *preimage under the normalization map $n : N\widetilde{\Gamma} \to \widetilde{\Gamma}$ of the intersection $\widetilde{\Gamma} \cap (\mathbb{C}P^1_z, \infty_w)$ except possibly for the point (∞_z, ∞_w), see Remark 5.10;*
- *preimage under the normalization map $n : N\widetilde{\Gamma} \to \widetilde{\Gamma}$ of the set of all affine essential critical points of $\widetilde{\Gamma}$.*

The principal polar locus is non-empty provided $\ell \geq 2$.

Remark 5.10. There is a simple criterion when the projectivization in $\mathbb{C}P^1_z \times \mathbb{C}P^1_w$ of an algebraic curve $\Gamma \subset \mathbb{C}_z \times \mathbb{C}_w$ given by a bivariate polynomial $\Phi(z, w) = 0$ passes through the point (∞_z, ∞_w). Obviously, this happens if and only if there is a branch of the algebraic function $w(z)$ which tends to ∞_w when $z \to \infty_z$. The latter condition can be reformulated in terms of the Newton polygon $\mathcal{N}_\Phi$ of $\Phi(z, w)$. Namely, this happens if and only if $\mathcal{N}_\Phi$ contains an edge with a negative slope and such that $\mathcal{N}_\Phi$ lies below the line spanned by this edge. This statement can be easily deduced from the known results of Section 38, Th. 63–66 in [4] and Ch. 4, Sections 3 and 4 in [24].

Proof of Lemma 5.9. Firstly we describe the poles of f and the zeros and poles of ω. Poles of f coincide with $f^{-1}(\infty_w)$ where $\mathbb{C}P^1_w = \mathbb{C}_w \cup \infty_w$. In other words, it is the set of points on $N\widetilde{\Gamma}$ where $\widetilde{\Gamma}$ intersects the projective line $\mathbb{C}P^1_z \times \infty_w \subset \mathbb{C}P^1_z \times \mathbb{C}P^1_w$.

Next, the singular points on $\widetilde{\Gamma}$ which project to the affine part $\mathbb{C}_z \subset \mathbb{C}P^1_z$ give rise to zeros of ω on $N\widetilde{\Gamma}$. We will show a bit later that the singularities of $\widetilde{\Gamma}$ projecting to $\infty_z \in \mathbb{C}P^1_z$ result in poles of ω.

Indeed each local singular branch of $\widetilde{\Gamma}$ near a point $z_0 \in \mathbb{C}_z$ can be represented by a Puiseux series

$$w(z) = \sum_{j \geq \ell} a_j (z - z_0)^{j/k}$$

where $k \geq 2$. This branch can be parameterized by $z(t) = z_0 + t^k$ and $w(t) = \sum_{j=\ell}^{\infty} a_j t^j$. Here t can be considered as a local chart on $N\widetilde{\Gamma}$ centered at the point mapped to the singularity of the singular branch of $\widetilde{\Gamma}$ under consideration. Since $z = z_0 + t^k$ one has that $dz = t^{k-1}dt$. Therefore ω being the pullback of dz acquires a zero of order $k - 1 > 1$ at this point.

Furthermore let $y = \frac{1}{z}$ be the local coordinate near $\infty_z \in \mathbb{C}P^1_z$. With respect to y, one has $dz = -\frac{1}{y^2}dy$. For any branch of $\widetilde{\Gamma}$ which is non-singular in a small neighborhood of ∞_z, i.e. $y = 0$ the pullback of dz to this branch will acquire a pole of order 2.

Finally, let us assume that $\widetilde{\Gamma}$ has a singular local branch near infinity whose Puiseux series is given by

$$w(y) = \sum_{j \geq \ell} a_j y^{j/k}.$$

We can parameterize this branch as $y = t^k$ and $w(t) = \sum_{j \geq \ell} a_j t^j$. The pullback of $dz = -\frac{1}{y^2}dy$ to the local coordinate t on $N\widetilde{\Gamma}$ equals $-\frac{k}{t^{k+1}}dt$, i.e. it has a pole of order $k + 1 > 2$ at the respective point.

Now by definition, $\mathcal{PPL}(\omega, f)$ consists of (a) poles of f that are not poles of ω and (b) zeros of ω at which f is not locally factorized by a primitive of ω. Poles of f which are also poles of ω are preimages of the point $(\infty_z, \infty_w) \in \mathbb{C}P^1_z \times \mathbb{C}P^1_w$.

If Γ has branches which tend to ∞_w when $z \to \infty_z$ then $\widetilde{\Gamma}$ contains (∞_z, ∞_w) and such points exist. Whether Γ has such branches is described in the above Remark.

Finally let us show that at each zero of ω the function f can not be locally factorized by a primitive of ω. As we have already shown zeros of ω come from the singularities of $\widetilde{\Gamma}$ which project to $\mathbb{C}_z$. Since ω is the pullback of dz and f is the pullback of the coordinate w the factorizability of f by a primitive of ω in simple words means that for the original branch of Γ, the algebraic function $w(z)$ is (locally) holomorphic in z which contradicts the assumption that we consider a singular branch.

In order to prove that the principal polar locus is non-empty, we just observe that for $\ell \geq 2$ the meromorphic function $f : N\widetilde{\Gamma} \to \mathbb{C}P^1_w$ is a ramified cover branched over at least two points of $\mathbb{C}P^1_w$. Since at least one of these two points is not ∞_w and $\ell \geq 2$, ω has at least one zero. □

Applying Theorem 1.5 we obtain that the zeros of $T^n_\omega(f)$ accumulate on the Voronoi diagram of the Riemann surface $N\widetilde{\Gamma}$ with the measure described in Section 2.5. The limit set of the algebraic function given by the curve Γ and the respective measure of this limit set on $\mathbb{C}P^1_z$ are obtained as the push-forward of the respective objects from $\mathcal{N}\widetilde{\Gamma}$ to $\mathbb{C}P^1_z$.

More explicitly, we get the following.

Corollary 5.11. *For any reduced algebraic curve $\Gamma \subset \mathbb{C}_z \times \mathbb{C}_w$, the affine limit set $\mathcal{F}_a(\Gamma) \subset \mathbb{C}_z$ coincides with the push-forward of the Voronoi diagram of the pair (ω, f) on $N\widetilde{\Gamma}$ under the projection π_z to $\mathbb{C}P^1_z$.*

Let us present some explicit examples of $\Gamma = \Gamma_0$, Γ_n for $n \geq 1$ and illustration of the zeros of consecutive derivatives of the occurring algebraic functions.

Example 1. Let $\Gamma_0 := \Gamma$ be the curve defined by $w^\ell = \frac{P(z)}{Q(z)}$ for some polynomials $P(z)$ and $Q(z)$.

Lemma 5.12. *The curve $\Gamma_n := \Gamma^{(n)}$ is given by the equation*

$$w^\ell = \frac{V_n^\ell(z)}{\ell^{n\ell} P(z)^{n\ell-1} Q(z)^{n\ell+1}} \tag{5.1}$$

where $V_n(z)$ is a polynomial determined by the recurrence relation

- $V_0(z) = 1$,
- $V_{n+1}(z) = \ell P(z)Q(z)V_n'(z) - ((n\ell-1)P'(z)Q(z) + (n\ell+1)P(z)Q'(z))V_n(z)$.

In particular, if P and Q has degrees d_1 and d_2, respectively, then

(1) $V_n(z)$ has degree $n(d_1 + d_2 - 1)$ and

(2) Γ_n has bidegree $(\max\{n\ell(d_1 + d_2) - n\ell, n\ell(d_1 + d_2) + (d_2 - d_1)\}, \ell)$.

Proof. The base case $n = 0$ is clear. Assuming Γ_n has the form in (5.1) and writing the denominator in (5.1) as W_n, we have

$$\ell (w^{(n)})^{\ell-1} w^{(n+1)} = \frac{V_n^{\ell-1}(\ell W_n V_n' - V_n W_n')}{W_n^2}$$

and

$$\begin{aligned}(w^{(n+1)})^\ell &= \left(\frac{W_n^{(\ell-1)/\ell}}{\ell V_n^{\ell-1}} \cdot \frac{V_n^{\ell-1}(\ell W_n V_n' - V_n W_n')}{W_n^2} \right)^\ell \\ &= \frac{(\ell W_n V_n' - V_n W_n')^\ell}{\ell^\ell W_n^{\ell+1}}.\end{aligned}$$

Substituting back W_n and simplifying yields

$$(w^{(n+1)})^\ell = \frac{(\ell^{n\ell}P^{n\ell-2}Q^{n\ell})^\ell(\ell PQV_n' - V_n((n\ell+1)PQ' + (n\ell-1)P'Q)^\ell}{\ell^\ell(\ell^{n\ell}P^{n\ell-1}Q^{n\ell+1})^{\ell+1}}$$
$$= \frac{(\ell PQV_n' - V_n((n\ell+1)PQ' + (n\ell-1)P'Q)^\ell}{\ell^{\ell(n+1)}P^{\ell(n+1)-1}Q^{\ell(n+1)+1}}.$$

The claims about the degrees follow directly. □

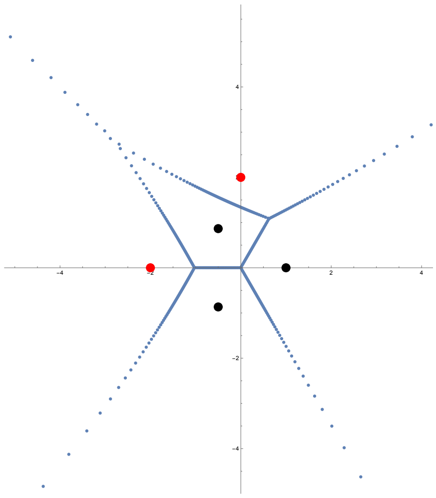

FIGURE 5. Root distribution of $V_{100}(z)$ for $P = (z+2)(z-2i)$, $Q = z^3 - 1$ and $\ell = 3$.

Remark 5.13. Fig. 5 illustrates the above Example 1. In this case the curve Γ is given by $(z^3-1)w^3 = (z+2)(z-2i)$. It is smooth and its projection onto $\mathbb{C}P^1_z$ has 6 branching points at the cubic roots of 1 (shown by black dots in Fig. 5), -2 and $2i$ (shown by red dots in Fig. 5), and ∞. All of them have multiplicity 2. The cubic roots of 1 are also poles of the algebraic function. From Riemann-Hurwitz formula follows that genus of $N\widetilde{\Gamma}$ is 4. The Voronoi diagram involves only five of the branching points since ∞ is a pole of ω.

Example 2. As a concrete special case of the above construction take $\Gamma := \Gamma_0$ as the unit circle given by $z^2 + w^2 = 1 \leftrightarrow w^2 = 1 - z^2$.

Lemma 5.12 implies the following.

Corollary 5.14. *The curve* $\Gamma^{(n)} := \Gamma_n$ *is given by the equation*

$$w^2 = -\frac{U_n^2(z)}{(z^2-1)^{2n-1}},$$

where the polynomial sequence $\{U_n(z)\}$ *is given by the recurrence relation:*

$$U_1(z) = z;\ U_n(z) = (2n-3)zU_{n-1}(z) + (1-z^2)U_{n-1}'(z) \text{ for } n \geq 2.$$

Remark 5.15. In the latter case the branching points are located at ± 1 and all roots of U_n are purely imaginary.

6. Further examples

In the previous sections we discussed the asymptotic root-counting measures of $T^n_\omega(f)$ which we applied to the study of consecutive derivatives of algebraic functions. Below we provide other types of examples.

6.1. Monomial linear operator applied to a function with one simple pole.

Assume that we have a first order linear differential operator of the form $T = R(z)\frac{d}{dz}$ where $R(z)$ is some rational function which we want to apply iteratively to another rational function $f(z)$ and study the asymptotic of the zeros of the iterates. If we can find a change of variable calling the new variable ϕ such that $\frac{d}{d\phi} = R(z)\frac{d}{dz}$ then we can in a sense reduce our problem to that about the usual derivative with respect to the variable ϕ. In fact this procedure corresponds to taking the appropriate Riemann surface defined by ϕ, its branched covering of $\mathbb{C}P^1$ and using Theorem 1.5, see the next example.

For any $\ell \in \mathbb{Z}$, set $T_\ell = z^\ell \frac{d}{dz}$ and $\omega_\ell = z^{-\ell}dz$. Let us calculate the root asymptotic of the sequence $\{T_\ell^n\left(-\frac{1}{z+a}\right)\}$ with $a \neq 0$ when $n \to \infty$.

Theorem 1.5 in the case under consideration claims the following. For any $\ell \geq 2$, $\omega = \frac{dz}{z^\ell} = d\phi$ where $\phi = \frac{z^{1-\ell}}{1-\ell}$. The principal polar locus $\mathcal{PPL}(\omega, \frac{1}{z+a})$ is $\{-a, \infty\}$. The image of the Voronoi diagram under ϕ is the bisector of $[\phi(\infty), \phi(-a)]$. It is a straight line L parametrized by $t \mapsto (\frac{1}{2} + it)\phi(-a)$ (where $t \in \mathbb{R}$).

The inverse image of line L under ramified cover ϕ is a planar algebraic curve of equation $\mathrm{Re}(\frac{z}{a})^{\ell-1} = \frac{(-1)^{\ell-1}}{2}$. It is a curve formed by $\ell - 1$ loops attached to 0 each contained in a cone of angle $\frac{2\pi}{\ell-1}$. The limit set $\mathcal{L}(T, \frac{1}{z+a})$ is the branch of the curve contained in the cone containing $-a$. An illustration of the above root distribution can be found in Fig. 6. (The explicit formula for the lemniscate is obtained from the above equation by using $\ell = 4$ and $a = 1$).

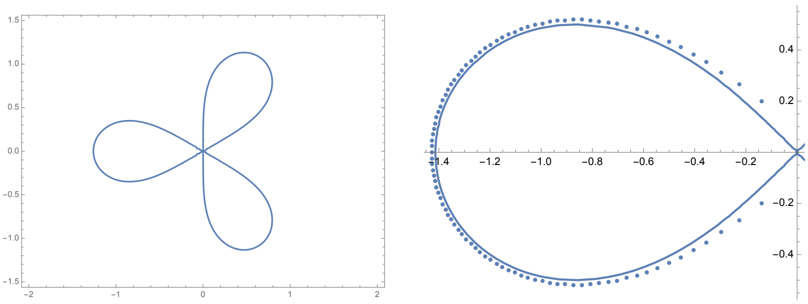

FIGURE 6. The lemniscate $x^6+3x^4y^2+3x^2y^4+y^6+2x^3-6xy^2 = 0$ (left) and its oval together with the roots of $T_4^{65}\left(-\frac{1}{z+1}\right)$ (right).

From the latter example it is not difficult to derive the asymptotic root distribution of $T_\ell^n\left(\sum_j \frac{\alpha_j}{z-z_j}\right)$ when $n \to \infty$.

6.2. Examples in genus one.

In the complex plane $\mathbb{C}$, we consider the lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, where $\tau = \exp(\pi i/3)$. Let $X = \mathbb{C}/\Lambda$ be the associated elliptic curve.

(1) Let $\omega = dz$ be the holomorphic 1-form and let $\wp(z)$ be the Weierstraß elliptic function on X, where z is the coordinate on $\mathbb{C}$. Since f has one pole of order 2 at 0 and ω has neither zeros nor poles, the principal polar locus $\mathcal{PPL}(T_\omega, \wp)$ in X is $\{0\}$. Using the primitive $\phi(z) = z$ of ω, it is easy to see that the Voronoi diagram in X is the quotient by Λ of the usual Voronoi diagram on $\mathbb{C}$ determined by the points in Λ, see Figure 7.

(2) For a more complicated example, we consider the derivative of the Weierstraß elliptic function on X, which we denote by $\wp_z$. Let $\omega = \wp_z(z)dz$ be the meromorphic 1-form and $f(z) = \wp_z(z)$ be the meromorphic function on X. It is a standard fact that $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ are the three simple zeros of ω in

X. These points become poles of $T_\omega^n(f)$ for all $n \geq 1$, so f is not locally factorized by a primitive of ω there by Lemma 2.1. Hence, they belong to $\mathcal{PPL}(T_\omega, f)$. Moreover, since the poles of ω and f are the same, we have that $\mathcal{PPL}(T_\omega, f)$ is $\{\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$.

The limit set $\mathcal{L}(T_\omega, f)$ on X is the preimage under the ramified cover $\wp$ of the Voronoi diagram in the $\mathbb{C}$-plane determined by $\wp(\mathcal{PPL}(T_\omega, f))$, where the ramification points are precisely the points in $\mathcal{PPL}(T_\omega, f)$, see Figure 8.

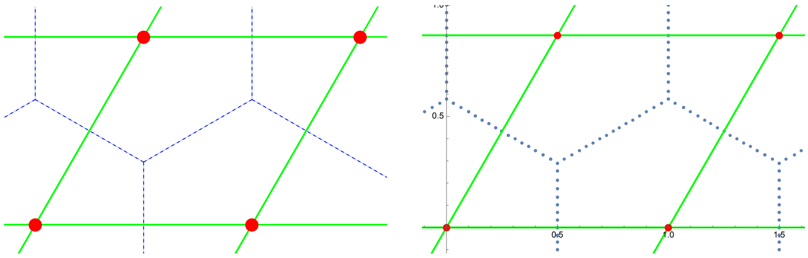

FIGURE 7. Let $\Lambda = \mathbb{Z} + e^{\frac{i\pi}{3}}\mathbb{Z}$. Left: The Voronoi diagram determined by the poles of $\wp(z)$ (red dots) consists of the blue dashed lines. Right: The zeros of $\wp^{(45)}$ are in blue dots. In this case, parallelograms cut out by the green lines are fundamental domains of the action of Λ on $\mathbb{C}$.

7. Outlook

1. (Open Riemann surfaces) The original shire theorem deals with $\mathbb{C}$ which is open and meromorphic functions on it. In particular, the main result of [8] shows that for an entire function of the form $R(z)e^{U(z)}$ with polynomial U a certain part of the total mass of the limiting root-counting measure will be placed at ∞. In the present paper we only consider compact Riemann surface X, but appropriate modifications of our results work in broader settings.

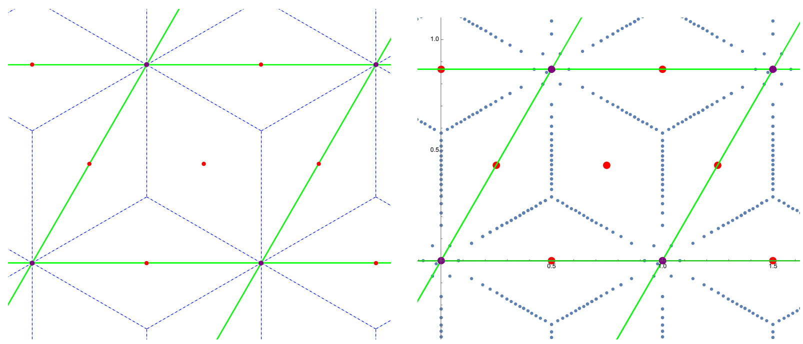

FIGURE 8. Let $\Lambda = \mathbb{Z} + e^{\frac{i\pi}{3}}\mathbb{Z}$. Left: The red dots are the points of $\mathcal{PPL}(\omega, f)$ where $\omega = \wp_z(z)dz$ and $f(z) = \wp_z(z)$, which determine the Voronoi diagram in blue dashes. The green lines cut out fundamental domains of the action of Λ on $\mathbb{C}$. Right: The dark blue dots are the zeros of $T_\omega^{25}(f)$.

The crucial hypothesis underlying the constructions of Section 2 is that our translation surfaces are metrically complete. It follows that the first part of Theorem 1.5 still holds for the class of meromorphic functions defined on the surface X^* punctured at the poles of ω (and having possibly essential singularities there). The latter class includes the functions studied by Ch. Hägg in [8].

2. (Global geometry of translation surfaces) An essential feature of the theory of translation surfaces is that the same objects have a complex-analytic side (a Riemann surface endowed with a holomorphic differential) and a geometric side (a polygon with pairs of sides identified by translations). Although these two descriptions are theoretically equivalent, going from one side to another is a delicate question in practice.

In a translation surface obtained by the gluing of a family of triangles, the lengths and the slopes of the edges are respectively the module and the argument of periods of the differential corresponding to these relative homology classes. However, starting with a complex structure (defined by a Fuchsian group for example) and an explicit holomorphic differential (in terms of modular forms), it is a difficult problem to determine which relative homology classes are represented by simple geodesic segments (in order to construct a triangulation).

In the current state of the art, the standard approach to obtain a geometric presentation of a translation surface is to discretize the circle of directions and integrate the differential equation corresponding to the differential form to find saddle connections.

If we can call the latter approach "classical", Theorem 1.5 suggests a "quantum" way from the complex-analytic data to the flat picture. For a given translation surface (X,ω), we consider a meromorphic function f with poles located at the zeros of ω. As $n \to +\infty$, zeros of $T_\omega^n(f)$ accumulate on the Voronoi diagram defined with respect to the zeros of ω. After an adequate number of iterations, the relative homology classes of the edges of the Delaunay polygonation (dual to the edges of the Voronoi tessellation) are characterized with an arbitrarily low error rate. In Figure 8, the Voronoi diagram of a nontrivial flat metric in genus one is obtained numerically using this very method.

3. (Fuchsian meromorphic connections) Generalization to an even broader settings can be made as follows. Let X be a compact Riemann surface with a Fuchsian meromorphic connection ∇ on a line bundle $\mathcal{L}$. We can investigate the limit set of a global meromorphic section of $\mathcal{L}$ under iteration of ∇.

Fuchsian meromorphic connections induce complex affine structures (see [14]) providing local coordinates where ∇ is conjugated with $\frac{d}{dz}$. We still have a meaningful notion of affine disk immersion so Voronoi diagrams can be defined. Besides, the definition of Cauchy measures in terms of angles is suitable for a generalization to complex affine structures (see Section 2.5). Nevertheless, an important difference with the current settings is that in most cases, meromorphic connections may fail to be geodesically complete, as in the case of the Hopf torus $\mathbb{C}^*/\langle z \mapsto 2z \rangle$.

A class of Fuchsian meromorphic connection can already be handled with the methods of the current paper. A k-differential ω is a global meromorphic section of the k^{th} tensor power $K_X^{\otimes k}$ of the canonical bundle. In local coordinates, it is a complex analytic object of the form $h(z)dz^k$ where h is a meromorphic function. A k^{th} root $\omega^{1/k}$ of ω can be thought as a global meromorphic section of a line bundle twisted by some character χ valued in the complex multiplicative group $(\mathbb{C}^*, \times)$. Operator $T : f \mapsto \frac{df}{\omega^{1/k}}$ acting on the space of global sections of a suitable line bundle coincides with a Fuchsian meromorphic connection.

For a compact Riemann surface X endowed with a k-differential ω, the canonical k-cover (see [1] for details) is the smallest ramified cover $\pi : (\tilde{X}, \tilde{\omega}) \to (X, \omega)$ such that ω is the k^{th} power of a globally defined meromorphic 1-form. This way, the limit set associated with operator T and some meromorphic section f of a line bundle $\mathcal{L}$ is the projection of the limit set associated with an operator $T_{\tilde{\omega}^{1/k}}$ and a section of $\pi^*\mathcal{L}$ defined on a surface of higher genus $\tilde{X}$. The latter is obtained using Theorem 1.5.

Appendix A.

A.1. **Extending a result of Orlov.** In [15, Theorem 1], Orlov proved pointwise asymptotics for the coefficients of a Taylor series representation of a branch of an algebraic function, which extends as Puiseux series at singularities. In the theorem and its corollary below, we will modify his proof to obtain uniform convergence for an arbitrary holomorphic function with one Puiseux-type singularity. This is crucial in Section 3, specifically in the proof of Lemma 3.2. To make the paper self-contained, we include a slight modification of Orlov's original proof.

We denote the open disk of radius $r > 0$ centered at $x \in \mathbb{C}$ by $D(x, r)$.

Theorem A.1. *Let f be a holomorphic function on an open disk $D(0, \rho)$ with ν singularities $a_1, \dots, a_\nu$ such that $|a_s| = \rho$ for $s = 1, \dots, \nu$. Suppose that f extends to a Puiseux series at each singular point and holomorphically at every other point of the circle of radius ρ. Then,*

$$\frac{f^{(k)}(0)}{k!} \sim \sum_{s=1}^{\nu} \tilde{b}_s(k) a_s^{-k} + O((\rho + \epsilon)^{-k}) \tag{A.1}$$

where $\epsilon > 0$ and, for each $s = 1, \dots, \nu$, the term $\tilde{b}_s(k)$ is of the form

$$\tilde{b}_s(k) = \frac{B_s}{\Gamma(-m_s)} a_s^{m_s} k^{-(m_s+1)} + o(k^{-(m_s+1)})$$

where B_s and m_s come from the term $B_s(a_s - z)^{m_s}$ with m_s being the smaller between the least negative exponent or the least non-integer positive exponent in the Puiseux representation of f at a_s.

Proof. Using Cauchy's integral formula, we have:

$$g(k) := \frac{f^{(k)}(0)}{k!} = \frac{1}{2i\pi} \int_C \frac{f(u)}{u^{k+1}} du \tag{A.2}$$

where C is the positively oriented circle of radius $0 < r < \rho$ centered at 0.

We will change the contour C of this integral to γ defined as in Figure 9. That is, the contour γ is the boundary ∂D of the domain $D = D(0, \rho+\epsilon) \setminus \bigcup_{s=1}^{\nu} D(a_s, \epsilon)$ for a choice of $\epsilon > 0$ such that the only singularities of f in $D(0, \rho + \epsilon)$ are $a_1, \dots, a_\nu$.

Without loss of generality, we assume that the a_s's are labeled counterclockwise. Then, the new contour γ decomposes into $\gamma_0 \cup \bigcup_{s=1}^{\nu} \gamma_s$ as follows

- γ_s is the boundary of $D(a_s, \epsilon)$ for $s = 1, \dots, \nu$;
- γ_0 is the union of the ν circular arcs of radius $\rho + \epsilon$ between the points $\frac{\rho+\epsilon}{\rho} a_s$ and $\frac{\rho+\epsilon}{\rho} a_{s+1}$, where the subscripts are read modulo ν.

On γ_0, we have $\left|\frac{f(u)}{u^{k+1}}\right| \leq \frac{K}{(\rho+\epsilon)^{k+1}}$ where K is some constant. Thus,

$$g(k) = \frac{1}{2i\pi} \sum_{s=1}^{\nu} \int_{\gamma_s} \frac{f(u)}{u^{k+1}} du + O((\rho + \epsilon)^{-k}). \tag{A.3}$$

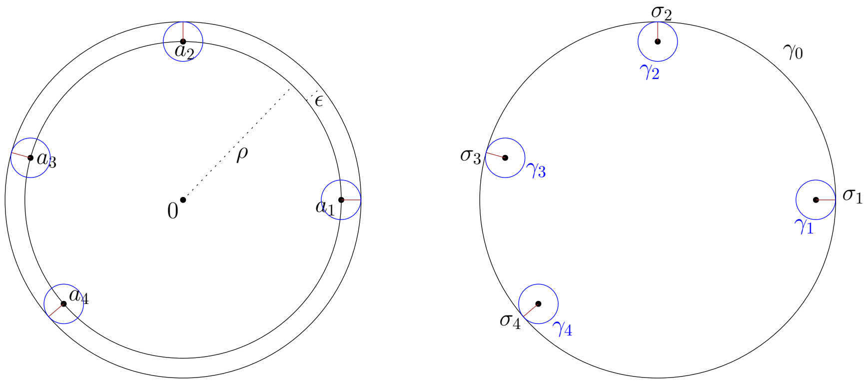

FIGURE 9. The contour α in Equation (A.2) can be replaced by $\gamma_0 \cup \bigcup_{s=1}^{\nu} \gamma_s$. (The figure displays a case $\nu = 4$). On the left, the singularities $a_1, \dots, a_4$ lie on the circle $|u| = \rho$, and f has no other singularities in $D(0, \rho + \epsilon)$. On the right, the new contour of the integral is the union $\gamma := \gamma_0 \cup \gamma_1 \cup \dots \cup \gamma_4$. The cuts σ_s are the straight segments joining a_s with $\frac{\rho+\epsilon}{\rho} a_s$.

For $1 \leq s \leq \nu$, we introduce $h_s(u)$ that is the Puiseux series of f at a_s restricted to the (finitely many) terms of negative exponents. Define $h(u) = \sum_{s=1}^{\nu} h_s(u)$ and take $F(u) = f(u) - h(u)$.

Denoting by σ_s the straight segment joining a_s to $\frac{\rho+\epsilon}{\rho} a_s$, we have that h and F are holomorphic functions on $D(0, \rho + \epsilon) \setminus \bigcup_{s=1}^{\nu} \sigma_s$. Thus, to find the asymptotics of $g(k)$ as in (A.3), we will study the asymptotic behaviors of the Taylor series coefficients of h and F.

In the rest of the proof, m will denote an arbitrary exponent of one of the considered Puiseux series. In other words, m will be an integer multiple of $\frac{1}{M}$, where M is the least common denominator of the exponents of the Puiseux series at the different points $a_1, \dots, a_s$.

Writing $h_s(u)$ as (a finite sum) $\sum_{m<0} c_{s,m}(a_s - u)^m$, with m_s^- being the least exponent, we can use the Gamma function to express

$$b_s(k) := \frac{h_s^{(k)}(0)}{k!}$$

explicitly as

$$\begin{aligned} b_s(k) &= \sum_{m<0} \frac{1}{k!} c_{s,m} (-1)^k (m \cdot (m-1) \cdot \dots \cdot (m - (k-1)) a_s^{m-k} \\ &= \sum_{m<0} \frac{c_{s,m}}{\Gamma(k+1)} \frac{\Gamma(k-m)}{\Gamma(-m)} a_s^{m-k}. \end{aligned}$$

Using the property of the Gamma function that $\lim_{k \to +\infty} k^{m+1} \frac{\Gamma(k-m)}{\Gamma(k+1)} = 1$, we represent $b_s(k)$ asymptotically by the following Puiseux series

$$b_s(k) \sim \sum_{m<0} \frac{c_{s,m}}{\Gamma(-m)} k^{-(m+1)} a_s^{m-k} \tag{A.4}$$

whose term with the least exponent equals

$$\frac{c_{s,m_s^-}}{\Gamma(-m_s^-)} k^{-(m_s^-+1)} a_s^{m_s^- - k}.$$

Next, to compute the contour integral $G_s(k) := \frac{1}{2i\pi} \int\limits_{\gamma_s} \frac{F(u)}{u^k} du$, we can replace the contour γ_s by $\sigma_s^+ \cup \sigma_s^-$, where both traverse along σ_s but with opposite orientations. Then, in $D(a_s, \epsilon)$, writing

$$F(u) = \sum_{s=1}^{\nu} \sum_{m \geq 0} d_{s,m} (a_s - u)^m, \tag{A.5}$$

we have

$$\frac{1}{2i\pi} \int\limits_{\gamma_s} \frac{F(u)}{u^{k+1}} du = \sum_{m \geq 0} \frac{1 - e^{2i\pi m}}{2i\pi} d_{s,m} \int\limits_{\sigma_s^+} \frac{(a_s - u)^m}{u^{k+1}} du.$$

We note that the integral is zero if m is a positive integer. One can transform the integrand

$$\frac{(a_s - u)^m}{u^{k+1}} = a_s^{m-(k+1)} \left(1 - \frac{u}{a_s}\right)^m \exp\left(-(k+1) \ln\left(\frac{u}{a_s}\right)\right)$$

and, with the change of variable $t = \frac{u}{a_s} - 1$, one has

$$\frac{1}{2i\pi} \int\limits_{\gamma_s} \frac{(a_s - u)^m}{u^{k+1}} du = \frac{1 - e^{2i\pi m}}{2i\pi} e^{\pi i m} d_{s,m} a_s^{m-k} \int\limits_{t=0}^{\delta} t^m e^{-(k+1)\ln(1+t)} dt$$

where $\delta = \frac{\epsilon}{\rho}$.

If we once again change the variable $x = \ln(1 + t)$, the problem then reduces to studying integrals of the form

$$I_\alpha(k) := \int_0^{\delta_0} x^\alpha \varphi_\alpha(x) e^{-kx} dx,$$

where $\delta_0 = \ln(1 + \delta)$ and $\varphi_\alpha(x) = \left(\frac{e^x - 1}{x}\right)^\alpha$. Since $\varphi(x)$ is analytic at $x = 0$, an application of Watson's Lemma yields

$$I_\alpha(k) \sim \sum_{n \geq 0} \frac{\varphi_\alpha^{(n)}(0)}{n!} \frac{\Gamma(\alpha + n + 1)}{k^{\alpha+n+1}}$$

as k tends to $+\infty$ and so $\frac{1}{2i\pi} \int\limits_{\gamma_s} \frac{F(u)}{u^k} du$ is equivalent to

$$\sum_{m \geq 0} \frac{1 - e^{2i\pi m}}{2i\pi} e^{i\pi m} d_{s,m} a_s^{m-k} \sum_{n \geq 0} \left(\frac{\varphi_m^n(0)}{n!} \frac{\Gamma(m + n + 1)}{k^{m+n+1}} \right).$$

From the identities $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}$ and

$$\frac{1 - e^{2i\pi m}}{2i\pi} = \frac{e^{-i\pi m}(e^{-i\pi m} - e^{i\pi m})}{2i\pi} = \frac{\sin \pi m}{\pi e^{i\pi m}},$$

we deduce that

$$\begin{aligned}\frac{1 - e^{2i\pi m}}{2i\pi} e^{i\pi m} \Gamma(m + n + 1) &= \frac{\sin(\pi m)}{\Gamma(-m - n) \sin\left[\pi(m + n + 1)\right]} \\ &= \frac{(-1)^{n+1}}{\Gamma(-m - n)}.\end{aligned}$$

After simplification, we obtain that $G_s(k)$ is asymptotic to

$$\sum_{m\geq 0} d_{s,m} a_s^{m-k} \sum_{n\geq 0} \left(\frac{\varphi_\alpha^n(0)}{n!} \frac{(-1)^{n+1}}{\Gamma(-m-n)k^{m+n+1}} \right)$$

as k tends to $+\infty$.

Summing over all contours γ_S, where $s = 1, \dots, \nu$, and reorganizing the latter series, we obtain that $G(k) := G_1(k) + \cdots + G_\nu(k)$ is asymptotic to

$$\sum_{s=1}^{\nu} \left(\sum_{m,n\geq 0} d_{s,m} a_s^m \frac{\varphi_\alpha^n(0)(-1)^{n+1}}{n!\Gamma(-m-n)} k^{-(m+n+1)} \right) a_s^{-k}. \tag{A.6}$$

Any given value of $m + n + 1$ is realized by at most finitely many tuples (m, n) where n is a non-negative integer while m is a non-negative multiple of $\frac{1}{M}$ that is not an integer. Therefore, for each s, the inner double series can be written as a series indexed by the increasing exponents of a_s whose term with with least exponent is

$$\frac{B_s}{\Gamma(-m_s^+)} k^{-(m_s^+ +1)} a_s^{m_s^+ -k}$$

where m_s^+ is the least fractional exponent of $(u - a_s)$ in $F(u)$ expressed as in Equation (A.5) and B_s is the coefficient of that term.

Since $f(u) = F(u) + h(u)$, we combine contributions from the holomorphic part and the Puiseux series part and find the following asymptotic representation of (A.3)

$$g(k) \sim \sum_{s=1}^{\nu} \tilde{b}_s(k) a_s^{-k} + O((\rho + \epsilon)^{-k}) \tag{A.7}$$

where $\tilde{b}_s(k)$ is a Puiseux series in k that is the sum of (A.4) and (A.5). It follows that the asymptotic behavior of the leading term of $\tilde{b}_s(k)$ can be given the form stated in the statement of the theorem. This finishes the proof. □

By tweaking Orlov's proof, we will show the uniform convergence that is needed for our purposes.

Corollary A.2. *Let f be a holomorphic function on an open disk $D(0, \rho)$ that extends to a Puiseux series at a unique singularity a with $|a| = \rho$ in $D(a, \epsilon')$ for some $\epsilon' > 0$, and holomorphically at every other point of the boundary circle $\partial D(0, \rho)$. Then, there exist $\delta > 0$ and $\epsilon > 0$ such that the formula*

$$\frac{f^{(k)}(z)}{k!} = b(k, z)(a - z)^{-k} + O((|a - z|(1 + \epsilon))^{-k}).$$

holds for any $z \in \overline{D}_\delta = \overline{D(0, \delta)}$. Here, we have

$$b(k, z) = K(a - z)^{m_0} k^{-(m_0+1)} (M + O(k^{-\frac{1}{N}} |a - z|^{\frac{1}{N}}))$$

where $K, M \in \mathbb{C} \setminus \{0\}$ and $N \in \mathbb{N}$ are constants depending only on f and δ while m_0 is the smaller between the least negative exponent or the exponent of the first non-integer term of the Puiseux series of f.

Proof. We follow the notations and strategy of the proof of the previous theorem. First, suppose that a is a pole of f. Then, as in (A.4), the singular part $h(z) =$

$\sum_{m_0 \le m < 0} c_m (a-z)^m$ of f has the following asymptotic description of its derivatives

$$\frac{h^{(k)}(z)}{k!} \sim \sum_{m_0 \le m < 0} \frac{c_m}{\Gamma(-m)} k^{-(m+1)} (a-z)^{m-k}$$
$$= (a-z)^{-k} \left[\frac{c_{m_0}}{\Gamma(-m_0)} (a-z)^{m_0} k^{-(m_0+1)} \left(1 + O(k^{-1}|a-z|^{-1})\right) \right]$$

throughout some $\overline{D}_\delta$, where $\delta > 0$ is chosen such that a is the only singular point of f closest to $\overline{D}_\delta$. The k-th derivatives of the holomorphic part of f vanish for k large enough, so the expression above is already of the desired form.

We may now assume that f has a pole-free Puiseux series expansion at $z = a$ given by

$$f(z) = \sum_{k \ge k_0} c_k (a-z)^{\frac{k}{M}}. \tag{A.8}$$

Again, choose $\delta > 0$ such that the closest singularity of f to any point in the closed disk $\overline{D}_\delta$ is a. Then, take $\alpha > 0$ strictly larger than the maximum distance from $z \in \overline{D}_\delta$ to a. Regard the curve γ in Figure 9 as a diagram for the curve $\gamma(z)$, determined by the center $z \ne a$ in $\overline{D}_\delta$, and choose $\epsilon(z) = \frac{|z-a|}{\alpha}\epsilon' < \epsilon'$. Moving the center in $\overline{D}_\delta$ changes the cut $\sigma(z)$, and also f through analytic continuation across $\sigma(z)$, but the assumption on f implies that the conditions of the theorem are still true. In particular, Equation (A.3)

$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \int_{\gamma_a(z)} \frac{f(u)}{(u-z)^{k+1}} du + O((|z-a|(1+\epsilon))^{-k}) \tag{A.9}$$

(with $\epsilon := \epsilon'/\alpha$) remains true for any $z \in \overline{D}_\delta$, and in fact uniformly, since the implicit constant in the principal term is bounded by the maximum value of (the continuations of) $|f|$ to any cut disk $B(z, |z-a| + \epsilon(z)),\ z \in \overline{D}_\delta$.

Fixing $z \in \overline{D}_\delta$, as in the proof of the theorem, we rewrite the integral

$$I := \int_{\gamma_a(z)} \frac{f(u)}{(u-z)^{k+1}} du = \int_{\sigma_a^+} \frac{f(u)}{(u-z)^{k+1}} du + \int_{\sigma_a^-} \frac{f(u)}{(u-z)^{k+1}} du. \tag{A.10}$$

Parametrizing $\sigma_a^+(z)$ as $u = a + s(a-z)$, $0 \le s \le \eta$, where $\eta = \epsilon(z)/|a-z|$, we have

$$\int_{\sigma_a^+(z)} \frac{f(u)}{(u-z)^{k+1}} du = \frac{1}{(a-z)^k} \int_0^\eta \frac{f(a+s(a-z))}{(1+s)^{k+1}} ds \tag{A.11}$$

Changing the variable again by setting $1 + s = e^v$ results in

$$\frac{1}{(a-z)^k} \int_0^{\eta'} f\left(a + (e^v - 1)(a-z)\right) e^{-kv} dv, \tag{A.12}$$

where $\eta' = \log(1+\eta) > 0$.

The Puiseux expansion in (A.8) implies that the function

$$\tilde{f}(t) := \sum_{k \ge k_0} c_k t^k$$

is analytic in a neighborhood of $t = 0$ and $f(z) = \tilde{f}((a-z)^{1/M})$ for a choice of $t = (a-z)^{1/M}$. The function $\tilde{f}$ is just the analytic continuation of f to the cover $t \mapsto t^M = a - z$. In particular, the analytic continuation of $f(z) = \tilde{f}((a-z)^{1/M})$ counterclockwise along a circle with center $z = a$ is given by $\tilde{f}(e^{2\pi i/M}(a-z)^{1/M})$.

Consider $\theta(v) := \frac{e^v - 1}{v}$, which is analytic and non-zero at $v = 0$. Then $\theta^{\frac{1}{M}}(v)$ is defined for $|v| < 2\pi$ in such a way that it restricts to the M-th real root for $v \ge 0$

real. Now, define

$$\tilde{g}_+(w) := \tilde{f}\left(w \cdot \theta^{\frac{1}{M}}((w^M/(z-a))\right)$$
$$\tilde{g}_-(w) := \tilde{f}\left(e^{\frac{2\pi i}{M}} w \cdot \theta^{\frac{1}{M}}((w^M/(z-a))\right).$$

Thus, along $\sigma_a^+(z)$ we have

$$\begin{aligned} f(a+(e^v-1)(a-z)) &= \tilde{f}(((v(z-a))^{\frac{1}{M}} \cdot \theta^{\frac{1}{M}}(v)) \\ &= \tilde{g}_+(((z-a)v)^{\frac{1}{M}}), \end{aligned} \tag{A.13}$$

and, similarly, the analytic continuation of f along $\sigma_a^-(z)$ is

$$f(a+e^{\frac{2\pi i}{M}}(e^v-1)(a-z)) = \tilde{g}_-(((z-a)v)^{\frac{1}{M}}). \tag{A.14}$$

By setting

$$g(v) = \tilde{g}_+(((z-a)v)^{\frac{1}{M}}) - \tilde{g}_-(((z-a)v)^{\frac{1}{M}}),$$

we can return to the integral I in (A.10). It equals by (A.12), (A.13), and (A.14)

$$I = \frac{1}{(a-z)^k}\int_0^{\eta'} g(v)e^{-kv}dv. \tag{A.15}$$

Applying Lagrange's error term for the first-order Maclaurin polynomial of $g(v)$ proves that the inequality

$$|g(v) - (1-e^{2k_0\pi i/M})c_{k_0}(a-z)^{m_0}v^{m_0}| \leq |(a-z)v|^{\frac{k_0+1}{M}} \tag{A.16}$$

holds for all $0 \leq v \leq \eta'$ and $z \in \overline{D}_\delta$, where $c_{k_0}(a-z)^{m_0}, m_0 = \frac{k_0}{M}$, is the first non-integer term of the Puiseux series of f, and D is an upper bound for g^{k_0+1} on the segment between 0 and our choice of $(a-z)^{1/M}$.

By (A.15) and (A.16),

$$|I-(a-z)^{m_0-k}\int_0^{\eta'} c_{k_0}v^{m_0}e^{-kv}dv| \leq |a-z|^{m_0+\frac{1}{M}-k}\int_0^{\eta'} Dv^{m_0+\frac{1}{M}}e^{-kv}dv. \tag{A.17}$$

Changing the variable $u = kv$, we obtain

$$\begin{aligned} &|I - k^{-(m_0+1)}(a-z)^{m_0-k}\int_0^{k\eta'} c_{m_0}u^{m_0}e^{-u}du| \\ &\leq k^{-(m_0+\frac{1}{M}+1)}|w-a|^{m_0+\frac{1}{M}-k}\int_0^{k\eta'} Du^{m_0+\frac{1}{M}}e^{-u}du. \end{aligned} \tag{A.18}$$

To obtain the desired asymptotics, we apply a standard proof of Watson's Lemma. Extending the interval of integration of the latter integral to $+\infty$, we have a bound

$$\int_0^{k\eta'} Du^{m_0+\frac{1}{M}}e^{-u}du \leq \int_0^{\infty} Du^{m_0+\frac{1}{M}}e^{-u}du = D\Gamma(m_0+1/M+1).$$

On the other hand, we have

$$\begin{aligned} \int_0^{\eta'} c_{m_0}u^{m_0}e^{-u}du &= c_{m_0}\left(\int_0^{+\infty} u^{m_0}e^{-u}du - \int_{k\eta'}^{+\infty} u^{m_0}e^{-u}du\right) \\ &= c_{m_0}(\Gamma(m_0+1) + O(e^{-k\eta'/2})) \end{aligned}$$

Here, the asymptotic term above arises as follows. Consider the integral $\int_{k\eta'}^{+\infty} u^{m_0}e^{-u}du$. Then, there is $r = r(m_0)$ such that $u^{m_0}e^{-u/2} \leq 1$ for all $u \geq r_0$. Hence, if $k\eta' \geq r_0$,

$$\int_{k\eta'}^{+\infty} u^{m_0}e^{-u}du \leq \int_{k\eta'}^{+\infty} e^{-u/2}du = 2e^{-k\eta'/2},$$

where η' is a constant depending on m_0. The corollary then follows from (A.18), noting that the term $O(e^{-k\eta'/2})$ is insignificant compared to the other asymptotic terms. □

A.2. **Laplacian of a Puiseux series.** It is known that the Laplacian of the logarithmic absolute value of a meromorphic function, thought of as a L^1_{loc}-function and hence a distribution, is (in algebraic-geometric terms) the divisor of the function expressed in terms of Dirac measures (see Theorem 3.7.8 in [20]). The same holds for a converging Puiseux series, as we now show.

Lemma A.3. *Let $f(z) = \sum_{k\geq r} a_k z^{\frac{k}{m}}$ be a converging Puiseux series where $a_r \neq 0$. Then, in some cut disc $U := D(0,R) \setminus \mathbb{R}^-$, we have the following equality between two L^1_{loc}-functions:*

$$\frac{2}{\pi}\frac{\partial^2}{\partial\bar{z}\partial z}\log|f| = \frac{r}{k}\delta_0.$$

Proof. Writing $f(z) = z^{r/k}g(z)$, we have $\log|f| = \log|g(z)| + \frac{r}{k}\log|z|$, so it suffices to check that $\frac{\partial^2 \log|g(z)|}{\partial\bar{z}\partial z}$ vanishes as a distribution.

Let ψ be a test function. Since $g(0) \neq 0$, there exists $M > 0$ such M is greater than $|\log|g(z)|| \cdot |\frac{\partial^2\psi}{\partial\bar{z}\partial z}|$ in the (cut) disc.

For an arbitrary $\eta > 0$, we introduce the strip $C(\eta)$ characterized by the inequalities $\operatorname{Re} z \leq \eta$ and $-\eta \leq \operatorname{Im} z \leq \eta$. Then, we divide the integral

$$\int_U \frac{\partial^2 \log|g(z)|}{\partial\bar{z}\partial z}\psi(z)\,dz\wedge d\bar{z}$$

into the integral on $U \setminus C(\eta)$, which vanishes, since $\log|g|$ is harmonic in this set, and the integral on $U \cap C(\eta)$. Since the absolute value of

$$\int_{U\cap C(\eta)} \frac{\partial^2 \log|g(z)|}{\partial\bar{z}\partial z}\psi(z)\,dz\wedge d\bar{z} = -\int_{U\cap C(\eta)} \log|g(z)|\frac{\partial^2\psi(z)}{\partial\bar{z}\partial z}\,dz\wedge d\bar{z}$$

is bounded by $2M\eta(R+\eta)$ for any $\eta > 0$, and the integral also has to vanish. Thus,

$$\int_U \frac{\partial^2 \log|g(z)|}{\partial\bar{z}\partial z}\psi(z)dz\wedge \bar{d}z = 0$$

for any test function ψ. The claim follows. □

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