MOSCOW MATHEMATICAL JOURNAL
Volume 21, Number 3, July–September 2021, Pages 613–637

GRASSMANN CONVEXITY AND MULTIPLICATIVE STURM THEORY, REVISITED

NICOLAU SALDANHA, BORIS SHAPIRO, AND MICHAEL SHAPIRO

To the late Vladimir Arnold, who started all this

Abstract. In this paper we settle a special case of the Grassmann convexity conjecture formulated by the second and the third authors about a decade ago. We present a conjectural formula for the maximal total number of real zeros of the consecutive Wronskians of an arbitrary fundamental solution to a disconjugate linear ordinary differential equation with real time. We show that this formula gives the lower bound for the required total number of real zeros for equations of an arbitrary order and, using our results on the Grassmann convexity, we prove that the aforementioned formula is correct for equations of orders 4 and 5.

2020 Math. Subj. Class. Primary: 34B05; Secondary: 52A55.

Key words and phrases. Disconjugate linear ordinary differential equations, Grassmann curves, osculating flags, Schubert calculus.

1. Introduction and Main Results

Our subject of study is related to the PhD theses of the second and third authors defended in the early 90s (see [12], [17]). Namely, the thesis of the second author contains Conjecture 1.2, see below, but the presented argument which is supposed to prove it is false. The statement itself is still open and (if proven) would be of fundamental importance to the general qualitative theory of linear ordinary differential equations with real time. The thesis of the third author contains a number of Schubert calculus problems relevant to Conjecture 1.2. Over the years the authors made several attempts to settle it and, in particular, worked out some reformulations and special cases. This paper contains a number of new results in that direction. (In what follows, we will label conjectures, theorems and lemmas borrowed from the existing literature by letters. Results and conjectures labelled by numbers are new).

Received May 30, 2019; in revised form June 6, 2020.

The first named author wants to acknowledge support of CNPq, CAPES and Faperj (Brazil) and to express his sincere gratitude to the Department of Mathematics, Stockholm University for the hospitality in November 2018. The second named author wants to acknowledge the financial support of his research provided by the Swedish Research Council grant 2016-04416. The third named author is supported by the NSF grant DMS-1702115.

DOI:10.17323/1609-4514-2021-21-3-613-637

We start with the following classical definition, see for example [5].

Definition 1.1. A linear ordinary homogeneous differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 \tag{1.1}$$

of order n with real-valued continuous coefficients $p_i(x)$ defined on an interval $I \subseteq \mathbb{R}$ is called *disconjugate on* I if any of its nontrivial solutions has at most $(n-1)$ zeros on I counting multiplicities. (The interval I can either be open or closed).

Conjecture 1.2 (Upper bound on the number of real zeros of a Wronskian)**.** *Given any equation* (1.1) *disconjugate on* I, *a positive integer* $1 \leqslant k \leqslant n-1$, *and an arbitrary* k*-tuple* $(y_1(x), y_2(x), \ldots, y_k(x))$ *of its linearly independent solutions, the number of real zeros of* $\det(W(y_1(x), y_2(x), \ldots, y_k(x)))$ *on* I *counting multiplicities does not exceed* $k(n-k)$.

Here

$$W(y_1(x), y_2(x), \ldots, y_k(x)) = \begin{pmatrix} y_1(x) & y_1'(x) & \ldots & y_1^{(k-1)}(x) \\ y_2(x) & y_2'(x) & \ldots & y_2^{(k-1)}(x) \\ \ldots & \ldots & \ldots & \ldots \\ y_k(x) & y_k'(x) & \ldots & y_k^{(k-1)}(x) \end{pmatrix}$$

is the Wronskian matrix of the k-tuple $(y_1(x), y_2(x), \ldots, y_k(x))$.

Cases $k = 1$ and $k = n-1$ of Conjecture 1.2 are straightforward, but not very illuminating. The simplest non-trivial case $k = 2$, $n = 4$ of Conjecture 1.2 has been settled in [13]. The first main result of the present paper extends these results.

Theorem 1. *Conjecture* 1.2 *holds for* $k = 2$ *and* $k = n-2$ *for any* $n \geqslant 3$.

The case $k = n-2$ follows easily from the case $k = 2$; see Remark 3.2.

We present an equivalent statement to Conjecture 1.2 in a different language. This version is used in this paper and is also in some references (including [7]). Let Lo_n^1 be the nilpotent Lie group of lower triangular $n \times n$ real matrices whose diagonal entries equal 1. Let $\mathcal{J}$ be the set of $n \times n$ real matrices A such that the entry A_{ij} is positive if $i = j+1$ and 0 otherwise. A smooth curve $\Gamma\colon [0, 1] \to \mathrm{Lo}_n^1$ is *flag-convex* (or, sometimes, just *convex*) if, for all $t \in [0, 1]$, $(\Gamma(t))^{-1}\Gamma'(t) \in \mathcal{J}$. For a flag-convex curve Γ and an integer k, $0 < k < n$, let

$$m_k\colon [0, 1] \to \mathbb{R}, \quad m_k(t) = \det(\mathrm{swminor}(\Gamma(t), k)). \tag{1.2}$$

Here $\mathrm{swminor}(L, k)$ is the southwest $k \times k$ submatrix of L, i.e., the submatrix formed by its last k rows and its first k columns. (If the curve Γ is not obvious from the context, we write $m_{\Gamma,k} = m_k$.) Conjecture 1.2 above is equivalent to saying that the number of zeroes $t \in [0, 1]$ of the smooth function $m_k\colon [0, 1] \to \mathbb{R}$ is at most $k(n-k)$; here, as above, zeroes are counted with multiplicities. This equivalence will be additionally clarified in §4.

Thus, Theorem 1 claims that, for any flag-convex function $\Gamma\colon I \to \mathrm{Lo}_n^1$, the functions m_2 and m_{n-2} have at most $2(n-2)$ zeroes each. Our second result is an inequality in the opposite sense; we state a related result in Corollary 2.16 below.

Theorem 2. *Consider a smooth flag-convex curve* $\Gamma_\bullet\colon I \to \mathrm{Lo}_n^1$ *(where* $I \subset \mathbb{R}$ *is a non-degenerate interval). Then, for any open subinterval* $I_1 \subset I$*, there exists a matrix* $L_1 \in \mathrm{Lo}_n^1$ *such that, for* $\Gamma_1(t) = L_1\Gamma_\bullet(t)$ *and* $m_k = m_{\Gamma_1,k}$*, the following properties hold*:

(1) *all roots of each* m_k *in* I *are simple and belong to* I_1;
(2) *they are distinct*: *if* $k_1 \neq k_2$ *then* m_{k_1} *and* m_{k_2} *have no common roots*;
(3) *for each* k*, the function* m_k *admits precisely* $k(n-k)$ *roots in* I.

In this case the total number of roots of all functions $m_k\colon I \to \mathbb{R}$ is $\frac{n^3-n}{6}$.

The structure of this paper is as follows. In Section 2 we provide context for the results above, compare Conjecture 1.2 above with the Grassmann convexity conjecture and obtain corollaries of our main results which, we hope, will further motivate their interest. We start Section 3 by reviewing and motivating the constructions above. We also recall the concept of *total positivity*, which will be used in the proof of both main theorems. There is of course a vast literature concerning the subject of total positivity (among many others, [3]). We define the set $\mathfrak{W}$ of *admissible cyclic words* and a correspondence from an open dense subset of Lo_n^1 to $\mathfrak{W}$. We define *admissible moves* in $\mathfrak{W}$ (Definition 3.9). The definition is combinatorial (and simple) and when following a flag-convex curve Γ we perform admissible moves (Lemma 3.10). We then introduce our main technical tool, the *rank function* (Definition 3.13). The rank function is integer valued and defined for admissible cyclic words; the definition is combinatorial and elementary, but long. Several basic properties are given, which admit simple but sometimes long proofs. For instance, the rank of the totally positive word is 0 and the rank of the totally negative word is $2(n-2)$; other words give intermediate values (Lemma 3.16). The crucial property is that, when following a flag-convex curve Γ, the rank of $\Gamma(t)$ is always non-increasing and is strictly decreasing at roots of m_2 (Proposition 3.17). The proof of Theorem 1 is now easy.

In Section 4 we prove Theorem 2. We first recall several concepts and results from [7]. In particular, we define the *multiplicity vector* $\mathrm{mult}(\sigma)$ of a permutation $\sigma \in S_n$. The definition of $\mathrm{mult}(\sigma)$ is combinatorial; however Theorem 4 from [7] provides an algebraic or geometric interpretation. Indeed, if Γ is flag-convex and $\Gamma(t_0) \in \mathrm{Bru}_\sigma$ then $\mathrm{mult}_k(\sigma)$ is the multiplicity of the zero $t = t_0$ of the function m_k. We use the notation Bru_σ for the Bruhat cell corresponding to the permutation σ (see again [7]). We first state a warm-up special case, Proposition 4.1. We then prove additional statements (Lemmas 4.2 and 4.5) concerning pairs of permutations $\sigma_0 \lhd \sigma_1 = \sigma_0 a_i$: here $\lhd$ denotes the Bruhat order and a_i is a transposition, a generator of S_n. The change in multiplicities between σ_0 and σ_1 is described by Lemma 2.4, also from [7]. Only the two results above are needed from [7]; duplicating the proof of these results here would lead us too far astray. A simple induction then settles Proposition 4.1 and Theorem 2 (the induction is described in Lemmas 4.3 and 4.6). Besides the references we already mentioned other relevant results can be found , for example, in [11] and [2].

We finish the introduction with the following tantalizing question:

Is it possible to extend the present approach (*see especially Section* 3) *from the case of* $G_{2,n}$ *to other Grassmannians?*

Acknowledgements. The authors thank Victor Goulart for his help with the revision of the text.

2. The Grassmann Convexity Conjecture

Conjecture 1.2 has an equivalent reformulation called the *Grassmann convexity conjecture* first suggested in [13, Main Conjecture 1.1]. To state it, we need some further definitions.

Definition 2.1. A smooth closed curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$ is called *locally convex* if, for any hyperplane $H \subset \mathbb{R}P^{n-1}$, the local multiplicity of the intersection of γ with H at any of the intersection points $p \in \gamma \cap H$ does not exceed $n-1 = \dim \mathbb{R}P^{n-1}$ and *globally convex* if the above condition holds for the sum of all local multiplicities, see for example [13].

Below we will often refer to globally convex curves simply as *convex*. The above notions are directly generalized to smooth non-closed curves, i.e., $\gamma\colon I \to \mathbb{R}P^{n-1}$.

Remark 2.2. Local convexity of γ is an easy requirement equivalent to the non-degeneracy of the osculating Frenet $(n-1)$-frame of γ, i.e., to the linear independence of $\gamma'(t), \ldots, \gamma^{(n-1)}(t)$ at all points $t \in \mathbb{S}^1$. Global convexity is a rather nontrivial property studied under different names since the beginning of the last century. (There exists a vast literature on convexity and the classical achievements are well summarized in [5]. For more recent developments see for example [1]).

Denote by $G_{k,n}$ the usual Grassmannian of real k-dimensional linear subspaces in $\mathbb{R}^n$ (or equivalently, of real $(k-1)$-dimensional projective subspaces in $\mathbb{R}P^{n-1}$).

Definition 2.3. Given an $(n-k)$-dimensional linear subspace $L \subset \mathbb{R}^n$, we define the *Grassmann hyperplane* $H_L \subset G_{k,n}$ *associated to* L as the set of all k-dimensional linear subspaces in $\mathbb{R}^n$ non-transversal to L.

Remark 2.4. The concept of Grassmann hyperplanes is well-known in Schubert calculus, (see for example [4] and [13].) More exactly, $H_L \subset G_{k,n}$ coincides with the union of all Schubert cells of positive codimension constructed using any complete flag containing L as a linear subspace. The complement $G_{k,n} \setminus H_L$ is the open Schubert cell isomorphic to the standard affine chart in $G_{k,n}$. By duality, $H_L \subset G_{k,n}$ is isomorphic to $H_{L'} \subset G_{n-k,n}$, where L' is a k-dimensional linear subspace in $\mathbb{R}^n$.

Remark 2.5. A usual hyperplane $H \subset \mathbb{R}P^{n-1}$ is a particular case of a Grassmann hyperplane if we interpret H as the set of all points non-transversal (i.e., belonging) to H. H itself can be considered as a point in $(\mathbb{R}P^{n-1})^\star$.

Definition 2.6. A smooth closed curve $\Gamma\colon \mathbb{S}^1 \to G_{k,n}$ is called *locally Grassmann-convex* if the local multiplicity of the intersection of Γ with any Grassmann hyperplane $H_L \subset G_{k,n}$ at any of its intersection points does not exceed $k(n-k) = \dim G_{k,n}$, and *globally Grassmann-convex* if the above condition holds for the sum of all local multiplicities, see [13].

Below we refer to globally Grassmann-convex curves as *Grassmann-convex*. The notions are directly generalized to smooth non-closed curves, i.e., $\Gamma\colon I \to G_{k,n}$.

Definition 2.7. Given a locally convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$ and a positive integer $1 \leqslant k \leqslant n-1$, we define its *kth osculating Grassmann curve* $\operatorname{osc}_k \gamma\colon \mathbb{S}^1 \to G_{k,n}$ as the curve formed by the $(k-1)$-dimensional projective subspaces osculating the initial γ.

For any $k = 1, \dots, n-1$, the curve $\operatorname{osc}_k \gamma$ is well-defined due to the local convexity of the curve γ.

Conjecture 2.8 (Grassmann convexity conjecture)**.** *For any convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$ (resp. $\gamma\colon I \to \mathbb{R}P^{n-1}$) and any $1 \leqslant k \leqslant n-1$, its osculating curve $\operatorname{osc}_k \gamma\colon \mathbb{S}^1 \to G_{k,n}$ (resp. $\operatorname{osc}_k \gamma\colon I \to G_{k,n}$) is Grassmann-convex.*

The equivalence of Conjectures 1.2 and 2.8 is straightforward and, in particular, is explained in [13].

To explain this equivalence we need the following. A curve $\gamma = (\gamma_1, \dots, \gamma_n)\colon I \to \mathbb{R}^n$ is called *non-degenerate* if at every point $t \in I$ its osculating frame $\{\gamma(t), \gamma'(t), \gamma''(t), \dots, \gamma^{n-1}(t)\}$ is non-degenerate. This is equivalent to the fact that its Wronski matrix $W(t) = W(\gamma_1(t), \dots, \gamma_n(t))$ has full rank.

Non-degenerate curves can be trivially identified with fundamental solutions of linear differential equations (1.1). In particular, we call a non-degenerate γ *disconjugate* if the corresponding equation (1.1) is disconjugate. On the other hand, it is obvious that γ is non-degenerate/disconjugate if and only if its projectivization is locally convex/convex.

Moreover, given a non-degenerate curve $\gamma = (\gamma_1, \dots, \gamma_n)\colon I \to \mathbb{R}^n$ and an integer $1 \leqslant k < n$, the zeros of the Wronskian $W(\gamma_1, \dots, \gamma_k)$ can be interpreted as the moments when the kth osculating Grassmann curve $\operatorname{osc}_k \gamma\colon I \to G_{k,n}$ intersects an appropriate Grassmann hyperplane; for more details on $k = 2$, see Section 3. Observe that Conjecture 2.8 is trivially satisfied for $k = 1$ and $k = n-1$.

Notice additionally that Theorem 1 admits the following natural interpretation, compare loc. cit.

Definition 2.9. Given a generic curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$, we define its *standard discriminant* $D_\gamma \subset \mathbb{R}P^{n-1}$ to be the hypersurface consisting of all subspaces of codimension 2 osculating γ. (Here 'generic' means having a non-degenerate osculating $(n-2)$-frame at every point.)

Definition 2.10. By the $\mathbb{R}$-*degree* of a real closed (algebraic or non-algebraic) hypersurface $\mathcal{H} \subset \mathbb{R}^n$ (resp. $\mathcal{H} \subset \mathbb{R}P^{n-1}$) without boundary we mean the supremum of the cardinality of $\mathcal{H} \cap L$ taken over all lines $L \subset \mathbb{R}^n$ (resp. $L \subset \mathbb{R}P^{n-1}$) such that L intersects $\mathcal{H}$ transversally. (Observe that the $\mathbb{R}$-degree of a hypersurface can be infinite. Discussions of this notion can be found in [9]).

Corollary 2.11. *For any closed convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$, the $\mathbb{R}$-degree of its discriminant D_γ equals $2n-4$.*

Basic notions of the multiplicative Sturm separation theory. Following [13], let us now recall the set-up of this theory, an early version of which can be found in [12].

Denote by Fl_n the space of complete real flags in $\mathbb{R}^n$. We say that two complete flags f_1, $f_2 \in \mathrm{Fl}_n$ are *transversal* if, for any $1 \leqslant i \leqslant n-1$, the intersection of the i-dimensional subspace of f_1 with the $(n-i)$-dimensional subspace of f_2 coincides with the origin. Otherwise the flags f_1, $f_2 \in \mathrm{Fl}_n$ are called *non-transversal*.

Definition 2.12. Given a locally convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$, define its *osculating flag curve* $\gamma_{\mathcal{F}}\colon \mathbb{S}^1 \to \mathrm{Fl}_n$ to be the curve formed by the complete flags osculating γ, see for example [12]. (The curve $\gamma_{\mathcal{F}}$ is well-defined due to the local convexity of γ; similar notion obviously exists for non-closed locally convex curves).

For a non-degenerate curve $\Gamma\colon I \to \mathbb{R}^n$ (or, equivalently, for its projectivization $\gamma\colon I \to \mathbb{R}P^{n-1}$) and any fixed flag $f \in \mathrm{Fl}_n$, denote by $\#_{\gamma,f}$ the number of *moments of non-transversality* between $\gamma_{\mathcal{F}}$ and f, where $t \in I$ is called a moment of non-transversality if the complete flags $\gamma_{\mathcal{F}}(t)$ and f are non-transversal. Define $\#_\gamma = \sup_{f\in\mathrm{Fl}_n} \#_{\gamma,f}$.

The following two lemmas provide criteria for (non-)disconjugacy of linear ordinary differential equation or, equivalently, (non-)convexity of projective curves, compare [10].

Lemma 2.13 (see [12])**.** *A locally convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$ (resp. $\gamma\colon I \to \mathbb{R}P^{n-1}$) is globally convex if and only if, for all $t_1 \neq t_2 \in \mathbb{S}^1$ (resp. $t_1 \neq t_2 \in I$), the flags $\gamma_{\mathcal{F}}(t_1)$ and $\gamma_{\mathcal{F}}(t_2)$ are transversal.*

Lemma 2.14 (see [12])**.** *A locally convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$ (resp. $\gamma\colon I \to \mathbb{R}P^{n-1}$) is not globally convex if and only if, for any complete flag $f \in \mathrm{Fl}_n$, there exists $t \in \mathbb{S}^1$ (resp. $t \in I$) such that f and $\gamma_{\mathcal{F}}(t)$ are non-transversal.*

The next claim appears to be new, but closely related to Conjecture 1.2.

Conjecture 2.15. *For any convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$ (resp. $\gamma\colon I \to \mathbb{R}P^{n-1}$), one has*

$$\#_\gamma = \frac{n^3-n}{6}.$$

Conjecture 2.15 is obvious for $n = 2$ and easy for $n = 3$; we shall soon see some further support for it. The following result follows directly from Theorem 2.

Corollary 2.16. *For any convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$ or $\gamma\colon I \to \mathbb{R}P^{n-1}$ we have*

$$\#_\gamma \geqslant \frac{n^3-n}{6}.$$

Proof. We may assume I open and γ associated to $\Gamma\colon I \to \mathrm{Lo}^1_n$, Γ flag-convex (this correspondence is discussed in the first paragraphs of Section 3). Apply Theorem 2 to obtain a matrix L_1 such that, for $L_1\Gamma$, the functions m_k have, among them, $\frac{n^3-n}{6}$ roots. The matrix L_1 defines a flag f_1: the moments of non-transversality beween γ and f_1 are precisely the roots of the functions m_k for $L_1\Gamma$, as above. $\square$

Combining Theorems 1 and Corollary 2.16 we get the following.

Corollary 2.17. *Conjecture 2.15 holds for $n = 4$ and $n = 5$.*

Proof. We use the notation of Theorem 2, particularly the functions m_k. We know that m_1 and m_{n-1} have at most $n-1$ zeroes each. From Theorem 1, we know that m_2 and m_{n-2} have at most $2(n-2)$ zeroes each. For $n \leqslant 5$, this covers all possible values of k. This implies that $\#_\gamma \leqslant \frac{n^3-n}{6}$ for any convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$. Corollary 2.16 gives us the other inequality. □

To finish this section, let us mention that it is well-known that, for general equations (1.1) of order exceeding 2, the number and location of the zeros of their different individual solutions can be quite arbitrary. On the other hand, for any equation (1.1), one can split its time interval I into maximal disjoint subintervals on each of which (1.1) is disconjugate. In order to get a meaningful comparison theory, instead of looking at the individual solutions we could compare different fundamental solutions of (1.1), i.e., count the number of moments of non-transverality of the flag curve of (1.1) with different complete flags. This approach leads to the following claim, which is a new type of generalization of the classical Sturm separation theorem from the case of linear ode of order 2 to the case of arbitrary order, comp. [12].

Conjecture 2.18 (see [13])**.** *For $n \geqslant 2$, let $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$ (resp. $\gamma\colon I \to \mathbb{R}P^{n-1}$) be a locally, but not globally convex curve. Then, for any pair of complete flags f_1 and f_2,*

$$\#_{\gamma,f_1} \leqslant \frac{n^3-n+6}{6} \cdot \#_{\gamma,f_2}. \tag{2.1}$$

Observe that (if settled) Conjecture 2.15 combined with Lemma 2.14 will imply Conjecture 2.18.

3. Proof of Theorem 1

In this section we follow the notation of [6], [7] and use matrix realizations of flag curves. (Such realizations were already used in the earlier papers by the authors).

Observe that we can assume that for any convex curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}P^{n-1}$ (or $\gamma\colon I \to \mathbb{R}P^{n-1}$), its osculating flag curve $\gamma_{\mathcal{F}}\colon \mathbb{S}^1 \to \mathrm{Fl}_n$ (resp. $\gamma_{\mathcal{F}}\colon I \to \mathrm{Fl}_n$) lies completely in some top-dimensional Schubert cell in Fl_n. To see that, depending on whether one considers the case of $\mathbb{S}^1$ or I, let us either fix an arbitrary point $\tau \in \mathbb{S}^1$ or the left endpoint $\tau \in I$. Take the flag $\gamma_{\mathcal{F}}(\tau) \in \mathrm{Fl}_n$ as the complete flag defining the top-dimensional Schubert cell in Fl_n. (In other words, we take all complete flags transversal to $\gamma_{\mathcal{F}}(\tau)$.) By Lemma 2.13, for any $\nu \in \mathbb{S}^1$ (resp. $\nu \in I$) different from τ, the flags $\gamma_{\mathcal{F}}(\tau)$ and $\gamma_{\mathcal{F}}(\nu)$ are transversal, which means that the latter flag lies in the top-dimensional Schubert cell in Fl_n with respect to the former flag. Thus the whole flag curve $\gamma_{\mathcal{F}}$, except for one point $\gamma_{\mathcal{F}}(\tau)$, lies in this top-dimensional cell.

Top-dimensional cells in Fl_n are standardly identified with Lo_n^1, where Lo_n^1 is the nilpotent Lie group of real lower triangular $n \times n$ matrices with all diagonal entries equal to 1. This group can be interpreted as the tangent space to Fl_n at any fixed chosen flag f. Alternatively, the usual LU and QR decompositions define diffeomorphisms $\mathbf{Q}\colon \mathrm{Lo}_n^1 \to \mathcal{U}_1$ and $\mathbf{L}\colon \mathcal{U}_1 \to \mathrm{Lo}_n^1$, where $\mathcal{U}_1 \subset \mathrm{Fl}_n$ is a top dimensional cell (see [7]).

Recall that the LU decomposition of an invertible matrix $A \in \mathrm{GL}_n$ is a pair of matrices L and U such that $A = \mathrm{LU}$, $L \in \mathrm{Lo}_n^1$ is lower triangular with diagonal entries equal to 1 and $U \in \mathrm{Up}_n$ is upper triangular (and invertible). There exists a neighborhood of the identity matrix where such a decomposition is smoothly and uniquely defined. The set $\mathcal{U}_1 \subset \mathrm{SO}_n$ is the intersection of this open neighborhood with the subgroup SO_n. We abuse notation by writing either $\mathcal{U}_1 \subset \mathrm{Fl}_n$, $\mathcal{U}_1 \subset \mathrm{SO}_n$ or $\mathcal{U}_1 \subset \mathrm{Spin}_n$ since the manifolds on the right hand side are locally identified: Spin_n is a double cover of SO_n and SO_n is a finite cover of Fl_n. Also, a QR decomposition of an invertible matrix $A \in \mathrm{GL}_n$ is a pair of matrices Q and R such that $A = \mathrm{QR}$, $Q \in O_n$ is orthogonal and $R \in \mathrm{Up}_n^+$ is upper triangular with positive diagonal entries: this decomposition is smoothly and uniquely defined in GL_n. These decompositions allow us to define the diffeomorphisms $\mathbf{Q}\colon \mathrm{Lo}_n^1 \to \mathcal{U}_1$ and $\mathbf{L}\colon \mathcal{U}_1 \to \mathrm{Lo}_n^1$.

The following statement can be found for example in [12], [6], [7]. Recall that $\mathcal{J}$ is the set of $n \times n$ real matrices A such that the entry A_{ij} is positive if $i = j+1$ and 0 otherwise. In other words, $A \in \mathcal{J}$ if and only if A has strictly positive subdiagonal entries (i.e., entries in positions $(j+1, j)$) and zero entries elsewhere.

Lemma 3.1. *Consider an interval $I \subseteq \mathbb{R}$ and a smooth curve $\Gamma\colon I \to \mathrm{Lo}_n^1$. Then Γ is the osculating flag curve of a convex projective curve $\gamma\colon I \to \mathbb{R}P^{n-1}$ (i.e., $\gamma_{\mathcal{F}} = \mathbf{Q} \circ \Gamma$, where $\mathbf{Q}\colon \mathrm{Lo}_n^1 \to \mathcal{U}_1 \subset \mathrm{Fl}_n$ is the above diffeomorphism) if and only if, for every $t \in I$, the logarithmic derivative $(\Gamma(t))^{-1}\Gamma'(t)$ belongs to the set $\mathcal{J}$.*

Let us call the osculating flag curves obtained by taking the flags osculating convex projective curves *flag-convex* (or sometimes just *convex*). In other words, a curve $\Gamma\colon I \to \mathrm{SO}_n$ is *convex* if and only if there exists a convex curve $\gamma\colon I \to \mathrm{SO}_n$ with $\Gamma = \gamma_{\mathcal{F}}$. It turns out that $\Gamma\colon I \to \mathrm{SO}_n$ is convex if and only if there exists $Q_0 \in \mathrm{SO}_n$ such that $Q_0\Gamma(t) \in \mathcal{U}_1$ for all t in the interior of I. (Compare with Lemma 2.13; see also [7]). We sometimes abuse notation by identifying Lo_n^1 with $\mathcal{U}_1$ (through the diffeomorphism $\mathbf{Q}$) and therefore Γ with $\mathbf{Q} \circ \Gamma$.

Given a flag-convex curve $\Gamma\colon I \to \mathrm{Lo}_n^1$, define the function $m_{\Gamma,k} = m_k\colon I \to \mathbb{R}$, given by

$$m_k(t) = \det(\mathrm{swminor}(\Gamma(t), k)).$$

Recall that $\mathrm{swminor}(L, k)$ is the $k \times k$ submatrix of L formed by its last k rows and its first k columns.

Observe now that if we interpret Lo_n^1 as the top-dimensional cell in Fl_n with respect to some fixed flag $g \in \mathrm{Fl}_n$, then the moments of non-transversality of the flag curve Γ with the $(n-k)$-dimensional linear subspace belonging to g are exactly the zeros of $m_k(t)$.

Thus, Conjecture 1.2 is equivalent to saying that for any n, for any $k \leqslant n$, and for any flag-convex curve $\Gamma\colon I \to \mathrm{Lo}_n^1$, the number of real zeroes of the function $m_k(t)$, where $t \in I$, is at most $k(n-k)$.

Remark 3.2. Consider a flag-convex curve $\Gamma\colon I \to \mathrm{Lo}_n^1$. Set $-I = \{t; -t \in I\}$; let P_η be the permutation matrix corresponding to the top permutation η so that

$(P_\eta)_{i,n+1-i} = 1$ $(1 \leqslant i \leqslant n)$, with the other entries equal to zero. Define

$$\Gamma_\star \colon -I \to \mathrm{Lo}^1_n, \quad \Gamma_\star(t) = P_\eta(\Gamma(-t))^{-\top} P_\eta;$$

a straightforward computation verifies that the curve $\Gamma_\star$ is also flag-convex. We also have

$$\begin{aligned} m_{\Gamma_\star,k}(t) &= \pm \det(\Gamma_\star(t)e_1, \dots, \Gamma_\star(t)e_k, e_1, \dots, e_{n-k}) = \\ &= \pm \det\big(e_1, \dots, e_k, (\Gamma_\star(t))^{-1}e_1, \dots, (\Gamma_\star(t))^{-1}e_{n-k}\big) = \pm m_{\Gamma,n-k}(-t). \end{aligned}$$

Thus, Conjecture 1.2 for $k = k_0$ implies the same conjecture for $k = n - k_0$.

Define the open and dense subset $\mathrm{Lo}^o_n \subset \mathrm{Lo}^1_n$ given by

$$\mathrm{Lo}^o_n = \{X \in \mathrm{Lo}^1_n \mid \forall k \in [1, n-1],\, m_k(X) \neq 0\}.$$

Remark 3.3. In the notation of [7], we have $\mathrm{Lo}^o_n = \bigsqcup_{q \in \mathrm{Quat}_n} \mathbf{Q}^{-1}[\mathrm{Bru}_{q\acute{\eta}}]$. The set Lo^o_n is a disjoint union of finitely many connected components. These connected components were counted in [16] and several follow-up papers. In particular, their number equals 2, 6, 20, 52 for $n = 2, 3, 4, 5$ resp. and it is equal to $3 \times 2^{n-1}$ for all $n \geqslant 6$.

We will specially distinguish two of these connected components. Recall that a matrix $L_0 \in \mathrm{Lo}^1_n$ is *totally positive* provided that, if a minor is nonzero for some $L \in \mathrm{Lo}^1_n$, then the corresponding minor is strictly positive for L_0 (see [14]); the set $\mathrm{Pos} \subset \mathrm{Lo}^1_n$ of totally positive matrices is a contractible connected component of Lo^o_n. Similarly, the set $\mathrm{Neg} \subset \mathrm{Lo}^1_n$ of totally negative matrices is another contractible connected component of Lo^o_n. For $L \in \mathrm{Lo}^1_n$, we have that $L \in \mathrm{Neg}$ if and only if $PLP \in \mathrm{Pos}$, where the diagonal matrix P is given by $P = \mathrm{diag}(1, -1, 1, \cdots, (-1)^{n-1})$. Equivalently, for $L \in \mathrm{Lo}^1_n$, $L \in \mathrm{Neg}$ if and only if $L^{-1} \in \mathrm{Pos}$. In Lemma 3.4 we provide an alternative characterization of the subsets $\mathrm{Pos}, \mathrm{Neg} \subset \mathrm{Lo}^1_n$; here $\mathrm{id} \in \mathrm{Lo}^1_n$ stands for the identity matrix.

Lemma 3.4 (see [12], [7]). *If $\Gamma \colon I \to \mathrm{Lo}^1_n$ is flag-convex and $\Gamma(0) = \mathrm{id}$ then $\Gamma(t) \in \mathrm{Pos}$ for $t > 0$ and $\Gamma(t) \in \mathrm{Neg}$ for $t < 0$. Conversely, if $L_1 \in \mathrm{Pos} \subset \mathrm{Lo}^1_n$ and $L_{-1} \in \mathrm{Neg} \subset \mathrm{Lo}^1_n$ then there exists a smooth flag-convex curve $\Gamma \colon \mathbb{R} \to \mathrm{Lo}^1_n$ such that $\Gamma(-1) = L_{-1}$, $\Gamma(0) = \mathrm{id}$ and $\Gamma(1) = L_1$.*

Recall that, for any locally convex curve $\gamma \colon I \to \mathbb{R}P^{n-1}$, we denote by $\gamma_{\mathcal{F}}(t)$ its osculating flag curve and by $\mathrm{osc}_2\,\gamma(t) \colon I \to G_{2,n}$ the osculating Grassmann curve obtained by taking the span of the first two columns of $\gamma_{\mathcal{F}}(t)$. Fix the subspace $L = \mathrm{span}\langle e_1, \dots, e_{n-2}\rangle \subset \mathbb{R}^n$ of codimension 2. Observe that the intersection of $\mathrm{osc}_2\,\gamma$ with H_L is given by the equation $m_2(t) = 0$. Here $H_L \subset G_{2,n}$ is the Grassmann hyperplane associate with the latter $(n-2)$-dimensional L.

In what follows, instead of considering the curve $\mathrm{osc}_2\,\gamma(t) \colon I \to G_{2,n}$ we present a related construction.

Let $\mathcal{C} = \mathrm{Lo}^1_{(2\times n)} \subset \mathbb{R}^{2\times n}$ be the space of real $(2 \times n)$ matrices X such that $X_{1,n-1} = X_{2,n} = 1$ and $X_{1,n} = 0$. There is a natural projection $\Pi \colon \mathrm{Lo}^1_n \to \mathcal{C}$ taking L to the submatrix formed by its last two rows: $(\Pi(L))_{i,j} = L_{i+n-2,j}$. Alternatively, $\Pi(L) = X_0 L$, where $X_0 = \Pi(\mathrm{id}) \in \mathcal{C}$ is the matrix whose only nonzero entries are

$(X_0)_{1,n-1} = (X_0)_{2,n} = 1$. Equivalently, let H_0, $H_1 \subset \mathrm{Lo}_n^1$ be the subgroups defined by

$$H_0 = \{L \in \mathrm{Lo}_n^1 \colon \forall (i,\, j),\ 1 \leqslant j < i \leqslant n,\ i > n-2 \to L_{i,j} = 0\};$$
$$H_1 = \{L \in \mathrm{Lo}_n^1 \colon \forall (i,\, j),\ 1 \leqslant j < i \leqslant n-2 \to L_{i,j} = 0\}.$$

If $L_0 \in H_0$, $L_1 \in H_1$ and $L = L_0 L_1$ then $L_{i,j} = (L_0)_{i,j}$ if $i \leqslant n-2$ and $L_{i,j} = (L_1)_{i,j}$ if $i > n-2$. Thus, any $L \in \mathrm{Lo}_n^1$ can be uniquely written as a product $L = L_0 L_1$ with $L_0 \in H_0$ and $L_1 \in H_1$. The restriction $\Pi|_{H_1} \colon H_1 \to \mathcal{C}$ is thus a bijection. The space $\mathcal{C}$ is naturally identified with $H_0 \backslash \mathrm{Lo}_n^1$, the set of right cosets of the form $H_0 L$, $L \in \mathrm{Lo}_n^1$; the map Π is now the natural quotient map $\mathrm{Lo}_n^1 \to H_0 \backslash \mathrm{Lo}_n^1$.

Below we will treat $X \in \mathcal{C}$ as an n-tuple of real column vectors: $X = (v_1,\, \ldots,\, v_n)$, $v_i \in \mathbb{R}^2$. In other words, $\mathcal{C} \subset (\mathbb{R}^2)^n$ is the set of n-tuples $X = (v_1,\, \ldots,\, v_n)$ satisfying

$$v_{n-1} = \begin{pmatrix} 1 \\ a \end{pmatrix}, \quad v_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for some $a \in \mathbb{R}$. Clearly, $m_2(L) = m_2(\Pi(L))$ for all $L \in \mathrm{Lo}_n^1$; the map m_2 is thus well-defined as $m_2 \colon \mathcal{C} \to \mathbb{R}$. For any set $Y = \{i < j\} \subset \{1,\, 2,\, \ldots,\, n\}$ (with $|Y| = 2$), we define a function $m_Y \colon \mathcal{C} \to \mathbb{R}$: for $X = (v_1,\, \ldots,\, v_n)$, set $m_Y(X) = \det(v_i,\, v_j)$. Notice that $m_2 = m_{\{1,2\}}$; for all such $Y = \{i,\, j\}$, set $\sum Y = i + j$.

A smooth curve $\Gamma_2 \colon I \to \mathcal{C}$ is called *flag-convex* if there exists a flag-convex curve $\Gamma \colon I \to \mathrm{Lo}_n^1$ such that $\Gamma_2 = \Pi \circ \Gamma$. The set $\mathcal{L}$ of flag-convex curves $\Gamma_2 \colon I \to \mathcal{C}$ has a smooth Banach manifold structure, inherited through Π from the set of flag-convex curves $\Gamma \colon I \to \mathrm{Lo}_n^1$. We are interested in proving that if Γ_2 is flag-convex then $m_2 \circ \Gamma_2$ has at most $2(n-2)$ real zeroes.

Define $\mathrm{Pos}_2 = \Pi[\mathrm{Pos}] \subset \mathcal{C}$ and $\mathrm{Neg}_2 = \Pi[\mathrm{Neg}] \subset \mathcal{C}$. Similarly, we say that $X \in \mathcal{C}$ is *totally positive* (resp. *negative*) if $X \in \mathrm{Pos}_2$ (resp. $X \in \mathrm{Neg}_2$). The following observation is straightforward.

Lemma 3.5. *A $2 \times n$-matrix $X \in \mathcal{C}$ lies in Pos_2 if and only if*

- *$X_{ij} > 0$ for all $i \in [1,\, 2]$, $j \in [1,\, n-2]$ and $i = 2$, $j = n-1$;*
- *$m_Y(X) > 0$ for all $Y \subset \{1,\, \ldots,\, n\}$, $|Y| = 2$.*

A $2 \times n$-matrix $X \in \mathcal{C}$ lies in Neg_2 if and only if

- *$(-1)^{(i+j)} X_{ij} > 0$ for all $i \in [1,\, 2]$, $j \in [1,\, n-2]$ and $i = 2$, $j = n-1$;*
- *$(-1)^{(1+\sum Y)} m_Y(X) > 0$ for all $Y \subset \{1,\, \ldots,\, n\}$, $|Y| = 2$.*

Interpret $\mathcal{C}$ as a set of n-tuples of vectors $v_i \in \mathbb{R}^2$. Let $\mathcal{C}_1 \subset \mathcal{C}$ be the open dense subset of such n-tuples such that, for all i, $v_i \neq 0$. Notice that the complement $(\mathcal{C} \backslash \mathcal{C}_1) \subset \mathcal{C}$ is a union of finitely many submanifolds of codimension 2. Consider the set $\mathcal{L}$ of flag-convex curves $\Gamma_2 \colon I \to \mathcal{C}$ and its subset $\mathcal{L}_1 \subset \mathcal{L}$ of curves with image contained in $\mathcal{C}_1$. Then $\mathcal{L}_1 \subset \mathcal{L}$ is also open and dense; moreover, the codimension of the complement is 1. In particular, an estimate on the number of zeroes of $m_2 \circ \Gamma_2$ for $\Gamma_2 \in \mathcal{L}_1$ automatically implies the same estimate for $\Gamma_2 \in \mathcal{L}$.

Let $\mathcal{C}_2 \subset \mathcal{C}_1$ be the open dense subset of n-tuples $X = (v_1,\, \ldots,\, v_n)$ such that $m_Y(X) = 0$ holds for at most one such set Y; let $\mathcal{C}_3 \subset \mathcal{C}_2$ be the open dense subset of n-tuples $X = (v_1,\, \ldots,\, v_n)$ such that, for all such sets Y, $m_Y(X) \neq 0$. In other words, for $X = (v_1,\, \ldots,\, v_n) \in \mathcal{C}_1$, we have $X \in \mathcal{C}_3$ if and only if the vectors $v_1,\, \ldots,\, v_n$ are pairwise linearly independent.

Notice that the complement of $\mathcal{C}_2$ is a union of finitely many submanifolds of codimension at least 2. Thus $\mathcal{C}_2$ is path-connected and generic flag-convex curves $\Gamma_2 \colon I \to \mathcal{C}$ are of the form $\Gamma_2 \colon I \to \mathcal{C}_2$. As we shall see, $\mathcal{C}_3$ has exactly $2^{n-2} \cdot (n-1)!$ connected components, all contractible. Connected components of $\mathcal{C}_3$ are labeled by signed cyclic words, as we explain below.

Namely, consider cyclic words w of length $2n$ in the alphabet $1, \ldots, n, 1', \ldots, n'$ such that w contains each letter exactly once. We say that such a word w is *admissible* (or *odd*) if, for every i, there are exactly $n-1$ other letters between the letters i and i': let $\mathfrak{W}$ denote the set of admissible words. For example, for $n = 3$, we have

$$\begin{aligned}\mathfrak{W} = \{&123'1'2'3,\ 12'3'1'23,\ 1'23'12'3,\ 1'2'3'123,\\ &213'2'1'3,\ 2'13'21'3,\ 21'3'2'13,\ 2'1'3'213\}.\end{aligned}$$

In general, we have $|\mathfrak{W}| = 2^{n-1} \cdot (n-1)!$ (fix n and n'; choose one among the $(n-1)!$ permutations of $\{1, \ldots, n-1\}$ to fill in the gap between n and n'; for i from 1 to $n-1$ choose the positions of i and i').

Words in $\mathfrak{W}$ should be imagined as written along a circle, always counter-clockwise. Given $i < j$, we say that we walk from i to j counter-clockwise in w in there are fewer than $n-1$ letters after i and before j: in this case we write $m_{\{i,j\}}(w) > 0$ (otherwise $m_{\{i,j\}}(w) < 0$). Equivalently, $m_{\{i,j\}}(w) > 0$ if and only if one encounters the triple $i, \ldots, j, \ldots, i'$ when reading w (counter-clockwise); of course, $m_{\{i,j\}}(w) < 0$ if and only if one encounters instead the triple $i, \ldots, j', \ldots, i'$. Thus, for instance, if $n = 5$ and $w = 13'4'25'1'342'5$ then $m_{\{1,2\}}(w) > 0$, $m_{\{1,3\}}(w) < 0$, $m_{\{1,4\}}(w) < 0$, $m_{\{1,5\}}(w) < 0$, $m_{\{2,3\}}(w) > 0$, $m_{\{2,4\}}(w) > 0$, $m_{\{2,5\}}(w) < 0$, $m_{\{3,4\}}(w) > 0$, $m_{\{3,5\}}(w) > 0$ and $m_{\{4,5\}}(w) > 0$. If w is a word, we assume that $m_Y(w) \in \{\pm 1\}$. Let $\mathfrak{W}^+ \subset \mathfrak{W}$ be the set of admissible words w for which $m_{\{n-1,n\}}(w) > 0$; we have $|\mathfrak{W}^+| = 2^{n-2} \cdot (n-1)!$.

We now show how to assign a word $w(X) \in \mathfrak{W}^+$ to each $X \in \mathcal{C}_3$. Given $X = (v_1, \ldots, v_n)$, set $\nu(X) = (\hat{v}_1, \ldots, \hat{v}_n) \in (\mathbb{S}^1)^n$, where $\hat{v}_i = v_i/|v_i| \in \mathbb{S}^1$. Let $\Omega_n = \nu[\mathcal{C}_3]$; in other words, $\Omega_n \subset (\mathbb{S}^1)^n$ is the set of configurations of n pairwise linearly independent labeled points on $\mathbb{S}^1$ such that point n is $(0, 1)$ and point $n-1$ has coordinates (x, y) with $x > 0$. Given $X = (v_1, \ldots, v_n)$, label the point $\hat{v}_i$ by i and the point $-\hat{v}_i$ by i'. Finally, traverse the unit circle $\mathbb{S}^1$ counter-clockwise, picking up the labels as you read, to obtain the desired word $w(X)$. Notice that $m_Y(X) > 0$ if and only if $m_Y(w(X)) > 0$; in particular, $w(X) \in \mathfrak{W}^+$, as desired.

Remark 3.6. The above discussion implies that the cyclic word $w(M)$ corresponding to any totally positive $2 \times n$-matrix M coincides with the cyclic word given by $(1, 2, \ldots, n, 1', 2', \ldots, n')$. We will call this particular cyclic word *totally positive*. The cyclic word corresponding to a totally negative matrix is obtained from the totally positive word by interchanging its every even entry with the opposite and reading it backwards. We will call this cyclic word *totally negative*.

Example 3.7. The cyclic word $(12341'2'3'4')$ is totally positive while the cyclic word $(43'21'4'32'1)$ is totally negative.

In all figures in the remaining part of this section the cyclic words should be read counter-clockwise along the circle.

Given $w_0 \in \mathfrak{W}^+$, let $\mathcal{C}[w_0] \subset \mathcal{C}_3$ be the set of matrices $X \in \mathcal{C}_3$ for which $w(X) = w_0$. Notice that $\mathcal{C}_3 = \bigsqcup_{w \in \mathfrak{W}^+} \mathcal{C}[w]$; the following lemma shows that these subsets are all well-behaved.

Lemma 3.8. *Given $w_0 \in \mathfrak{W}^+$, the set $\mathcal{C}[w_0]$ is contractible (and nonempty).*

Proof. We proceed by induction on n; the cases $n \leqslant 3$ are easy. In this proof, we write $\mathfrak{W}_n^+$ in order to avoid confusion.

Given a word $w_0 \in \mathfrak{W}_n^+$, let $w_1 \in \mathfrak{W}_{n-1}^+$ be obtained by removing 1 and $1'$ from w_0 and then subtracting 1 from each remaining label (thus, for instance, if $w_0 = 13'452'1'34'5'2$ then $w_1 = 2'34'1'23'4'1$). Similarly, given $X_0 = (v_1, v_2, \ldots, v_n) \in \mathrm{Lo}^1_{(2\times n)}$ we obtain $X_1 = (v_2, \ldots, v_n) \in \mathrm{Lo}^1_{(2\times(n-1))}$ by removing the first column.

By induction hypothesis, $\mathcal{C}[w_1]$ is contractible. Given $X_1 \in \mathcal{C}[w_1]$, the set of vectors $v_1 \in \mathbb{R}^2$ which can be placed at the left of X_1 to obtain $X_0 \in \mathcal{C}[w_0]$ is a convex cone. Thus $\mathcal{C}[w_0]$ is also contractible, as desired. $\square$

Given a flag-convex curve $\Gamma_2 \colon I \to \mathcal{C}_2$, we are now interested in following the sequence of words w corresponding to the sets $\mathcal{C}[w]$ traversed by Γ_2. We first present a combinatorial description. Let us now define *admissible moves* on the set of all admissible cyclic words. (Below $\ldots$ stands for an arbitrary sequence of labels in an admissible word.)

Definition 3.9. Below, we denote by α either the label j or its opposite label j'. If $\alpha = j$, then $-\alpha = j'$, if $\alpha = j'$ then $-\alpha = j$. For every $k > 2$, the following two moves are called *admissible*:

1) clockwise rotation of label $(k-1)$ toward point k

$$\ldots, k', \ldots, k-1, j, \ldots, k, \ldots, (k-1)', j', \ldots \\ \to \ldots, k', \ldots, j, k-1, \ldots, k, \ldots, j', (k-1)', \ldots$$

2) counter-clockwise rotation of label $(k-1)$ toward label k

$$\ldots, k, \ldots, j, k-1, \ldots, k', \ldots, j', (k-1)', \ldots \\ \to \ldots, k, \ldots, k-1, j, \ldots, k', \ldots (k-1)', j', \ldots$$

The first admissible move describes the change of a cyclic word when the label $(k-1)$ rotates clockwise toward the label k and passes the position of the label j (or j') while the second admissible move describes the similar change when the label $(k-1)$ rotates counterclockwise.

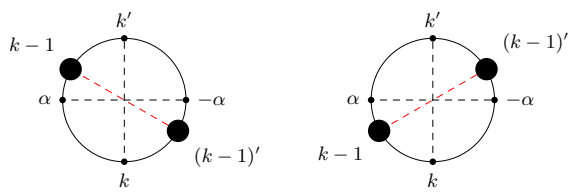

FIGURE 1. The first admissible move. The label $k-1$ rotates counter-clockwise towards the label k.

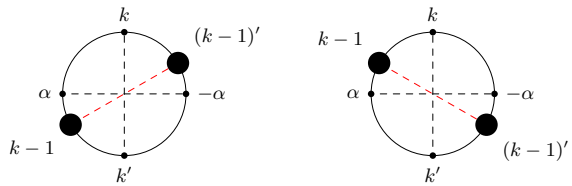

FIGURE 2. The second admissible move. The label $k-1$ rotates clockwise towards the label k.

For any flag-convex curve $\Gamma_2\colon I \to \mathcal{C}_2$ there are finitely many $t \in I$ for which $\Gamma_2(t) \notin \mathcal{C}_3$. Indeed, a flag-convex curve transversally intersects any hypersurface which is a connected component of $\mathcal{C}_2 \setminus \mathcal{C}_3$; a stronger result is proved in [7].

Lemma 3.10. *Given a flag-convex curve $\Gamma_2\colon I \to \mathcal{C}_2$, consider the sequence of words $w \in \mathfrak{W}^+$ for which Γ_2 traverses $\mathcal{C}[w]$. Then this sequence of words consists of admissible moves.*

Proof. By Lemma 3.1, the tangent vector to the curve Γ_2 at t_0 belongs to the cone spanned by the vectors $\Gamma_2(t_0)\mathfrak{l}_j$, where $\mathfrak{l}_j$ is the $n \times n$ matrix whose only nonzero entry is located at the position $(j+1, j)$ and is equal to 1. Note that the right multiplication of an arbitrary $k \times n$-matrix by $(\mathrm{id} + s\,\mathfrak{l}_j)$ acts as a column operation adding s times the $(j+1)$st column to the jth column. This corresponds to moving (infinitesimally) the point labelled j towards the point labelled $j+1$ along the shortest of the two arcs of $\mathbb{S}^1$ connecting them. Since any infinitesimal motion of the point configuration induced by the curve Γ_2 is represented as a positive linear combination of such infinitesimal elementary moves we can approximate the whole time evolution of the point configuration as a sequence of consecutive elementary moves described in Definition 3.9. $\square$

Example 3.11. Set

$$\Gamma(t) = L_0 \exp(tN), \qquad L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 1/8 & 1/5 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The curve $\Gamma\colon \mathbb{R} \to \mathrm{Lo}_4^1$ is flag-convex. A simple computation verifies that the flag-convex curve $\Gamma_2\colon \mathbb{R} \to \mathcal{C}$ is of the form $\Gamma_2\colon \mathbb{R} \to \mathcal{C}_2$. Indeed, the values of t for which $m_Y(\Gamma_2(t)) = 0$ for some Y are: $t_1 \approx -0.63$ (for $Y = \{2, 3\}$); $t_2 = 0$ (for $Y = \{2, 4\}$); $t_3 \approx 0.26$ (for $Y = \{1, 2\}$); $t_4 \approx 0.63$ (for $Y = \{2, 3\}$); $t_5 \approx 0.77$ (for $Y = \{1, 3\}$); $t_6 \approx 1.11$ (for $Y = \{1, 2\}$). The corresponding sequence of words is: $w_0 = 143'21'4'32'$, $w_1 = 1423'1'4'2'3$, $w_2 = 31243'1'2'4'$, $w_3 = 32143'2'1'4'$, $w_4 = 23142'3'1'4'$, $w_5 = 21342'1'3'4'$, $w_6 = 12341'2'3'4'$; moves are admissible, as expected.

As discussed above, the set of all admissible words labels the set of all connected components of $\mathcal{C}_3$ (or of Ω_n). The connected components are separated by codimension 1 walls in $\mathcal{C}_2 \setminus \mathcal{C}_3$. Admissible moves correspond to crossing walls between connected components following flag-convex curves.

Remark 3.12. We explained above that when a (locally) convex curve intersects the divisor H_L, where $L = \operatorname{span}\langle e_1, \dots, e_{n-2}\rangle \subset \mathbb{R}^n$, the respective admissible configuration of labelled points on $\mathbb{S}^1$ is acted upon by an admissible move which either interchanges the relative order of the points labelled 1 and 2 or the points 1 and $2'$.

For an admissible cyclic word w and any of its two distinct entries a and b (belonging to $\{1, \dots, n, 1', \dots, n'\}$), we denote by $[\![a, b]\!]$ the shortest closed arc in $\mathbb{S}^1$ starting at the point labeled a and ending at the point labeled b. In other words, of two possible arcs connecting a and b, we choose the one whose length does not exceed half a turn (or $(n-1)$ positions in the word w).

We call an arc $[\![a, b]\!]$ (*increasing*) *decreasing* if $a, b \in \{1, \dots, n\}$, $a < b$ and b is obtained from a by a rotation through less than a half-turn in the (counter-)clockwise direction. Given $i < j$, we say that an admissible cyclic word w *contains a monotone subsequence* $[\![i \dots j]\!] = [\![i, i+1, i+2, \dots, j]\!]$ if, for $i \leqslant k \leqslant j-1$, all of the arcs $[\![k, k+1]\!]$ are either simultaneously increasing or simultaneously decreasing. A monotone sequence $[\![i \dots j]\!]$ can be interpreted as an immersed arc $I \to \mathbb{S}^1$, $I = [i, j]$, taking $i \in I$ to the point of $\mathbb{S}^1$ labeled i and taking $j \in I$ to the point labeled j; notice that, unlike the subarcs $[\![k, k+1]\!]$, such an immersed arc can be (much) longer than a half-turn. The *content* of an immersed arc in $\mathbb{S}^1$ is the number of complete half-turns contained in the arc; we denote the content of a monotone sequence $[\![i \dots j]\!]$ by $\operatorname{Cont}([\![i \dots j]\!])$.

We call a monotone subsequence $[\![i \dots j]\!]$ *maximal* if neither $[\![i-1 \dots j]\!]$ for $i > 1$ nor $[\![i \dots j+1]\!]$ for $j < n$ is monotone. An admissible word w can be interpreted as the concatenation of its maximal monotone arcs $[\![1 \dots k_1]\!]$, $[\![k_1 \dots k_2]\!]$, $\dots$, $[\![k_{s-1} \dots n]\!]$; here $k_0 = 1$, $k_s = n$ and s is the total number of maximal monotone subsequences in w. The *total content* $\operatorname{Cont}(w)$ of an admissible word w is $\operatorname{Cont}(w) = \operatorname{Cont}([\![1 \dots k_1]\!]) + \dots + \operatorname{Cont}([\![k_{s-1} \dots n]\!])$, the sum of the contents of all maximal monotone subsequences of w.

Definition 3.13. We define $\operatorname{rk}(w)$, the *rank* of the admissible word w, by

$$\operatorname{rk}(w) = 2\operatorname{Cont}(w) + s - 1.$$

For any matrix $X \in \mathcal{C}_3$, we define its rank $\operatorname{rk}(X) = \operatorname{rk}(w(X))$.

Example 3.14. Consider the following cyclic words:

(1) for $w_1 = 123451'2'3'4'5'$, we get $s = 1$, $\operatorname{Cont}(w) = 0$, and $\operatorname{rk}(w) = 0$;
(2) for $w_2 = 15'43'21'54'32'$, we get $s = 1$, $\operatorname{Cont}(w) = 3$, and $\operatorname{rk}(w) = 6$;
(3) in $w_3 = 145231'4'5'2'3'$ there are $s = 3$ maximal monotone subsequences: $[\![1, 2, 3]\!]$, $[\![3, 4]\!]$ and $[\![4, 5]\!]$. Hence, $\operatorname{Cont}(w) = 0$, and $\operatorname{rk}(w) = 2$;
(4) for $w_4 = 415234'1'5'2'3'$, one gets $s = 3$, $\operatorname{Cont}(w) = 0$ and $\operatorname{rk}(w) = 2$.

Remark 3.15. The word w_1 is totally positive, while w_2 is totally negative. The word w_4 is obtained from w_3 by one admissible move which shifts 1 closer to 2; w_3

can be obtained from w_4 by an admissible move which shifts 4 closer to 5. Observe that $\mathrm{rk}(w_3) = \mathrm{rk}(w_4)$.

Lemma 3.16. *Fix n so that admissible words $w \in \mathfrak{W}^+$ have length $2n$. For the totally positive word $w_+ \in \mathfrak{W}^+$, we have $\mathrm{rk}(w_+) = 0$. For the totally negative word $w_- \in \mathfrak{W}^+$, we have $\mathrm{rk}(w_-) = 2(n-2)$. For any other admissible cyclic word $w \in \mathfrak{W}^+ \setminus \{w_+, w_-\}$, we have $0 < \mathrm{rk}(w) < 2(n-2)$. Furthermore, $m_{\{1,2\}}(w) > 0$ if and only if $\mathrm{rk}(w)$ is even.*

Proof. The proof is by induction on n; the cases $n \leqslant 3$ are easy.

Given $w_0 \in \mathfrak{W}_n^+$, let $w_1 \in \mathfrak{W}_{n-1}^+$ be obtained by removing 1 and $1'$ and by decreasing by 1 the remaining labels (as in the proof of Lemma 3.8). Let $[\![1 \dots k_1]\!]$ be the first maximal monotonic subarc for w_0. Notice that $k_1 = 2$ if and only if $m_{\{1,2\}}(w_0) \neq m_{\{2,3\}}(w_0)$ (recall that $m_Y(w) \in \{\pm 1\}$). On the other hand, if $m_{\{1,2\}}(w_0) = m_{\{2,3\}}(w_0)$ then the first maximal monotonic arc for the word w_1 is $[\![1 \dots (k_1 - 1)]\!]$. Let $c_0 = \mathrm{Cont}([\![1 \dots k_1]\!])$ and $c_1 = \mathrm{Cont}([\![2 \dots k_1]\!])$ (both for w_0). Notice that $c_1 = \mathrm{Cont}([\![1 \dots (k_1 - 1)]\!])$ for w_1. We have either $c_0 = c_1$ (if, for w_0, the points labeled k_1 and k_1' are both outside the arc $[\![1, 2]\!]$) or $c_0 = c_1 + 1$ (otherwise). We thus have

$$\mathrm{rk}(w_0) = \mathrm{rk}(w_1) + \begin{cases} 0, & m_{\{1,2\}}(w_0) = m_{\{2,3\}}(w_0),\ c_0 = c_1, \\ 1, & m_{\{1,2\}}(w_0) \neq m_{\{2,3\}}(w_0), \\ 2, & m_{\{1,2\}}(w_0) = m_{\{2,3\}}(w_0),\ c_0 = c_1 + 1; \end{cases}$$

this provides us with the desired induction step. □

The next statement is the most important technical step in our proof of Theorem 1. The argument is simple but a little long, and is done case by case; it is presented and illustrated by the series of ten figures shown below.

Proposition 3.17. *Consider $w_0, w_1 \in \mathfrak{W}_n^+$. Assume that an admissible move takes w_0 to w_1: then $\mathrm{rk}(w_1) \leqslant \mathrm{rk}(w_0)$. Furthermore, if $m_{\{1,2\}}(w_1) \neq m_{\{1,2\}}(w_0)$ then $\mathrm{rk}(w_1) < \mathrm{rk}(w_0)$.*

Notice that the last claim follows from the first claim together with the parity remark in Lemma 3.16.

Proof. Below we present all possible types of elementary moves and, for each of them, we analyze what happens with the rank of w. Observe that during the evolution of the point configuration in $(\mathbb{S}^1)^n$ following some curve Γ_2 the rank of configuration does not change until two or more configuration points collide. We consider admissible moves and list all cases when a moving point labelled i collides with one of the remaining points labelled j or j'. Detailed consideration of all possible cases led us to their subdivision into the following types of collisions. (This subdivision is an artifact of our proof).

Type Ia: the moving point 1 collides with the point k, $k > 2$;
Type Ib: the moving point 1 collides with the point k', $k > 2$;

Type IIa: the moving point i, $i > 1$ collides with the point 1;
Type IIb: the moving point i, $i > 1$ collides with the point $1'$;
Type IIIa: the moving point i collides with j when both i, $j > 1$, $j \neq i - 1$;

If $j = i - 1$ the case needs to be subdivided into two subcases by the location of point $i - 2$ in one of the following two intervals:
Type IIIb: the moving point i collides with $j = i - 1$, $i > 1$ and the point $i - 2$ belongs to the shortest arc between $(i - 1)'$ and i;
Type IIIc: the moving point i collides with $j = i - 1$, $i > 1$ and the point $i - 2$ belongs to the shortest arc between $i - 1$ and i'.
Type IVa: the moving point i collides with j', i, $j > 1$, $j \neq i - 1$;

If $j = i - 1$ then case needs to be subdivided into two subcases also by the location of point $i - 2$ in one of the following two intervals:
Type IVb: the moving point i collides with $j = i - 1$, $i > 1$ and the point $i - 2$ belongs to the shortest arc between $(i - 1)$ and i;
Type IVc: the moving point i collides with $j = i - 1$, $i > 1$ and the point $i - 2$ belongs to the shortest arc between $(i - 1)'$ and i'.

The above types exhaust all possible situations of collision and we discuss below what happens with the rank function under these collisions.

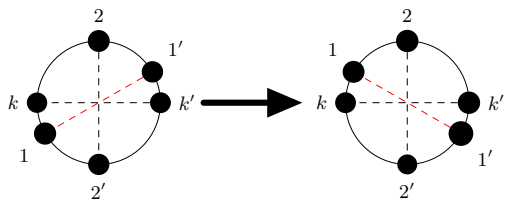

FIGURE 3. Elementary admissible move of type Ia with $k > 2$.

The move changes the relative order of points 1 and k in cyclic word w. Rank $\mathrm{rk}(w)$ does not change.

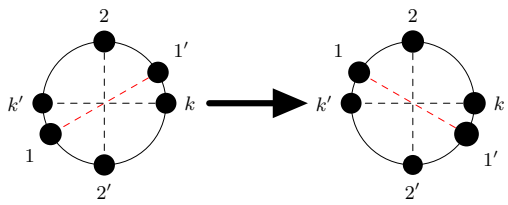

FIGURE 4. Elementary admissible move of type Ib.

The move changes the relative order of points 1 and k' in the word w. If the maximal element k_1 of the first maximal monotone subsequence $[1, 2, \ldots, k_1]$ is different from k, then $\mathrm{rk}(w)$ does not change. If $k_1 = k$, then $\mathrm{rk}(w)$ decreases by 1.

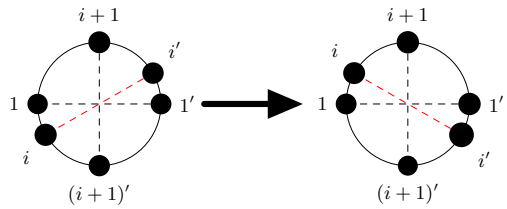

FIGURE 5. Elementary admissible move of type IIa.

If $i > 2$ and the first monotone subsequence is $\mathbf{seq}_1 = [1, 2, \ldots, k_1]$ with $k_1 \neq i$, then $\mathrm{rk}(w)$ does not change. If $i > 2$ and $k_1 = i$, then $\mathbf{seq}_1$ is increasing (since otherwise, $k_1 \geqslant i+1$) and $\mathrm{Cont}(w)$ decreases by 1, hence $\mathrm{rk}(w)$ drops by 2. Finally, if $i = 2$, then $\mathrm{rk}(w)$ drops by 1.

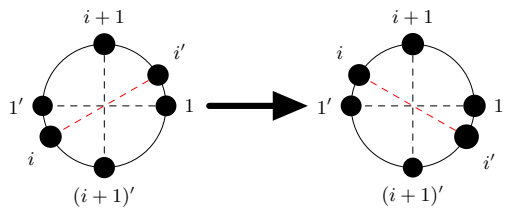

FIGURE 6. Elementary admissible move of type IIb.

As in type III, if $i > 2$ and the first monotone subsequence is $\mathbf{seq}_1 = [1, 2, \ldots, k_1]$ with $k_1 \neq i$, then $\mathrm{rk}(w)$ does not change. If $i > 2$ and $k_1 = i$, then $\mathbf{seq}_1$ is increasing (since otherwise, $k_1 \geqslant i+1$) and $\mathrm{rk}(w)$ drops by 2. Finally, if $i = 2$, then $\mathrm{rk}(w)$ drops by 1.

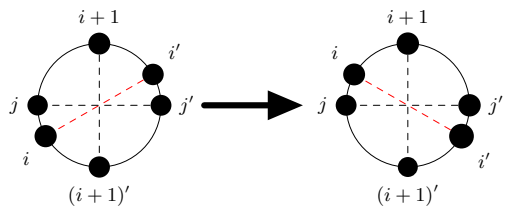

FIGURE 7. Elementary admissible move of type IIIa with $i, j > 1$ and $j \notin \{i, i+1\}$.

In this case $\mathrm{rk}(w)$ can only change if either $i < j$ and there exists an increasing maximal monotone subsequence $[i, i+1, \ldots, j]$ or if $i > j$ and there exists a decreasing maximal monotone subsequence $[j, j+1, \ldots, i]$. In both cases, $\mathrm{Cont}(w)$ decreases by 1 and $\mathrm{rk}(w)$ decreases by 2. The remaining situation $j = i-1$ is considered in detail below.

We split the case $j = i-1$ in Figure 7 into several subcases according to the relative position of $i-2$. The point $i-2$ can be located either in the interval $((i-1)', i)$ or in $(i-1, i')$, as below.

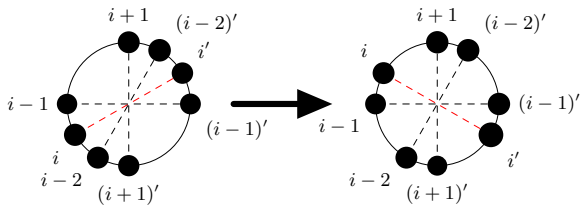

FIGURE 8. Elementary admissible move of type IIIb.

The point $i-2$ belongs to $((i-1)', i)$. $\mathrm{Cont}(w)$ does not change. Both the number of maximal monotone subsequences and $\mathrm{rk}(w)$ decrease by 2.

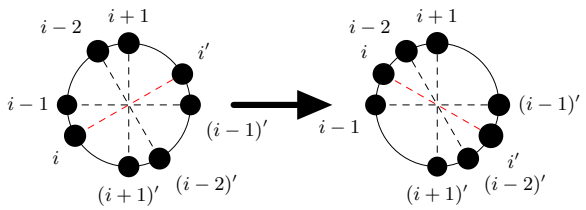

FIGURE 9. Elementary admissible move of type IIIc.

The point $i-2$ belongs to $(i-1, i')$. The rank $\mathrm{rk}(w)$ does not change.

Finally,

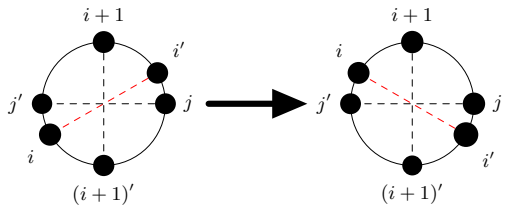

FIGURE 10. Elementary admissible move of type IVa.

Here $i, j > 1$, and j is not in $\{i, i+1\}$. If $j \neq i-1$, then $\mathrm{rk}(w)$ does not change unless either $i < j$ and there exists a maximal decreasing subsequence $[i, i+1, \ldots, j]$ or $i > j$ and there exists a maximal increasing subsequence $[j, j+1, \ldots, i]$. In both cases $\mathrm{Cont}(w)$ decreases by 1 and $\mathrm{rk}(w)$ decreases by 2. The remaining sitiuation $j = i-1$ is considered in detail below.

As above, we split the case $j = i-1$ in the last figure into several subcases according to the relative position of $i-2$. The point $i-2$ can be located either in the interval $((i-1), i)$ or in the interval $((i-1)', i')$, as below.

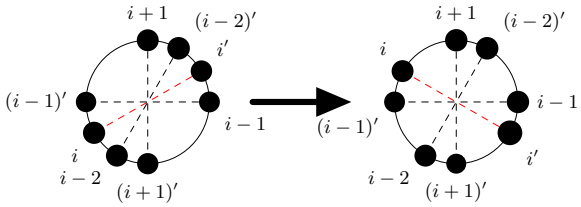

FIGURE 11. Elementary admissible move of type IVb.

The point $i-2$ belongs to $(i-1, i)$. $\mathrm{Cont}(w)$ and the number s of maximal monotone subsequences do not change. Hence, $\mathrm{rk}(w)$ does not change either.

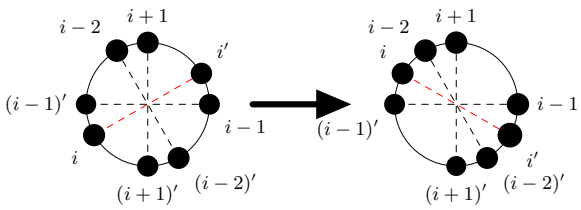

FIGURE 12. Elementary admissible move of type IVc.

The point $i-2$ belongs to $((i-1)', i')$. Either $\mathrm{rk}(w)$ does not change or it decreases by 2.

We have analyzed all the possible types of admissible elementary moves and concluded that, for each admissible move which changes the sign of $m_{\{1,2\}}$, $\mathrm{rk}(w)$ decreases. □

Recall the unipotent matrix N in Example 3.11; notice that

$$(\exp(tN))_{i,j} = \begin{cases} 0, & i < j; \\ t^{i-j}/(i-j)!, & i \geqslant j. \end{cases}$$

Lemma 3.18. *For any $n \times n$-matrix $G \in \mathrm{Lo}_n^1$,*

(1) *there exists $t_+ > 0$ such that $G\exp(tN)$ is totally positive for any $t > t_+$;*
(2) *there exists $t_- < 0$ such that $G\exp(tN)$ is totally negative for any $t < t_-$.*

Proof. Write $T(t) = \exp(tN)$. Note that $G_{ij} = 1$ if $i = j$ and $G_{ij} = 0$ if $i < j$. Hence,

$$(GT(t))_{ij} = T_{ij}(t) + \text{lower order terms in } t.$$

Any minor m_T of $T(t)$ equals $m_T(t) = at^d$ for some positive a and d. The corresponding minor m_G of $G \cdot T(t)$ equals $m_G(t) = at^d + p_{m,G}(t)$, where $p_{m,G}$ is a polynomial of degree strictly less than d. Hence, for t such that $|t| \gg 0$ the sign of $m_G(t)$ coincides with that of $m_T(t)$. It remains to notice that $T(t)$ is totally positive for positive t and totally negative for negative t and the lemma follows. □

Corollary 3.19. *Let $\gamma\colon I \to \mathbb{R}P^{n-1}$ be a globally convex curve and $\gamma_{\mathcal{F}}\colon I \to \mathrm{Lo}_n^1$ be its osculating flag curve (considered in the appropriate open Schubert cell identified with Lo_n^1). Then, γ can be extended to a globally convex $\tilde{\gamma}\colon [a,\, b] \to \mathbb{R}P^{n-1}$, $I \subset [a,\, b]$ such that $\tilde{\gamma}_{\mathcal{F}}\colon [a,\, b] \to \mathrm{Lo}_n^1$, $\tilde{\gamma}_{\mathcal{F}}(a) \in \mathrm{Neg}$, $\tilde{\gamma}_{\mathcal{F}}(b) \in \mathrm{Pos}$.*

Proof. Take $I = [s,\, f]$ and set $b = f + t_+(\gamma_{\mathcal{F}}(f))$, $a = s + t_-(\gamma_{\mathcal{F}}(s))$. Define

$$\tilde{\gamma}_{\mathcal{F}}(t) = \begin{cases} \gamma_{\mathcal{F}}(t), & \text{for } t \in I; \\ \gamma_{\mathcal{F}}(f) \cdot T(t-f), & \text{for } t \in [f,\, b]; \\ \gamma_{\mathcal{F}}(s) \cdot T(t-s), & \text{for } t \in [a,\, s], \end{cases}$$

where $T(t)$ is defined in Lemma 3.18.

We define $\tilde{\gamma}(t)$ by taking the first column of the matrix $\tilde{\gamma}_{\mathcal{F}}(t)$. Lemma 3.1 implies that the curve $\tilde{\gamma}(t)$ is globally convex.

Finally, by definition of t_- and t_+, one has that $\tilde{\gamma}_{\mathcal{F}}(a) \in \mathrm{Neg}$, $\tilde{\gamma}_{\mathcal{F}}(b) \in \mathrm{Pos}$.

Notice that the curve $\tilde{\gamma}_{\mathcal{F}}$ is usually not smooth. The curve can be perturbed to become smooth or we can work with a larger space of curves. These issues are amply discussed in several papers, including [6], [8]. □

Proof of Theorem 1. Consider a flag-convex curve $\Gamma_2\colon [a,\, b] \to \mathcal{C}_2$. From the previous results we may assume that $\Gamma_2(a) \in \mathrm{Neg}$, $\Gamma_2(b) \in \mathrm{Pos}$. Consider the associated word as a function of t: as t increases, we perform admissible moves. The rank of the associated word starts as $2(n-2)$ and ends as 0. The rank never increases and decreases by 1 at every zero of m_2; it may also decrease at other points. Thus, the number of zeros of m_2 is at most $2(n-2)$, as desired. □

4. Proof of Theorem 2

We start by introducing a special class of matrix curves. Namely, given a pair of matrices $(N_0,\, L_0)$, where N_0 is a nilpotent lower triangular matrix with positive subdiagonal entries and zero entries elsewhere, and $L_0 \in \mathrm{Lo}_n^1$, define the curve $\Gamma_{N_0,L_0}\colon \mathbb{R} \to \mathrm{Lo}_n^1$ as given by

$$\Gamma_{N_0,L_0}(t) = L_0 \exp(tN_0). \tag{4.1}$$

One can easily see that Γ_{N_0,L_0} is flag-convex and, for $i > j$, its entry $(i,\, j)$ is a polynomial of degree $i-j$. We call such flag-convex curves *polynomial*. (They are closely related to the fundamental solutions of the simplest differential equation $y^{(n)} = 0$.)

For a polynomial curve Γ_{N_0,L_0}, the function $m_k(t)$ is indeed a real polynomial of degree $k(n-k)$ in t. So for polynomial curves, Conjecture 1.2 trivially holds.

In this section we will first prove Proposition 4.1, a warm-up result, and then prove Theorem 2. Proposition 4.1 shows that there exist polynomial flag-convex curves which are non-transversal to the reference flag at $(n^3-n)/6$ distinct points, which implies that the estimates in Theorems 2 (and therefore also in Corollary 2.16) hold for polynomial curves.

Proposition 4.1. *Choose a nilpotent lower triangular matrix N_0 with positive subdiagonal entries and zero entries elsewhere. Then there exists $L_0 \in \mathrm{Lo}_n^1$ such that,*

for the polynomial curve Γ_{N_0,L_0} *given by Equation* (4.1) *and every* $k = 1, \dots, n-1$, *all roots of* $m_k(t)$ *are real and simple. Furthermore,* L_0 *can be taken so that all such roots are distinct, implying that there are totally exactly* $(n^3 - n)/6$ *such roots, all real and distinct.*

To settle Proposition 4.1 we need more notation. As in [7] (see especially Sections 2 and 7), let S_n be the symmetric group with generators $a_i = (i \quad i+1)$. The symmetric group is endowed with the usual Bruhat order. The top permutation (or the Coxeter element) of S_n is denoted by η (another common notation is w_0). For a permutation $\sigma \in S_n$, define its *multiplicity vector* $\mathrm{mult}(\sigma) \in \mathbb{Z}^{n-1}$ with coordinates $\mathrm{mult}_k(\sigma) = (1^\sigma + \cdots + k^\sigma) - (1 + \cdots + k)$; thus, $\mathrm{mult}_k(\eta) = k(n-k)$. If $\sigma_0 \triangleleft \sigma_1 = \sigma_0 a_j = (i_0 i_1)\sigma_0$, then

$$\mathrm{mult}_k(\sigma_1) = \begin{cases} \mathrm{mult}_k(\sigma_0) + 1, & i_0 \leqslant k < i_1, \\ \mathrm{mult}_k(\sigma_0), & \text{otherwise;} \end{cases} \tag{4.2}$$

this is Lemma 2.4 in [7].

For $\rho \in S_n$, the permutation matrix P_ρ has nonzero entries in positions (i, i^ρ) so that $e_i^\top P_\rho = e_{i^\rho}^\top$. Apply the Bruhat factorization to decompose Lo_n^1 as a disjoint union of subsets Bru_ρ, $\rho \in S_n$. More precisely, for $L \in \mathrm{Lo}_n^1$, write $L \in \mathrm{Bru}_\rho$ if and only if there exist upper triangular matrices U_1 and U_2 such that $L = U_1 P_\rho U_2$. In particular, $\mathrm{Bru}_e = \{I\}$ and Bru_η is open and dense. If $\Gamma\colon I \to \mathrm{Lo}_n^1$ is smooth and flag-convex, $\Gamma(0) \in \mathrm{Bru}_\rho$ and $\sigma = \eta\rho$ then $t = 0$ is a root of multiplicity $\mathrm{mult}_k(\sigma)$ of $m_k(t) = 0$; this is Theorem 4 in [7].

Recall that $\mathfrak{l}_j$ is the matrix whose only nonzero entry equals 1 in position $(j+1, j)$. Let $\lambda_j(t) = \exp(t\mathfrak{l}_j) \in \mathrm{Lo}_n^1$ so that $\lambda_j(t)$ has an entry equal to t in position $(j+1, j)$; the remaining entries equal 1 (on the main diagonal) and 0 (elsewhere). If $\rho_0 \triangleleft \rho_1 = \rho_0 a_j$, $L_0 \in \mathrm{Bru}_{\rho_0}$ and $t \neq 0$, then $L_1 = L_0 \lambda_j(t) \in \mathrm{Bru}_{\rho_1}$ (see Section 5 in [7]).

Consider N_0 arbitrary but fixed, as in the statement of Proposition 4.1. Given $L_0 \in \mathrm{Lo}_n^1$, construct the curve $\Gamma_{N_0,L_0}(t) = L_0 \exp(tN_0)$ and the real polynomials $m_k(t) \in \mathbb{R}[t]$ as above. We say that a matrix L_0 is *ρ-good* if and only if $L_0 \in \mathrm{Bru}_\rho \subset \mathrm{Lo}_n^1$ and, for all k, all nonzero roots of m_k are real and simple. Notice that Id is (vacuously) e-good.

Lemma 4.2. *Consider* $\rho_0, \rho_1 \in S_n$, $\rho_0 \triangleleft \rho_1 = \rho_0 a_j$. *Let* $L_0 \in \mathrm{Bru}_{\rho_0} \subset \mathrm{Lo}_n^1$ *be a* ρ_0*-good matrix. Then there exists* $\epsilon > 0$ *such that, for all* $\tau \in \mathbb{R}$ *satisfying the restriction* $0 < |\tau| < \epsilon$ *one has that* $L_1 = L_0 \lambda_j(\tau)$ *is* ρ_1*-good.*

Proof. As above, we have $L_1 \in \mathrm{Bru}_{\rho_1}$. For τ near 0, nonzero real simple roots of m_k remain nonzero, real and simple.

Let $\sigma_0 = \eta\rho_0$ and $\sigma_1 = \eta\rho_1$ so that $\sigma_1 \triangleleft \sigma_0 = \sigma_1 a_j = (i_0 i_1)\sigma_1$. As in Equation 4.2, $\mathrm{mult}_k(\sigma_1) = \mathrm{mult}_k(\sigma_0) - 1$ for $i_0 \leqslant k < i_1$ and $\mathrm{mult}_k(\sigma_1) = \mathrm{mult}_k(\sigma_0)$ otherwise. Originally (i.e., for L_0) the root $t = 0$ has multiplicity $\mathrm{mult}_k(\sigma_0)$; after perturbation (i.e., for L_1) it has multiplicity $\mathrm{mult}_k(\sigma_1)$. Thus, for $k < i_0$ or $k \geqslant i_1$ no new root is born and we are done. For $i_0 \leqslant k < i_1$ exactly one new root is born: it must therefore be real and, for small $|\tau|$, simple. $\square$

Lemma 4.3. *For all $\rho \in S_n$ there exist ρ-good matrices.*

Proof. Consider a reduced word $\rho = a_{i_1} \cdots a_{i_l}$, where $l = \mathrm{inv}(\rho)$ is the number of inversions of ρ. For $k \leqslant l$, define $\rho_k = a_{i_1} \cdots a_{i_k}$; in particular, $\rho_0 = e$ and $\rho_l = \rho$. As mentioned above, I is ρ_0-good. Apply Lemma 4.2 to deduce that if there exists a ρ_k-good matrix then there exists a ρ_{k+1}-good matrix. The result follows by induction. □

Proof of Proposition 4.1. By Lemma 4.3, there exists an η-good matrix L_0. The roots of every polynomial $m_k(t)$ are real and simple. The same holds for any $\tilde{L}_0 \in A$, where A is some sufficiently small open neighborhood of L_0. It suffices to show that for some such $\tilde{L}_0$ all roots are distinct.

Let $\rho \in S_n$ be different from η and ηa_i, $1 \leqslant i < n$. Then $\mathrm{Bru}_\rho \subset \mathrm{Lo}_n^1$ is a submanifold of codimension at least 2. Define

$$X_\rho = \{L \exp(tN_0);\, L \in \mathrm{Bru}_\rho,\, t \in \mathbb{R}\}, \qquad Y = \mathrm{Lo}_n^1 \setminus \bigcup_{\rho \in S_n \setminus \{\eta, \eta a_1, \ldots, \eta a_{n-1}\}} X_\rho;$$

each set X_ρ has measure zero. The set Y has total measure and is therefore dense. Take $\tilde{L}_0 \in A \cap Y$. We claim that all roots of the polynomials m_k are real, simple and distinct, as desired.

Indeed, assume by contradiction that $m_{k_1}(t_0) = m_{k_2}(t_0) = 0$, $k_1 < k_2$. Take $\rho \in S_n$ such that $\Gamma_{N_0,L_0}(t_0) = \tilde{L}_0 \exp(t_0 N_0) \in \mathrm{Bru}_\rho$; set $\sigma = \eta\rho$. We have that $\mathrm{mult}_{k_1}(\sigma) \geqslant 1$ and $\mathrm{mult}_{k_2}(\sigma) \geqslant 1$ whence $\sigma \notin \{e, a_1, \ldots, a_{n-1}\}$ and therefore $\rho \in S_n \setminus \{\eta, \eta a_1, \ldots, \eta a_{n-1}\}$. Thus $\tilde{L}_0 = \Gamma_{N_0,L_0}(t_0) \exp(-t_0 N_0) \in X_\rho$ and therefore $\tilde{L}_0 \notin Y$, a contradiction. □

Example 4.4. For $n = 5$, let N_0 be the matrix with subdiagonal entries equal to 1. Write $\eta = a_1 a_2 a_1 a_3 a_4 a_3 a_2 a_1 a_3 a_2$, an arbitrary reduced word. The matrices

$$\lambda_1(1), \quad \lambda_1(1)\lambda_2(-1), \quad \lambda_1(1)\lambda_2(-1)\lambda_1(1)$$

are easily seen to be a_1-, a_1a_2- and $a_1a_2a_1$-good, respectively (signs are chosen in an arbitrary manner). If we thus proceed from left to right, at each step taking a number of sufficiently small absolute value, we obtain the following example of an (ηb)-good matrix:

$$L_0 = \lambda_1(1)\lambda_2(-1)\lambda_1(1)\lambda_3(\tfrac{1}{8})\lambda_4(-\tfrac{1}{8})\lambda_3(\tfrac{1}{64})\lambda_2(\tfrac{1}{512})\lambda_1(-\tfrac{1}{512})\lambda_3(-\tfrac{1}{4096}).$$

For $\Gamma_{N_0,L_0}(t) = L_0 \exp(tN_0)$, all roots of the polynomials m_k are real, simple and distinct (see also Remark 4.7).

Let $\mathrm{Pos} \subset \mathrm{Lo}_n^1$ (resp. $\mathrm{Neg} \subset \mathrm{Lo}_n^1$) be the open subset of totally positive (resp. negative) matrices. If $\Gamma\colon I \to \mathrm{Lo}_n^1$ is flag-convex and $\Gamma(t_0) \in \overline{\mathrm{Pos}}$ then $\Gamma(t) \in \mathrm{Pos}$ for all $t > t_0$; similarly, if $\Gamma(t_0) \in \overline{\mathrm{Neg}}$ then $\Gamma(t) \in \mathrm{Neg}$ for all $t < t_0$: see Lemma 3.4 above and Lemma 5.7 from [7].

We are almost ready to prove Theorem 2. We may assume without loss of generality that $0 \in I_1 \subset I$ and that $\Gamma_\bullet(0) = \mathrm{Id}$. Take $I_2 = [t_-, t_+] \subseteq I_1$ with $t_- < 0 < t_+$. Notice that $t = 0$ is the only root of $m_{\Gamma_\bullet;k}$ in I: also, for $t > 0$ we have $\Gamma(t) \in \mathrm{Pos}$ and for $t < 0$ we have $\Gamma(t) \in \mathrm{Neg}$. Also, $t = 0$ is a root of multiplicity $k(n-k) = \mathrm{mult}_k(\eta)$ of $m_{\Gamma_\bullet;k}$. Given $L_1 \in \mathrm{Lo}_n^1$, set $\Gamma_1(t) = L_1\Gamma_\bullet(t)$;

write $m_{L_1;k} = m_{\Gamma_1;k}$. Thus, if $L_1 \in \mathrm{Bru}_\rho$ and $\sigma = \eta\rho$ then $t = 0$ is a root of multiplicity $\mathrm{mult}_k(\sigma)$ of $m_{L_1;k}$.

For $\rho = \eta\sigma \in S_n$, a matrix L_1 is *ρ-good* (for $\Gamma_\bullet$ and $I_2 = [t_-, t_+]$ fixed) if and only if, for $\Gamma_1(t) = L_1\Gamma_\bullet(t)$, we have: $L_1 \in \mathrm{Bru}_\rho \subset \mathrm{Lo}_n^1$, $\Gamma_1(t_+) \in \mathrm{Pos}$, $\Gamma_1(t_-) \in \mathrm{Neg}$ and, for all k, the function $m_{L_1;k}$ admits precisely $k(n-k) - \mathrm{mult}_k(\sigma)$ nonzero roots in $I \setminus \{0\}$, all in the interior of I_2 and all simple. Recall that $L_1 \in \mathrm{Bru}_\rho$ implies that $t = 0$ is a root of $m_{L_1,k}$ of multiplicity $\mathrm{mult}_k(\sigma)$. Notice that Id is e-good.

Lemma 4.5. *Consider ρ_0, $\rho_1 \in S_n$, $\rho_0 \triangleleft \rho_1 = \rho_0 a_j$. Let $L_0 \in \mathrm{Bru}_{\rho_0} \subset \mathrm{Lo}_n^1$ be a ρ_0-good matrix. Then there exists $\epsilon > 0$ such that, for all $\tau \in \mathbb{R}$, if $0 < |\tau| < \epsilon$, then $L_\tau = L_0\lambda_j(\tau)$ is ρ_1-good.*

Proof. Let $\sigma_0 = \eta\rho_0$ and $\sigma_1 = \eta\rho_1$ so that $\sigma_1 \triangleleft \sigma_0 = \sigma_1 a_j = (i_0 i_1)\sigma_1$. As above, we have $L_\tau \in \mathrm{Bru}_{\rho_1}$ for $\tau \neq 0$. Write $\mu_k = \mathrm{mult}_k(\sigma_0)$. As in Equation 4.2, $\mathrm{mult}_k(\sigma_1) = \mu_k - 1$ for $i_0 \leqslant k < i_1$, and $\mathrm{mult}_k(\sigma_1) = \mu_k$ otherwise.

Since Pos and Neg are open sets, for τ near 0 we have $\Gamma_\tau(t_+) \in \mathrm{Pos}$ and $\Gamma_\tau(t_-) \in \mathrm{Neg}$ (where $\Gamma_\tau(t) = L_\tau\Gamma_\bullet(t)$). This condition will be assumed from now on.

For τ near 0, the $k(n-k) - \mathrm{mult}_k(\sigma_0)$ nonzero simple roots of $m_{L_\tau,k}$ remain nonzero and simple; from the previous paragraph, they are in the interior of I_2. By compactness, for small $|\tau|$, there are no new roots away from a small neighborhood of $t = 0$.

The root $t = 0$ has multiplicity μ_k for $m_{L_0;k}$. Let $s_k \in \{\pm 1\}$ be the sign of $m_{L_0;k}^{(\mu_k)}(0) \neq 0$ so that $s_k m_{L_0;k}^{(\mu_k)}(t) > 0$ in a small neighborhood $I_0 \subset I_2$ of $t = 0$. For small $|\tau|$, we likewise have $s_k m_{L_\tau;k}^{(\mu_k)}(t) > 0$ in I_0. For $\tau \neq 0$, the root $t = 0$ has multiplicity $\mathrm{mult}_k(\sigma_1)$. For $k < i_0$ or $k \geqslant i_1$, we have $\mathrm{mult}_k(\sigma_1) = \mu_k$, and therefore $t = 0$ is the only root in I_0 and we are done. For $i_0 \leqslant k < i_1$ we have $\mathrm{mult}_k(\sigma_1) = \mu_k - 1$. The signs of m_k at the extrema of I_0 together with the sign of $m_k^{(\mu_k)}$ and the multiplicity of the zero at $t = 0$ imply that, for small $|\tau|$, there is exactly one new nonzero root of $m_{L_\tau;k}$ in $I_0 \setminus \{0\}$; this root is simple, as desired. □

Lemma 4.6. *Consider a flag-convex curve $\Gamma_\bullet \colon I \to \mathrm{Lo}_n^1$ and intervals $I_2 \subseteq I$ fixed, as above. For all $\rho \in S_n$ there exist ρ-good matrices.*

Proof. Consider a reduced word $\rho = a_{i_1} \cdots a_{i_l}$, where $l = \mathrm{inv}(\rho)$ is the number of inversions of ρ. For $k \leqslant l$ define $\rho_k = a_{i_1} \cdots a_{i_k}$; in particular, $\rho_0 = e$ and $\rho_l = \rho$. As remarked, Id is ρ_0-good. Apply Lemma 4.5 to deduce that if there exists a ρ_k-good matrix, then there exists a ρ_{k+1}-good matrix. The result follows by induction. □

Proof of Theorem 2. For $L \in \mathrm{Lo}_n^1$, set $\Gamma_L(t) = L\Gamma_0(t)$ and $m_{L;k} = m_{\Gamma_L;k}$. By Lemma 4.6, there exists an η-good matrix L_0. Each function $m_{L_0;k}$ has exactly $k(n-k)$ roots in the interior of I_2, all simple. Since $\Gamma_{L_0}(t_+) \in \mathrm{Pos}$ and $\Gamma_{L_0}(t_-) \in \mathrm{Neg}$, there are no other roots. These conditions are open and therefore hold for the functions $m_{\tilde{L}_0;k}$ for any $\tilde{L}_0 \in A$, where A is some sufficiently small open neighborhood of L_0. It suffices to show that, for some such $\tilde{L}_0$, all roots are distinct.

Consider the quotient map $\mathrm{Lo}_n^1 \to \mathbb{R}$ taking L to $L_{2,1} + L_{3,2} + \cdots + L_{n,n-1}$; pre-images of points form a family of parallel hyperplanes. Let $S \subset A$, $L_0 \in S$, be a convex neighborhood of L_0 in its hyperplane; notice that S is transversal to

$\Gamma'_{L_0}(0)$. The function $\Phi\colon S \times I_1 \to \mathrm{Lo}^1_n$ defined by $\Phi(L, t) = \Gamma_L(t)$ is a tubular neighborhood of the image $\Gamma_{L_0}[I_1]$. This may require replacing the original interval I_1 by a smaller interval, still with 0 in the interior: notice that this is allowed.

Let $\rho \in S_n$ be different from η and ηa_i, $1 \leqslant i < n$. Then $\mathrm{Bru}_\rho \subset \mathrm{Lo}^1_n$ is a submanifold of codimension at least 2, and therefore so is $\Phi^{-1}[\mathrm{Bru}_\rho] \subset S \times I_1$. Let $X_\rho \subset S$ be its image under the projection onto S: the subset $X_\rho \subset S$ has measure zero. Let

$$Y = S \setminus \bigcup_{\rho \in S_n \setminus \{\eta, \eta a_1, \ldots, \eta a_{n-1}\}} X_\rho :$$

the subset $Y \subseteq S$ has total measure and is therefore dense. Notice that since $Y \subseteq S \subset A$, if $\tilde{L}_0 \in Y$, then the function $m_{\tilde{L}_0;k}$ has precisely $k(n-k)$ roots in I_1, all simple and all in the interior of I_1. We claim that in this case all roots of the functions $m_{\tilde{L}_0;k}$ are also distinct, as desired.

Indeed, assume by contradiction that $m_{k_1}(t_0) = m_{k_2}(t_0)$, $k_1 < k_2$. Take $\rho \in S_n$ such that $\Gamma_{\tilde{L}_0}(t_0) = \Phi(\tilde{L}_0, t_0) \in \mathrm{Bru}_\rho$; set $\sigma = \eta\rho$. We have $\mathrm{mult}_{k_1}(\sigma) \geqslant 1$ and $\mathrm{mult}_{k_2}(\sigma) \geqslant 1$ whence $\sigma \notin \{e, a_1, \ldots, a_{n-1}\}$; thus, $\rho \in S_n \setminus \{\eta, \eta a_1, \ldots, \eta a_{n-1}\}$. Thus $(\tilde{L}_0, t_0) \in \Phi^{-1}[\mathrm{Bru}_\rho]$ and therefore $\tilde{L}_0 \in X_\rho$ and therefore $\tilde{L}_0 \notin Y$, a contradiction. □

Remark 4.7. In [8], we introduce the concept of *itinerary* $\mathrm{iti}(\Gamma)$ of a locally convex curve Γ. The itinerary is a word with letters in $S_n \setminus \{e\}$ which gives important information about the curve: essentially, it lists the moments of non-transversality according to Bruhat cell. The construction applies to flag-convex curves $\Gamma\colon I \to \mathrm{Lo}^1_n$. In the proof of Theorem 2, we start with a curve $\Gamma_\bullet$ with itinerary $\mathrm{iti}(\Gamma_\bullet) = \eta$ (a word with a single letter). We construct a small perturbation Γ_1 of the original curve. The itinerary is now a word of length $\frac{n^3-n}{6}$, each letter being a generator a_k of S_n. Each letter a_k appears $k(n-k)$ times. There are many different reduced words for η and even for a given word we may have more that one itinerary. The itinerary of the curve Γ_{N_0,L_0} constructed in Example 4.4 is $\mathrm{iti}(\Gamma_{N_0,L_0}) = dcbcdabacbcbabdcbadc$. Conjecture 1.2 is equivalent to the statement that for any flag-convex curve $\Gamma\colon I \to \mathrm{Lo}^1_n$ the letter a_k appears at most $k(n-k)$ times in the itinerary $\mathrm{iti}(\Gamma)$.

References

[1] V. Arnold, *On the number of flattening points on space curves*, Sinai's Moscow Seminar on Dynamical Systems, Amer. Math. Soc. Transl. Ser. 2, vol. 171, Amer. Math. Soc., Providence, RI, 1996, pp. 11–22. MR 1359089

[2] I. Bárány, J. Matoušek, and A. Pór, *Curves in $\mathbb{R}^d$ intersecting every hyperplane at most $d+1$ times*, Computational geometry (SoCG'14), ACM, New York, 2014, pp. 565–571. MR 3382339

[3] A. Berenstein, S. Fomin, and A. Zelevinsky, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math. **122** (1996), no. 1, 49–149. MR 1405449

[4] G. Canuto, *Associated curves and Plücker formulas in Grassmannians*, Invent. Math. **53** (1979), no. 1, 77–90. MR 538685

[5] W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, Berlin-New York, 1971. MR 0460785

[6] V. Goulart and N. Saldanha, *Combinatorialization of spaces of nondegenerate spherical curves*, preprint arXiv:1810.08632 [math.GT].

[7] V. Goulart and N. C. Saldanha, *Locally convex curves and the Bruhat stratification of the spin group*, Isr. J. Math. (2021). DOI: 10.1007/s11856-021-2127-z

[8] V. Goulart and N. C. Saldanha, *Stratification by itineraries of spaces of locally convex curves*, preprint arXiv:1907.01659 [math.GT].

[9] L. Lang, B. Shapiro, and E. Shustin, *On the number of intersection points of the contour of an amoeba with a line*, preprint arXiv:1905.08056 [math.AG]; to appear in Indiana Univ. Math. J.

[10] A. J. Levin, *The non-oscillation of solutions of the equation* $x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0$, Uspehi Mat. Nauk **24** (1969), no. 2 (146), 43–96 (Russian). MR 0254328. English translation: Russian Math. Surveys **24** (1969), no. 2, 43–99.

[11] V. Sedykh and B. Shapiro, *On two conjectures concerning convex curves*, Internat. J. Math. **16** (2005), no. 10, 1157–1173. MR 2182213

[12] B. Shapiro, *Spaces of linear differential equations and flag manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 1, 173–187, 223 (Russian). MR 1044054. English translation: Math. USSR-Izv. **36** (1991), no. 1, 183–197.

[13] B. Shapiro and M. Shapiro, *Projective convexity in* $\mathbf{P}^3$ *implies Grassmann convexity*, Internat. J. Math. **11** (2000), no. 4, 579–588. MR 1768174

[14] B. Shapiro and M. Shapiro, *On the boundary of totally positive upper triangular matrices*, Linear Algebra Appl. **231** (1995), 105–109. MR 1361102

[15] B. Shapiro and M. Shapiro, *Linear ordinary differential equations and Schubert calculus*, Proceedings of the Gökova Geometry-Topology Conference 2010, Int. Press, Somerville, MA, 2011, pp. 79–87. MR 2931881

[16] B. Shapiro, M. Shapiro, and A. Vainshtein, *Connected components in the intersection of two open opposite Schubert cells in* $\mathrm{SL}_n(\mathbb{R})/B$, Internat. Math. Res. Notices (1997), no. 10, 469–493. MR 1446839

[17] M. Shapiro, *Nonoscillating differential equations*, Ph.D. thesis, Moscow State University, 1992.

Departamento de Matemática, PUC-Rio R. Mq. de S. Vicente 225, Rio de Janeiro, RJ 22451-900, Brazil
E-mail address: saldanha@puc-rio.br

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
E-mail address: shapiro@math.su.se

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027
E-mail address: mshapiro@math.msu.edu