LINEAR RECURRENCES AND VANDERMONDE VARIETIES. I.

RALF FRÖBERG AND BORIS SHAPIRO

Abstract. Motivated by the famous Skolem-Mahler-Lech theorem we initiate in this paper the study of a natural class of determinantal varieties which we call *Vandermonde varieties*. They are closely related to the varieties of linear recurrence relations of a given order possessing a non-trivial solution vanishing at a given set of integers. In the regular case, i.e. when the dimension of a Vandermonde variety is the expected one we present its free resolution, obtain its degree and the Hilbert series. Many open problems and conjectures are posed.

1. Introduction

In the present paper we study a circle of problems motivated by the famous Skolem-Mahler-Lech theorem. Let us briefly recall its formulation. A linear recurrence relation with constant coefficients of order k is an equation of the form

$$u_n + \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + \cdots + \alpha_k u_{n-k} = 0,\ n \geq k \tag{1}$$

where the coefficients $(\alpha_1, ..., \alpha_k)$ are complex numbers and $\alpha_k \neq 0$. (Equation (1) is often referred to as a linear homogeneous difference equation with constant coefficients.)

The left-hand side of the equation

$$t^k + \alpha_1 t^{k-1} + \alpha_2 t^{k-2} + \cdots + \alpha_k = 0 \tag{2}$$

is called the *characteristic polynomial* of recurrence (1). Denote the roots of (2) (listed with possible repetitions) by $x_1, \ldots, x_k$ and call them the *characteristic roots* of (1).

Notice that all x_i are non-vanishing since $\alpha_k \neq 0$. To obtain a concrete solution of (1) one has to prescribe additionally an initial k-tuple $(u_0, ...u_{k-1})$ which can be chosen arbitrarily. Then $u_n,\ n \geq k$ are obtained by using the relation (1). A solution of (1) is called *non-trivial* if not all of its entries vanish. In case of all distinct characteristic roots a general solution of (1) can be given by

$$u_n = c_1 x_1^n + c_2 x_2^n + ... + c_k x_k^n$$

where $c_1, ..., c_k$ are arbitrary complex numbers. In the general case of multiple characteristic roots a similar formula can be found in e.g. [16].

An arbitrary solution of a linear homogeneous difference equation with constant coefficients of order k or, equivalently, of a linear homogeneous differential equation with constant coefficients of order k is called an *exponential polynomial of order* k. One usually substitutes $x_i \neq 0$ by e^{γ_i} and considers the obtained function in $\mathbb{C}$ instead of $\mathbb{Z}$ or $\mathbb{N}$. (Other terms used for exponential polynomials are *quasipolynomials* or *exponential sums*.)

The most fundamental fact about the structure of integer zeros of exponential polynomials is the well-known Skolem-Mahler-Lech theorem given below. It was first proved for recurrence sequences of algebraic numbers by K. Mahler [11] in the 30's, based upon an idea of T. Skolem [15]. Then, C. Lech [9] published the result for general recurrence sequences in 1953. In 1956 Mahler published the same result,

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apparently independently (but later realized to his chagrin that he had actually reviewed Lech's paper some years earlier, but had forgotten it).

Theorem 1 (The Skolem-Mahler-Lech theorem)**.** *If $a_0, a_1, \dots$ is a recurrences sequence, then the set of all k such that $a_k = 0$ is the union of a finite (possibly empty) set and a finite number (possibly zero) of full arithmetic progressions. (Here, a full arithmetic progression means a set of the form $r, r+d, r+2d, \dots$ with $0 < r < d$.)*

A simple criterion guaranteeing the absence of arithmetic progressions is that no quotient of two distinct characteristic roots of the recurrence relation under consideration is a root of unity. A recurrence relation (1) satisfying this condition is called *non-degenerate*.

In 1977 J. H. Loxton and A. J. van der Poorten formulated an important conjecture (Conjecture $1'$ of [10]) claiming that there exists a constant μ_k such that any integer recurrence of order k either has at most μ_k integer zeros or has infinitely many zeros.

This conjecture was first settled by W. M. Schmidt in 1999, see [14] and also by J. H. Evertse and H. P. Schlickewei, see [7].

The upper bound for μ_k obtained in [14] was

$$\mu_k < e^{e^{e^{3k \log k}}},$$

which was later improved by the same author to

$$\mu_k < e^{e^{e^{20k}}}.$$

Apparently the best known at the moment upper bound for μ_k was obtained in [1] and is given by

$$\mu_k < e^{e^{k^{\sqrt{11k}}}}.$$

Although the known upper bounds are at least double exponential it seems plausible that the realistic upper bounds should be polynomial. The only known nontrivial lower bound for μ_k was found in [2] and is given by

$$\mu_k \geq \binom{k+1}{2} - 1.$$

One should also mention the non-trivial exact result of F. Beukers showing that for sequences of rational numbers obtained from recurrence relations of length 3 one has $\mu_3 = 6$, see [3].

The main objects of our study are the following two varieties $V_{k;I}$ and $Vd_{k;I}$ which one can associate to an arbitrary pair $(k; I)$ where $k \geq 2$ is a positive integer and $I = \{i_0 < i_1 < i_2 < \dots < i_{m-1}\}$, $m \geq k$ is a sequence of integers. In what follows we will always assume that $\gcd(i_1 - i_0, \dots, i_{m-1} - i_0) = 1$ to avoid unnecessary freedom related to the time rescaling in (1).

To define $V_{k;I}$ denote by L_k the space of all linear recurrence relations (1) of order at most k with constant coefficients (which we identify with the affine space of the coefficients $(\alpha_1, \dots, \alpha_k)$) and denote by $L_k^* = L_k \setminus \{\alpha_k = 0\}$ the subset of all linear recurrence of order exactly k.

Main notation. Denote by $V_{k;I} \subset L_k^*$ the set of all linear recurrences of order exactly k such that they have a non-trivial solution vanishing at all points of I and denote by $\overline{V_{k;I}}$ the (set-theoretic) closure of $V_{k;I}$ in L_k. We call $V_{k;I}$ (resp. $\overline{V}_{k;I}$) the open (resp. closed) *linear recurrence variety* associated to the pair $(k; I)$.

Notice that since for $m \leq k-1$ one has $V_{k;I} = L_k^*$ and $\overline{V}_{k;I} = L_k$ the latter case does not require special consideration. A more important observation is that due to translation invariance of (1) for any integer l and any pair $(k; I)$ the variety

$V_{k;I}$ (resp. $\overline{V}_{k;I}$) coincides with the variety $V_{k;I+l}$ (resp. $\overline{V}_{k;I+l}$) where the set of integers $I+l$ is obtained by adding l to all entries of I.

So far we defined $\overline{V}_{k;I}$ and $V_{k;I}$ as sets. However, the following statement holds.

Proposition 2. *For any pair $(k;I)$ the set $\overline{V}_{k;I}$ is an affine algebraic variety. Therefore, $V_{k;I}=\overline{V}_{k;I}|_{L_k^*}$ is a quasi-affine variety.*

Notice that this fact is not completely obvious since if we, for example, instead of a set of integers choose as I an arbitrary subset of real or complex numbers then the similar subset of L_n will, in general, only be analytic.

Many questions related to the Skolem-Mahler-Lech theorem translate immediately into questions about $V_{k;I}$. For example, one can name the following formidable challenges.

Problem 1. For which pairs $(k;I)$ the variety $V_{k;I}$ is empty/non-empty? More generally, what is the dimension of $V_{k;I}$?

Problem 2. For which pairs $(k;I)$ any solution of a linear recurrence vanishing at I must have an additional integer root outside I? More specifically, for which pairs $(k;I)$ any solution of a linear recurrence vanishing at I must vanish infinitely many times in $\mathbb{Z}$? In other words, for which pairs $(k;I)$ the set of all integer zeros of the corresponding solution of any recurrence relation from $V_{k;I}$ must necessarily contain an arithmetic progression?

For example, in case $k=3,\ m=4$ the first situation occurs for 4-tuples $(0,1,4,6)$ and $(0,1,4,13)$ which both force a non-trivial solution of a third order recurrence vanishing at them to vanish at the 6-tuple $(0,1,4,6,13,52)$, see [3]. The second situation occurs if in a 4-tuple $I=\{0,i_1,i_2,i_3\}$ two differences between its entries coincide, see [3]. But this condition is only sufficient in this case and no systematic information is available. Notice that for any pair $(k;I)$ the variety $\overline{V}_{k;I}$ is weighted-homogeneous where the coordinate $\alpha_i,\ i=1,..,k$ has weight i. (This action corresponds to the scaling of the characteristic roots of (2).) From the point of view of algebraic geometry one can ask what is a natural defining radical ideal of $\overline{V}_{k;I}$, is it equidimensional, irreducible, what is its Hilbert series etc.

Now we define the Vandermonde variety associated with a given pair $(k;I)$, $I=\{0\le i_0<i_1<i_2<...<i_{m-1}\},\ m\ge k$. In fact, we define three slightly different but closely related versions of this variety as follows. For a given k and $I=\{0\le i_0<i_1<i_2<...<i_{m-1}\},\ m\ge k$ consider first the set $M_{k;I}$ of (generalized) Vandermonde matrices

$$M_{k;I}=\begin{pmatrix} x_1^{i_0} & x_2^{i_0} & \cdots & x_k^{i_0}\\ x_1^{i_1} & x_2^{i_1} & \cdots & x_k^{i_1}\\ \cdots & \cdots & \cdots & \cdots\\ x_1^{i_{m-1}} & x_2^{i_{m-1}} & \cdots & x_k^{i_{m-1}} \end{pmatrix}, \tag{3}$$

where $(x_1,...,x_k)\in\mathbb{C}^k$. In other words, for a given pair $(k;I)$ we consider the map $M_{k;I}:\mathbb{C}^k\to Mat(m,k)$ given by (3) where $Mat(m,k)$ is the space of all $m\times k$-matrices with complex entries and $(x_1,...,x_k)$ are chosen coordinates in $\mathbb{C}^k$.

Main notation. Given a pair $(k;I)$ with $|I|\ge k$ define the *coarse Vandermonde variety* $Vd^{\mathbf{c}}_{k;I}$ as the subset of $M_{k;I}$ of all degenerate matrices, i.e. whose rank is smaller than k. This set is obviously an algebraic variety whose defining ideal $\mathcal{I}_I$ is generated by all $\binom{m}{k}$ maximal minors of $M_{k;I}$. Denote the quotient ring by $\mathcal{R}_I=\mathcal{R}/\mathcal{I}_I$.

Denote by $\mathcal{A}_k\subset\mathbb{C}^k$ the standard Coxeter arrangement (of the Coxeter group A_{k-1}) consisting of all diagonals $x_i=x_j$ and by $\mathcal{BC}_k\subset\mathbb{C}^k$ the Coxeter arrangement (of the Coxeter group BC_k??) consisting of all $x_i=x_j$ and $x_i=0$. Obviously,

$\mathcal{BC}_k \supset \mathcal{A}_k$. Notice that $Vd^{\mathbf{c}}_{k;I}$ always includes the arrangement $\mathcal{BC}_k$ (some of the hyperplanes with multiplicities) which is often inconvenient. Namely, with very few exceptions this means that $Vd^{\mathbf{c}}_{k;I}$ is not equidimensional, not CM, not reduced etc. For applications to linear recurrences as well as questions in combinatorics and geometry of Schur polynomials it seems more natural to consider the localizations of $Vd^{\mathbf{c}}_{k;I}$ in $\mathbb{C}^k \setminus \mathcal{A}_k$ and in $\mathbb{C}^k \setminus \mathcal{BC}_k$.

Main notation. Define the $\mathcal{A}_k$-localization $Vd^{\mathcal{A}}_{k;I}$ of $Vd^{\mathbf{c}}_{k;I}$ as the contraction of $Vd^{\mathbf{c}}_{k;I}$ to $\mathbb{C}^k \setminus \mathcal{A}_k$. Its is easy to obtain the generating ideal of $Vd^{\mathcal{A}}_{k;I}$. Namely, recall that given a sequence $J = (j_1 < j_2 < \cdots < j_k)$ of nonnegative integers one defines the associated Schur polynomial $S_J(x_1, ..., x_k)$ as given by

$$S_J(x_1,\dots,x_k) = \begin{vmatrix} x_1^{j_1} & x_2^{j_1} & \cdots & x_k^{j_2} \\ x_1^{j_2} & x_2^{j_2} & \cdots & x_k^{j_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{j_k} & x_2^{j_k} & \cdots & x_k^{j_k} \end{vmatrix} / W(x_1,\dots,x_k),$$

where $W(x_1,\dots,x_k)$ is the usual Vandermonde determinant. Given a sequence $I = (0 \le i_0 < i_1 < i_2 < \cdots < i_{m-1})$ with $\gcd(i_1 - i_0, \dots, i_{m-1} - i_0) = 1$ consider the set of all its $\binom{m}{k}$ subsequences J_κ of length k where the index κ runs over the set of all subsequences of length k among $\{1, 2, ..., m\}$. Take the corresponding Schur polynomials $S_{J_\kappa}(x_1,\dots,x_k)$ and form the ideal $\mathcal{I}^{\mathcal{A}}_I$ in the polynomial ring $\mathbb{C}[x_1,\dots,x_k]$ generated by all $\binom{m}{k}$ Schur polynomials $S_{J_\kappa}(x_1,\dots,x_k)$. One can easily see that the Vandermonde variety $Vd^{\mathcal{A}}_{k;I} \subset \mathbb{C}^k$ is generated by $\mathcal{I}^{\mathcal{A}}_I$. Denote the quotient ring by $\mathcal{R}^{\mathcal{A}}_I = \mathcal{R}/\mathcal{I}^{\mathcal{A}}_I$ where $\mathcal{R} = \mathbb{C}[x_1, ..., x_k]$. Analogously, to the coarse Vandermonde variety $Vd_{k;I}$ the variety $Vd^{\mathcal{A}}_{k;I}$ often contains irrelevant coordinate hyperplanes which prevents it from having nice algebraic properties. For example, if $i_0 > 0$ then all coordinate hyperplanes necessarily belong to $Vd^{\mathcal{A}}_{k;I}$ ruining equidimensionality etc. On the other hand, under the assumption that $i_0 = 0$ the variety $Vd^{\mathcal{A}}_{k;I}$ often has quite reasonable properties presented below.

Main notation. Define the $\mathcal{BC}_k$-localization $Vd^{BC}_{k;I}$ of $Vd^{\mathbf{c}}_{k;I}$ as the contraction of $Vd^{\mathbf{c}}_{k;I}$ to $\mathbb{C}^k \setminus \mathcal{BC}_k$. Again it is simple to find the generating ideal of $Vd^{BC}_{k;I}$. Namely, given a sequence $J = (0 \le j_1 < j_2 < \cdots < j_k)$ of nonnegative integers define the reduced Schur polynomial $\hat{S}_J(x_1, ..., x_k)$ as given by

$$\hat{S}_J(x_1,\dots,x_k) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1^{j_2-j_1} & x_2^{j_2-j_1} & \cdots & x_k^{j_2-j_1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{j_k-j_1} & x_2^{j_k-j_1} & \cdots & x_k^{j_k-j_1} \end{vmatrix} / W(x_1,\dots,x_k).$$

In other words, $\hat{S}_J(x_1, ..., x_k)$ is the usual Schur polynomial corresponding to the sequence $(0, j_2 - j_1, ..., j_k - j_1)$. Given a sequence $I = (0 \le i_0 < i_1 < i_2 < \cdots < i_{m-1})$ with $\gcd(i_1 - i_0, \dots, i_{m-1} - i_0) = 1$ consider as before the set of all its $\binom{m}{k}$ subsequences J_κ of length k where the index κ runs over the set of all subsequences of length k. Take the corresponding reduced Schur polynomials $\hat{S}_{J_\kappa}(x_1,\dots,x_k)$ and form the ideal $\mathcal{I}^{BC}_I$ in the polynomial ring $\mathbb{C}[x_1,\dots,x_k]$ generated by all $\binom{m}{k}$ Schur polynomials $\hat{S}_{J_\kappa}(x_1,\dots,x_k)$. One can easily see that the Vandermonde variety $Vd^{BC}_{k;I} \subset \mathbb{C}^k$ as generated by $\mathcal{I}^{BC}_I$. Denote the quotient ring by $\mathcal{R}^{BC}_I = \mathcal{R}/\mathcal{I}^{BC}_I$.

Conjecture 3. *If* $\dim(Vd^{BC}_{k;I}) \ge 2$ *then* $\mathcal{I}^{BC}_I$ *is a radical ideal.*

Notice that considered as sets the restrictions to $\mathbb{C}^k \setminus \mathcal{BC}_k$ of all three varieties $Vd^{\mathbf{c}}_{k;I}$, $Vd^{\mathcal{A}}_{k;I}$, $Vd^{BC}_{k;I}$ coincide with what we call the *open Vandermonde variety* $Vd^{op}_{k;I}$ which is the subset of all matrices of the form $M_{k;I}$ with three properties: (i) rank is smaller than k; (ii) all x_i's are non-vanishing; (iii) all x_i's are pairwise distinct. Thus all the differences between the three Vandermonde varieties are concentrated on the hyperplane arrangement $\mathcal{BC}_k$. Also from the above definitions it is obvious that $Vd^{op}_{k;I}$ and $Vd^{BC}_{k;I}$ are invariant under addition of an arbitrary integer to I.

The relation between the linear recurrence variety $V_{k;I}$ and the open Vandermonde variety $Vd^{op}_{k;I}$ is quite straight-forward. Namely, consider the standard Vieta map:

$$Vi : \mathbb{C}^k \to L_k \tag{4}$$

sending an arbitrary k-tuple $(x_1, ..., x_k)$ to the polynomial $t^k+\alpha_1 t^{k-1}+\alpha_2 t^{k-2}+\cdots+\alpha_k$ whose roots are $x_1, ..., x_k$. Inverse images of the Vieta map are exactly the orbits of the standard S_k-action on $\mathbb{C}^k$ by permutations of coordinates. Thus, the Vieta map sends a homogeneous and symmetric polynomial to a weighted homogeneous polynomial.

Define the *open linear recurrence variety* $V^{op}_{k;I} \subseteq V_{k;I}$ of a pair $(k; I)$ as consisting of all recurrences in $V_{k;I}$ with all distinct characteristic roots distinct. The following statement is obvious.

Lemma 4. *The map Vi restricted to $Vd^{op}_{k;I}$ gives an unramified $k!$-covering of the set $V^{op}_{k;I}$.*

Unfortunately at the present moment the following natural question is still open.

Problem 3. Is it true that for any pair $(k; I)$ one has that $\overline{V^{op}_{k;I}} = V_{k;I}$ where $\overline{V^{op}_{k;I}}$ is the set-theoretic closure of $V^{op}_{k;I}$ in L^*_k? If 'not', then under what additional assumptions?

The remaining part of the paper is devoted to the study of the Vandermonde varieties $Vd^{\mathcal{A}}_{k;I}$ and $Vd^{BC}_{k;I}$. We start with the $\mathcal{A}_k$-localized variety $Vd^{\mathcal{A}}_{k;I}$. Notice that if $m = k$ the variety $Vd^{\mathcal{A}}_{k;I} \subset \mathbb{C}^k$ is an irreducible hypersurface given by the equation $S_I = 0$ and its degree equals $\sum_{j=0}^{k-1} i_j - \binom{k}{2}$. We will need the following alternative description of the ideal $\mathcal{I}^{\mathcal{A}}_I$ in the general case. Namely, using the Jacobi-Trudi identity for the Schur polynomials we get the following statement.

Lemma 5. *For any pair $(k; I)$, $I = \{i_0 < i_1 < ... < i_{m-1}\}$ the ideal $\mathcal{I}^{\mathcal{A}}_I$ is generated by all $k \times k$-minors of the $m \times k$-matrix*

$$H_{k;I} = \begin{pmatrix} h_{i_0-(k-1)} & h_{i_0-(k-2)} & \cdots & h_{i_0} \\ h_{i_1-(k-1)} & h_{i_1-(k-2)} & \cdots & h_{i_1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_{m-1}-(k-1)} & h_{i_{m-1}-(k-2)} & \cdots & h_{i_{m-1}} \end{pmatrix}. \tag{5}$$

Here h_i denotes the complete symmetric function of degree i, and $h_i = 0$ if $i < 0, h_0 = 1$.

In particular, Lemma 5 shows that $Vd^{\mathcal{A}}_{k;I}$ is an actual determinantal variety. When working with $Vd^{\mathcal{A}}_{k;I}$ and unless the opposite is explicitly mentioned we will assume that $I = \{0 < i_1 < ... < i_{m-1}\}$, i.e. that $i_0 = 0$ and that $\gcd(i_1, ..., i_{m-1}) = 1$. Let us first study some properties of $Vd^{\mathcal{A}}_{k;I}$ in the so-called *regular case* when its dimension coincides with the expected one. Namely, consider the set $\Omega_{m,k} \subset Mat(m, k)$ of all $m \times k$-matrices having positive corank. It is well-known that $\Omega_{m,k}$ has codimension equal to $m-k+1$. Since $Vd^{\mathfrak{c}}_{k;I}$ coincides with the pullback of $\Omega_{m,k}$ under the map $M_{k;I}$ and $Vd^{\mathcal{A}}_{k;I}$ is closely related to it (but with trivial pathology on $\mathcal{A}_k$ removed) the expected codimension of $Vd^{\mathcal{A}}_{k;I}$ equals $m - k + 1$. We call a pair $(k; I)$ *$\mathcal{A}$-regular* if $k \le m \le 2k - 1$ (implying that the expected dimension of

$Vd^{\mathcal{A}}_{k;I}$ is positive) and the actual codimension of $Vd^{\mathcal{A}}_{k;I}$ coincides with its expected codimension. We now describe the Hilbert series of the quotient ring $\mathcal{R}^{\mathcal{A}}_I$ in the case of a arbitrary regular pair $(k;I)$ using the well-known resolution of determinantal ideals of Eagon-Northcott [6].

To explain the notation in the following theorem, we introduce two gradings, tdeg and deg, on $\mathbb{C}[t_0,\dots,t_{m-1}]$. The first one is the usual grading induced by $\mathrm{tdeg}(t_i)=1$ for all i, and a second one is induced by $\deg(t_i)=-i$. In the next theorem M denotes a monomial in $\mathbb{C}[t_0\dots,t_{m-1}]$.

Theorem 6. *In the above notation, and with $I'=\{i_1,\dots,i_{m-1}\}$, one has that*

(a) *the Hilbert series $Hilb^{\mathcal{A}}_I(t)$ of $\mathcal{R}^{\mathcal{A}}_I=\mathcal{R}/\mathcal{I}^{\mathcal{A}}_I$ is given by*
$$Hilb^{\mathcal{A}}_I(t)=\frac{1-\sum_{i=0}^{m-k}((-1)^{i+1}\sum_{J\subseteq I',|J|=k+i}t^{s_J}\sum_{M\in N_i}t^{\deg(M)})}{(1-t)^k},$$
where $s_J=\sum_{j\in J}j-\binom{k}{2}$ and $N_i=\{M;\mathrm{tdeg}(M)=i\}$.

(b) *The degree of $\mathcal{R}^{\mathcal{A}}_I$ is $T^{(m-k+1)}(1)(-1)^{m-k+1}/(m-k+1)!$, where $T(t)$ is the numerator in* (a).

An alternative way to calculate $\deg(Vd^{\mathcal{A}}_{k;I})$ is to use the Giambelli-Porteous formula, see e.g. [8]. The next result corresponded to the authors by M. Kazarian explains how to do that.

Proposition 7. *Assume that $Vd^{\mathcal{A}}_{k;I}$ has the expected codimension $m-k+1$. Then its degree (taking multiplicities of the components into account) is equal to the coefficient of t^{m-k+1} in the Taylor expansion of the series*
$$\frac{\prod_{j=1}^{m-1}(1+i_jt)}{\prod_{j=1}^{k-1}(1+jt)}.$$
More explicitly,
$$\deg(Vd^{\mathcal{A}}_{k;I})=\sum_{j}^{m-k+1}\sigma_j(I)u_{m-k+1-j}$$
where σ_j is the jth elementary symmetric function of the entries $(i_1,...,i_{m-1})$ and $u_0,u_1,u_2,...$ are the coefficients in the Taylor expansion of $\prod_{j=1}^{k-1}\frac{1}{1+jt}$, i.e. $u_0+u_1t+u_2t^2+...=\prod_{j=1}^{k-1}\frac{1}{1+jt}$. In particular, $u_0=1$, $u_1=-\binom{k}{2}$, $u_2=\binom{k+1}{3}\frac{3k-2}{4}$, $u_3=-\binom{k+2}{4}\binom{k}{2}$, $u_4=\binom{k+3}{5}\frac{15k^3-15k^2-10k+8}{48}$.

In the simplest non-trivial case $m=k+1$ one can obtain more detailed information about $Vd^{\mathcal{A}}_{k;I}$. Notice that for $m=k+1$ the $k+1$ Schur polynomials generating the ideal $\mathcal{I}^{\mathcal{A}}_I$ are naturally ordered according to their degree. Namely, given an arbitrary $I=\{0<i_1<i_1<...<i_k\}$ with $\gcd(i_1,...,i_k)=1$ denote by $S_j,\ j=0,...,k$ the Schur polynomial obtained by removal of the (j)-th row of the matrix $M_{k;I}$. (Pay attention that here we enumerate the rows starting from 0.) Then, obviously, $\deg S_k<\deg S_{k-1}<....<\deg S_0$. Using presentation (5) we get the following.

Theorem 8. *For any integer sequence $I=\{0=i_0<i_1<i_2<...<i_k\}$ of length $k+1$ with $\gcd(i_1,...,i_k)=1$ the following facts are valid.*

(i) *$codim(Vd^{\mathcal{A}}_{k;I})=2$;*

(ii) *The quotient ring $\mathcal{R}^{\mathcal{A}}_I$ is Cohen-Macaulay;*

(iii) *The Hilbert series* $Hilb_I^{\mathcal{A}}(t)$ *of* $\mathcal{R}_I^{\mathcal{A}}$ *is given by the formula*

$$Hilb_I^{\mathcal{A}}(t) = \left(1 - \sum_{j=1}^{k} t^{N-i_j-\binom{k}{2}} + \sum_{j=1}^{k-1} t^{N-j-\binom{k}{2}}\right) /(1-t)^k,$$

where $N = \sum_{j=1}^{k} i_j$*;*

(iv) $\deg(Vd_{k;I}^{\mathcal{A}}) = \sum_{1\le j<l\le k} i_j i_l - \binom{k}{2}\sum_{j=1}^{k} i_j + \binom{k+1}{3}(3k-2)/4$*;*

(v) *The ideal* $\mathcal{I}_I^{\mathcal{A}}$ *is always generated by* k *generators* $S_k, ..., S_1$ *(i.e. the last generator* S_0 *always lies in the ideal generated by* $S_k, ..., S_1$*.) Moreover, if for some* $1 \le n \le k-2$ *one has* $i_n \le k-n$ *then* $\mathcal{I}_I^{\mathcal{A}}$ *is generated by* $k-n$ *elements* $S_k, ..., S_{n+1}$*. In particular, it is generated by two elements* S_k, S_{k-1} *(i.e. is a complete intersection) if* $i_{k-2} \le k-1$*.*

That the ring is CM follows from the fact that the ideal is generated by the maximal minors of a $t \times m$-matrix in the ring of Laurent polynomials. To prove the theorem it suffices to prove the relations between the Schur polynomials. Unfortunately we have managed to do that only for $k = 3$.

Remark 9. *For* $k = 3$ *the ring is a CI if and only if for* $(0, a, b, c)$ *we have* $a \le 2$ *or* $b - a \le 2$*.*

Remark 10. *The theorem gives lots of relations between Schur polynomials. Let the generators be* $S_k = s_{i_{k-1}-k+1, i_{k-2}-k+2, ..., i_1-1}, S_{k-1}, \dots, S_0 = s_{i_k-(k-1), i_{k-1}-(k-2), ..., i_1}$ *in degree increasing order. For* $s = 0, 1 \dots, k-1$ *we have*

$$h_{i_k-s}S_k - h_{i_{k-1}-s}S_{k-1} + \dots + (-1)^{k-1}h_{i_1-s}S_1 + (-1)^k h_{-s}S_0 = 0.$$

Here $h_i = 0$ *if* $i < 0$*.*

The following interesting fact was observed by the authors in numerous experiments with Macaulay. Consider the sequence of ideals $\mathcal{I}_I^{(0)} \subseteq \mathcal{I}_I^{(1)} \subseteq \mathcal{I}_I^{(2)} \subseteq .... \subseteq \mathcal{I}_I^{(k)} = \mathcal{I}_I^{\mathcal{A}}$, where $\mathcal{I}_I^{(l)}$ is generated by $S_k, S_{k-1}, ..., S_{k-l}$. The following strange-looking conjecture seems to hold.

Conjecture 11. *For an arbitrary* $I = \{0 = i_0 < i_1 < i_1 < ... < i_k\}$ *with* $\gcd(i_1, ..., i_k) = 1$ *one has*

$$\deg \mathcal{I}_I^{(2)} = \deg \mathcal{I}_I^{(3)} = ... = \deg \mathcal{I}_I^{(k)} = \sum_{1\le j<l\le k} i_j i_l - \binom{k}{2}\sum_{j=1}^{k} i_j + \binom{k+1}{3}(3k-2)/4.$$

Remark 12. *Notice that the ideals* $\mathcal{I}_I^{(l)}$*,* $l = 2, .., k-1$ *are not necessarily Cohen-Macaulay which explains why the latter conjecture might be valid.*

In connection with Theorems 6 and 8 the following question is completely natural.

Problem 4. Under the assumptions $i_0 = 0$ and $\gcd(i_1, ..., i_{m-1}) = 1$ which pairs $(k; I)$ are $\mathcal{A}$-regular?

Theorem 8 shows that for $m = k+1$ the condition $\gcd(i_1, ..., i_k) = 1$ guarantees regularity of any pair $(k; I)$ with $|I| = k+1$. On the other hand, our computer experiments with Macaulay show that for $m > k$ regular cases are rather seldom. In particular, we were able to prove the following.

Theorem 13. *If* $m > k$ *a necessary (but insufficient) condition for* $Vd_{k;I}^{\mathcal{A}}$ *to have the expected codimension is* $i_1 = 1$*.*

So far a complete (conjectural) answer to Problem 4 is only available in the first non-trivial case $k = 3,\ m = 5$. Namely, for a 5-tuple $I = \{0, 1, i_2, i_3, i_4\}$ to be regular one needs the corresponding the Vandermonde variety $Vd^{\mathcal{A}}_{3;I}$ to be a complete intersection. This is due to the fact that in this situation the ideal $\mathcal{I}^{\mathcal{A}}_I$ is generated by the Schur polynomials S_4, S_3, S_2 of the least degrees in the above notation. Notice that $S_4 = h_{i_2-2},\ S_3 = h_{i_3-2},\ S_2 = h_{i_4-2}$. Thus $Vd^{\mathcal{A}}_{3;I}$ has the expected codimension (equal to 3) if and only if $\mathbb{C}[x_1, x_2, x_3]/\langle h_{i_2-2}, h_{i_3-2}, h_{i_4-2}\rangle$ is a complete intersection or, in other words, $h_{i_2-2}, h_{i_3-2}, h_{i_4-2}$ is a regular sequence. Exactly this problem (along with many other similar questions) was considered in an intriguing paper [4] where the authors formulated the following claim, see Conjecture 2.17 of [4].

Conjecture 14. *Let $A = \{a, b, c\}$ with $a < b < c$. Then h_a, h_b, h_c in three variables is a regular sequence if and only if the following conditions are satisfied:*

(1) $abc \equiv 0 \mod 6$;
(2) $\gcd(a + 1, b + 1, c + 1) = 1$;
(3) *For all $t \in \mathbb{N}$ with $t > 2$ there exists $d \in A$ such that $d + 2 \not\equiv 0, 1 \mod t$.*

In fact, our experiments allow us to strengthen the latter conjecture in the following way.

Conjecture 15. *In the above set-up if the sequence h_a, h_b, h_c with $a > 1$ in three variables is not regular, then h_c lies in the ideal generated by h_a and h_b. (If $(a, b, c) = (1, 4, 3k + 2)$, $k \geq 1$, then h_a, h_b, h_c neither is a regular sequence, nor $h_c \in (h_a, h_b)$.) We note that if we extend the set-up of* [4] *by allowing Schur polynomials $s(r, s, t)$ instead of just complete symmetric functions then if $t > 0$ in all three of them the sequence is never regular.*

Conjecture 14 provides a criterion which agrees with our calculations of $\dim(Vd^{\mathcal{A}}_{3;I})$, see partial information in Table 2. Finally, we made experiments checking how $\dim(Vd^{\mathcal{A}}_{k;I})$ depends on the last entry i_{m-1} of $I = \{0, 1, i_2, ..., i_{m-1}\}$ while keeping the first $m - 1$ entries fixed.

Conjecture 16. *For any given $I = (0, 1, i_2, ..., i_{m-1})$ one has that $\dim(Vd^{\mathcal{A}}_{k;I})$ depends periodically on i_{m-1} for all i_{m-1} sufficiently large.*

Notice that Conjecture 16 follows from Conjecture 14 in the special case $k = 3,\ m = 5$. Unfortunately, we do not have a complete description of the length of this period in terms of the fixed part of I at the moment and it might be quite tricky.

For the $\mathcal{BC}_k$-localized variety $Vd^{BC}_{k;I}$ we have only conjectures, supported by many calculations.

Conjecture 17. *For any integer sequence $I = \{0 = i_0 < i_1 < i_2 < ... < i_k\}$ of length $k + 1$ with $\gcd(i_1, ..., i_k) = 1$ the following facts are valid.*

(i) *$codim(Vd^{BC}_{k;I}) = 2$;*
(ii) *The quotient ring $\mathcal{R}^{BC}_I$ is Cohen-Macaulay;*
(iii) *There is a $\mathcal{C}[x_1, \ldots, x_n] = R$-resolution of $\mathcal{R}^{BC}_I$ of the form*

$$0 \to \oplus_{j=0}^{k-1} R[-N+j+\binom{k}{2}] \to \oplus_{j=1}^{k} R[-N+i_j+\binom{k}{2}] \oplus R[-N+ki_1] \to R \to R \to \mathcal{R}^{BC}_I \to 0$$

(iv) *The Hilbert series $Hilb^{BC}_I(t)$ of $\mathcal{R}^{BC}_I$ is given by the formula*

$$Hilb^{BC}_I(t) = \left(1 - \sum_{j=1}^{k} t^{N-j-\binom{k}{2}} - t^{N-ki_1} + \sum_{j=1}^{k-1} t^{N-i_1-j} - \binom{k}{2}\right) / (1-t)^k$$

where $N = \sum_{j=1}^{k} i_j$;

(v) $\deg(Vd^{BC}_{k;I}) = \sum_{1\le j<l\le k} i_j i_l - \binom{k}{2}\sum_{j=1}^k i_j + \binom{k+1}{3}(3k-2)/4 - \binom{k}{2} i_1(i_1-1)$;

(vi) *The ideal* $\mathcal{I}^{BC}_I$ *is always generated by* k *generators. It is generated by two elements (i.e. is a complete intersection) if* $i_1 \le k-1$.

(vii) *Let* $S_k, \dots, S_1$ *be as in Theorem 6 and* $G_0 = s_{i_k-i_1-k+1,\dots,i_2-i_1-1}$. *Then, for* $s = 0, \dots, k-1$ *we have*

$$h_{i_k-i_1-s}S_k - h_{i_{k-1}-i_1-s}S_{k-1} + \cdots + (-1)^{k-2}h_{i_2-i_1-s}S_2 + (-1)^{k-1}h_{-s}S_1 +$$

$$(-1)^k s_{i_1-1,\dots,(i_1-1)^{k-1},k-1-s}G_0.$$

Here $h_i = 0$ *if* $i < 0$ *and* $h_{i,\dots,i,j} = 0$ *if* $j > i$, *and* $(i_1-1)^{k-1}$ *means* $i_1-1, \dots, i_1-1$ *(*$k-1$ *times).*

The structure of the paper is as follows. In § 2 we prove our main results on $V_{k;I}$ and $Vd^{\mathcal{A}}_{K;I}$. In § 3 we prove our main results on $Vd^{BC}_{k;I}$. In § 4 we formulate a number of related projects and further directions. Finally, in Appendix we present supporting tables obtained by using Macaulay.

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2. Results about $V_{k;I}$ and $Vd^{\mathcal{A}}_{k;I}$

We first prove Proposition 2.

Proof. We will show that for any pair $(k;I)$ the variety $\overline{V}_{k;I}$ of linear recurrences is constructible. Since it is by definition closed in the usual topology of $L_k \simeq \mathbb{C}^k$ it is an algebraic, see [12], I.10 Corollary 1 claiming that if $Z \subset X$ is a constructible subset of a variety, then the Zarisky closure and the strong closure of Z are the same. Instead of showing that $\overline{V}_{k;I}$ is constructible we prove that $V_{k;I} \subset L^*_k$ is constructible. Namely, we can use an analog of Lemma 4 to construct a natural stratification of $V_{k;I}$ into the images of quasi-affine sets under appropriate Vieta maps. Namely, we stratify $V_{k;I}$ as $V_{k;I} = \bigcup_{\lambda\vdash k} V^\lambda_{k;I}$ where $\lambda \vdash k$ is an arbitrary partition of k and $V^\lambda_{k;I}$ is the subset of $V_{k;I}$ consisting of all recurrence relations of length exactly k which has a non-trivial solution vanishing at each point of I and whose characteristic polynomial determines the partition λ of its degree k. In other words, if $\lambda = (\lambda_1, ..., \lambda_s)$, $\sum_{j=1}^s \lambda_j = k$ then the characteristic polynomial should have s distinct roots of multiplicities $\lambda_1, ..., \lambda_s$ resp. Notice that any of these $V^\lambda_{k;I}$ can be empty including the whole $V_{k;I}$ in which case there is nothing to prove. Let is now show that each $V^\lambda_{k;I}$ is the image of a set similar to the open Vandermonde variety under the appropriate Vieta map. Recall that if $\lambda = (\lambda_1, ..., \lambda_s)$, $\sum_{j=1}^s \lambda_j = k$ and $x_1, .., x_s$ are the distinct roots with the multiplicities $\lambda_1, ..., \lambda_s$ respectively of the linear recurrence (1) then the general solution of (1) has the form

$$u_n = P_{\lambda_1}(n)x_1^n + P_{\lambda_2}(n)x_2^n + ... + P_{\lambda_s}(n)x_s^n$$

where $P_{\lambda_1}(n), ..., P_{\lambda_s}(n)$ are arbitrary polynomials in the variable n of degrees $\lambda_1 - 1, \lambda_2 - 1, ..., \lambda_{s-1}$ resp. Now, for a given $\lambda \vdash k$ consider the set of matrices

$$M^\lambda_{k;I} =$$

$$\begin{pmatrix} x_1^{i_0} & i_0x_1^{i_0} & \dots & i_0^{\lambda_1-1}x_1^{i_0} & \dots & x_s^{i_0} & i_0x_s^{i_0} & \dots & i_0^{\lambda_s-1}x_s^{i_0} \\ x_1^{i_1} & i_1x_1^{i_1} & \dots & i_1^{\lambda_1-1}x_1^{i_1} & \dots & x_s^{i_1} & i_1x_s^{i_1} & \dots & i_1^{\lambda_s-1}x_s^{i_1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_1^{i_{m-1}} & i_{m-1}x_1^{i_{m-1}} & \dots & i_{m-1}^{\lambda_1-1}x_1^{i_{m-1}} & \dots & x_s^{i_{m-1}} & i_1x_s^{i_{m-1}} & \dots & i_{m-1}^{\lambda_s-1}x_s^{i_{m-1}} \end{pmatrix}.$$

That is we are taking the fundamental solution $x_1^n, nx_1^n, ..., n^{\lambda_1-1}x_1^n$, x_2^n, nx_2^n, $..., n^{\lambda_2-1}x_2^n, ...,$ $x_s^n, nx_s^n, ..., n^{\lambda_s-1}x_s^n$ of (1) under the assumption that the characteristic polynomial gives a partition λ of k and are evaluating each function in this system at the points $i_0, i_1, ..., i_{m-1}$ resp. We now define the variety $Vd_{k;I}^\lambda$ as the subset of matrices of the form $M_{k;I}^\lambda$ such that: (i) the rank of such a matrix is smaller than k; (ii) all x_i are distinct; (iii) all x_i are non-vanishing. Obviously, $Vd_{k;I}^\lambda$ is a quasi-projective variety in $\mathbb{C}^s$. Define the analog $Vi_\lambda : \mathbb{C}^s \to L_k$ which sends an s-tuple $(x_1, ..., x_s) \in \mathbb{C}^s$ to the polynomials $\prod_{j=1}^s (x - x_j)^{\lambda_j} \in L_k$ of the Vieta map Vi. One can easily see that Vi_λ maps $Vd_{k;I}^\lambda$ onto $V_{k;I}^\lambda$. Applying this construction to all partitions $\lambda \vdash k$ we will obtain that $V_{k;I} = \bigcup_{\lambda\vdash k} V_{k;I}^\lambda$ is constructible which finishes the proof. □

Let us settle Lemma 5.

Proof. It follows directly from the standard Jacobi-Trudi identity for the Schur polynomials, see e.g. [17]. □

We now prove Theorem 6.

Proof. According to [6] provided that $\mathcal{I}_I^\mathcal{A}$ has the expected codimension $m - k + 1$, it is known to be Cohen-Macaulay and it has a resolution of the form

$$0 \to F_{m-k+1} \to \cdots \to F_1 \to \mathcal{R} \to \mathcal{R}_I^\mathcal{A} \to 0, \tag{6}$$

where F_j is free module over $\mathcal{R} = \mathbb{C}[x_1, \dots, x_k]$ of rank $\binom{m}{k+j-1}\binom{k+j-2}{k-1}$. We denote the basis elements of F_j by $M_I T$, where $I \subseteq \{i_0, \dots . i_{m-1}\}$, $|I| = k+j-1$, and T is an arbitrary monomial in $\{t_0, \dots, t_{k-1}\}$ of degree $j - 1$. If $M_I = \{i_{l_1}, \dots, i_{l_{k+j-1}}\}$ and $T = t_{j_1}^{s_1} \cdots t_{j_r}^{s_r}$ with $s_i > 0$ for all i and $\sum_{i=1}^r s_i = j - 1$, then, in our situation, $d(M_I T) = \sum_{i=1}^r (\sum_{l=1}^{k+j-1} (-1)^{k+1} h_{i_{k_l} - j_i} M_{I \setminus \{i_{k_l}\}}) T/t_{j_i}$. Here $\deg(M_I) = \sum_{n=1}^{k+j-1} i_{l_j} - \binom{k}{2}$ and $\deg(t_i) = -i$. (Note that $\mathrm{tdeg}(t_i) = 1$ but $\deg(t_i) = -i$.) Thus $\deg(M_I T) = \sum_{n=1}^{k+j-1} i_{l_n} + \sum_{i=1}^r s_i j_i - \binom{k}{2}$ if $M_I = \{i_{l_1}, \dots, i_{l_{k+j-1}}\}$ and $T = t_{j_1}^{s_1} \cdots t_{j_r}^{s_r}$. Observe that this resolution is never minimal. Indeed, for any sequence $I = \{0 = i_0 < i_1 \cdots < i_{m-1}\}$, we only need the Schur polynomials coming from subsequences starting with 0, so $\mathcal{I}_I^\mathcal{A}$ is generated by at most $\binom{m-1}{k-1}$ Schur polynomials instead of totally $\binom{m}{k}$, see discussions preceding the proof of Theorem 8 below. Now, if $\mathcal{J}$ is an arbitrary homogeneous ideal in $\mathcal{R} = \mathbb{C}[x_1, \dots, x_k]$ and $\mathcal{R}/\mathcal{J}$ has a resolution

$$0 \to \oplus_{i=1}^{\beta_r} \mathcal{R}(-n_{r,i}) \to \cdots \to \oplus_{i=1}^{\beta_1} \mathcal{R}(-n_{1.i}) \to \mathcal{R} \to \mathcal{R}/\mathcal{J} \to 0,$$

then the Hilbert series of $\mathcal{R}/\mathcal{J}$ is given by

$$\frac{1 - \sum_{i=1}^{\beta_1} t^{n_{1,i}} + \cdots + (-1)^r \sum_{i=1}^{\beta_r} t^{n_{r,i}}}{(1-t)^k}.$$

For the resolution (6), all terms coming from $M_I T$ with $t_0 | T$ cancel. Thus we get the claimed Hilbert series. If the Hilbert series is given by $T(t)/(1-t)^k = P(t)/((1-t)^{\dim(\mathcal{R}/\mathcal{I}_I)}$, then the degree of the corresponding variety equals $P(1)$. We have $T(t) = (1-t)^{m-k+1} P(t)$, or after differentiating the latter identity $m-k+1$ times we get $P(1) = T^{(m-k+1)}(1)(-1)^{m-k+1}/(m-k+1)!$. □

Example 18. *For the case* 3×5 *with* $I=\{0,i_1,i_2,i_3,i_4)$ *, if the ideal* $Vd^{\mathcal{A}}_{k;I}$ *has the right codimension, we get that its Hilbert series equals* $T(t)/(1-t)^3$*, where*

$$T(t)=1-t^{i_1+i_2-2}-t^{i_1+i_3-2}-t^{i_1+i_4-2}-t^{i_2+i_3-2}-t^{i_2+i_4-2}-t^{i_3+i_4-2}+t^{i_1+i_2+i_3-3}+ t^{i_1+i_2+i_4-3}+t^{i_1+i_3+i_4-3}+t^{i_2+i_3+i_4-3}+t^{i_1+i_2+i_3-4}+t^{i_1+i_2+i_4-4}+t^{i_1+i_3+i_4-4}+ t^{i_2+i_3+i_4-4}-t^{i_1+i_2+i_3+i_4-3}-t^{i_1+i_2+i_3+i_4-4}-t^{i_1+i_2+i_3+i_4-5}$$

and the degree of $Vd^{\mathcal{A}}_{k;I}$ *equals*

$$i_1i_2i_3+i_1i_2i_4+i_1i_3i_4+i_2i_3i_4-3(i_1i_2+i_1i_3+i_1i_4+i_2i_3+i_2i_4+i_3i_4)+7(i_1+i_2+i_3+i_4)-15.$$

Let us now settle Proposition 7.

Proof. In the Giambelli formula setting, we consider a "generic" family of $(n\times l)$-matrices $A=||a_{p,q}||,\ 1\le p\le n,\ 1\le q\le l$, whose entries are homogeneous functions of degrees $\deg(a_{p,q})=\alpha_p-\beta_q$ in parameters $(x_1,...,x_k)$ for some fixed sequences $\beta=(\beta_1,...,\beta_l)$ and $\alpha=(\alpha_1,...,\alpha_n)$. Denote by Σ^r the subvariety in the parameter space $\mathbb{C}^k$ determined by the condition that the matrix A has rank at most $l-r$, that is, the linear operator $A:\mathbb{C}^l\to\mathbb{C}^n$ has at least a r-dimensional kernel. Then the expected codimension of the subvariety Σ^r is equal to

$$\mathrm{codim}(\Sigma^r)=r(n-l+r),$$

and in case when the actual codimension coincides with the expected one its degree is computed as the following $r\times r$-determinant:

$$\deg(\Sigma^r)=\det||c_{n-l+r-i+j}||_{1\le i,\ j\le r}, \tag{7}$$

where the entries c_i's are defined by the Taylor expansion

$$1+c_1t+c_2t^2+...=\frac{\prod_{p=1}^n(1+\alpha_pt)}{\prod_{q=1}^l(1+\beta_qt)}.$$

There is a number of situations where this formula can be applied. Depending on the setting, the entries α_p,β_q can be rational numbers, formal variables, first Chern classes of line bundles or formal Chern roots of vector bundles of ranks n and l, respectively. In the situation of Theorem 6 we should use the presentation (5) of $Vd^{\mathcal{A}}_{k;I}$ from Lemma 5. Then we have $n=m,\ l=k,\ r=1,\ \alpha=I=(0,i_1,...,i_{m-1}),\ \beta=(k-1,k-2,...,0)$. Under the assumptions of Theorem 6 the degree of the Vandermonde variety $Vd^{\mathcal{A}}_{k;I}$ will be given by the 1×1-determinant of the Giambelli-Porteous formula (7), that is, the coefficient c_{m-k+1} of t^{m-k+1} in the expansion of

$$1+c_1t+c_2t^2+...=\frac{\prod_{j=0}^{m-1}(1+i_jt)}{\prod_{j=1}^{k}(1+(k-j)t)}=\frac{\prod_{j=1}^{m-1}(1+i_jt)}{\prod_{j=1}^{k-1}(1+jt)},$$

which gives exactly the stated formula for $\deg(Vd^{\mathcal{A}}_{k;I})$. □

To prove Theorem 8 notice that in the case $m=k+1$ the ideal $\mathcal{I}^{\mathcal{A}}_I$ always has the expected codimension 2 unless it coincides with the whole ring $\mathbb{C}[x_1,...,x_k]$, since Schur polynomials are irreducible [5], so vanishing of any two Schur polynomials lowers the dimension by two. (Recall that we assume that $\gcd(i_1,...,i_k)=1$.) On the other hand, as we mentioned in the introduction the codimension of $Vd^{\mathcal{A}}_{k;I}$ in this case is at most 2. For $m=k+1$ one can present a very concrete resolution of the quotient ring $\mathcal{R}^{\mathcal{A}}_I$.

Namely, given a sequence $I=\{0=i_0<i_1<\dots<i_k\}$ we know that the ideal $\mathcal{I}^{\mathcal{A}}_I$ is generated by the $k+1$ Schur polynomials $S_l=s_{a_k,a_{k-1},\dots,a_1},\ l=0,\dots,k$, where

$$(a_k,\dots,a_1)=(i_k,i_{k-1},\dots,i_{l+1},\hat{i_l},i_{l-1},\dots,i_0)-(k-1,k-2,\dots,1,0).$$

Obviously, S_l has degree $\sum_{j=1}^k i_j - i_l - \binom{k}{2}$ and by the Jacobi-Trudi identity is given by

$$S_l = \begin{vmatrix} h_{i_0-(k-1)} & h_{i_0-(k-2)} & \cdots & h_{i_0} \\ h_{i_1-(k-1)} & h_{i_1-(k-2)} & \cdots & h_{i_1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_{l-1}-(k-1)} & h_{i_{l-1}-(k-2)} & \cdots & h_{i_{l-1}} \\ h_{i_{l+1}-(k-1)} & h_{i_{l+1}-(k-2)} & \cdots & h_{i_{l+1}} \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_{k-1}-(k-1)} & h_{i_{k-1}-(k-2} & \cdots & h_{i_{k-1}} \\ h_{i_k-(k-1)} & h_{i_k-(k-2)} & \cdots & h_{i_k} \end{vmatrix}$$

where (as above) h_j denotes the complete symmetric function of degree j in $x_1, ..., x_k$. (We set $h_j = 0$ if $j < 0$ and $h_0 = 1$.) Consider the $(k+1) \times k$-matrix $H = H_{k;I}$ given by

$$H = \begin{pmatrix} h_{i_0-(k-1)} & h_{i_0-(k-2)} & \cdots & h_{i_0} \\ h_{i_1-(k-1)} & h_{i_1-(k-2)} & \cdots & h_{i_1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_{k-1}-(k-1)} & h_{i_{k-1}-(k-2} & \cdots & h_{i_{k-1}} \\ h_{i_k-(k-1)} & h_{i_k-(k-2)} & \cdots & h_{i_k} \end{pmatrix}$$

Let H_l be the $(k+1) \times (k+1)$-matrix obtained by extending H with l-th column of H. Notice that $\det(H_l) = 0$, and expanding it along the last column we get for $0 \leq l \leq k-1$ the relation

$$0 = \det(H_l) = h_{i_k-(k-l)} S_k - h_{i_{k-1}-(k-l)} S_{k-1} + \cdots + (-1)^{k-1} h_{i_1-(k-l)} S_1.$$

For $l = k$ we get

$$h_{i_k} S_k - h_{i_{k-i}} S_{k-1} + \cdots + (-1)^k h_{i_0} S_0 = 0$$

which implies that S_0 always lie in the ideal generated by the remaining $S_1, \ldots, S_k$.

We now prove Theorem 8.

Proof. Set $N = \sum_{j=1}^k i_j$. For an arbitrary $I = \{0, i_1, ..., i_k)$ with $\gcd(i_1, ..., i_k) = 1$ we get the following resolution of the quotient ring $\mathcal{R}_I^{\mathcal{A}} = \mathcal{R}/\mathcal{I}_I^{\mathcal{A}}$

$$0 \longrightarrow \oplus_{l=1}^k \mathcal{R}(-N + \binom{k}{2} + l) \longrightarrow \oplus_{l=1}^{k-1} \mathcal{R}(-N + i_l + \binom{k}{2}) \longrightarrow \mathcal{R} \longrightarrow \mathcal{R}_I^{\mathcal{A}} \longrightarrow 0$$

where $\mathcal{R} = \mathbb{C}[x_1, ..., x_k]$. Simple calculation with this resolution implies that the Hilbert series $Hilb_I^{\mathcal{A}}(t)$ of $\mathcal{R}_I^{\mathcal{A}}$ is given by

$$Hilb_I^{\mathcal{A}}(t) = \left(1 - \sum_{l=1}^k t^{N-i_l-\binom{k}{2}} + \sum_{l=1}^{k-1} t^{N-\binom{k}{2}-l}\right) / (1-t)^k$$

and the degree of $Vd_{k;I}^{\mathcal{A}}$ is given by

$$\deg(Vd_{k;I}^{\mathcal{A}}) = \sum_{1 \leq r < s \leq k} i_r i_s - \binom{k}{2} \sum_{r=1}^k i_r + \binom{k+1}{3} (3k-2)/4.$$

Notice that the latter resolution might not be minimal, since the ideal might have fewer than k generators. To finish proving Theorem 8 notice that if conditions of (v) are satisfied then a closer look at the resolution reveals that the Schur polynomials $S_0, \ldots, S_{k-n}$ lie in the ideal generated by $S_{k-n+1} \ldots, S_k$. □

We now prove Theorem 13.

Proof. If $i_1 \geq 2$, then $i_{k-2} \geq k-1$. This means that the ideal is generated by Schur polynomials $s_{a_0,\dots,a_{k-1}}$ with $a_{k-2} \geq 1$. Multiplying these up to degree n gives linear combinations of Schur polynomials $s_{b_1,\dots,b_{k-1}}$ with $b_{k-2} \geq 1$. Thus we miss all Schur polynomials with $b_{k-2} = 0$. The number of such Schur polynomials is the number of partitions of n in at most $k-2$ parts. The number of partitions of n in exactly $k-2$ parts is approximated with $n^{k-3}/((k-2)!(k-1)!)$. Thus the number of elements of degree n in the ring is at least cn^{k-3} for some positive c, so the ring has dimension $\geq k-2$. The expected dimension is $\leq k-3$, which implies a contradiction. □

2.1. **Further results for $k = 3$.** For $k = 3$ and any 4-tuple $I = \{0, i_1, i_2, i_3\}$ with $\gcd(i_1, i_2, i_3) = 1$ the variety $Vd^{\mathcal{A}}_{3;I}$ is a finite collection of lines in $\mathbb{C}^3$ with multiplicities. These lines can either coincide with one of coordinate axes (and then have no meaning in terms of recurrences) or lie on coordinate planes with the coordinate axes removed (and then correspond to recurrence relations of order 2) or lie in the complement to coordinate planes (and then correspond to the actual recurrence relations of order 3).

Let us first present results about the coordinate axes and lines the in coordinate planes which are guessed from our computer experiments, see Table 1 below which contains complete information for all 4-tuples with $i_3 \leq 13$.

Conjecture 19. *For any $I = \{0 < i_1 < i_2 < i_3\}$ the sum of the multiplicities of the three coordinate axes in $Vd^{\mathcal{A}}_{3;I}$ equals $3((i_1-1)(i_2-1) + \gcd(i_2-i_1, i_3-i_2))$.*

First we show that for I as above the coordinate axes belong to $Vd^{\mathcal{A}}_{3;I}$ unless $i_1 = 1$, see Table 1. To simplify the notation set $i_1 = a,\ i_2 = b,\ i_3 = c$ and $x_1 = x,\ x_2 = y,\ x_3 = z$. Due to S_3-symmetry we check the statement for the x-axis only. Notice that $Vd^{\mathcal{A}}_{3;I}$ coincides with the set of degenerate matrices of the form:

$$H_{3;I}(x,y,z) = \begin{pmatrix} 0 & 0 & 1 \\ h_{a-2}(x,y,z) & h_{a-1}(x,y,z) & h_a(x,y,z) \\ h_{b-2}(x,y,z) & h_{b-1}(x,y,z) & h_b(x,y,z) \\ h_{c-2}(x,y,z) & h_{c-1}(x,y,z) & h_c(x,y,z) \end{pmatrix} \tag{8}$$

If we set $y = z = 0$ we get

$$H_{3;I}(x,0,0) = \begin{pmatrix} 0 & 0 & 1 \\ x^{a-2} & x^{a-1} & x^a \\ x^{b-2} & x^{b-1} & x^b \\ x^{c-2} & x^{c-1} & x^c \end{pmatrix}.$$

In case $i_1 > 1$ the latter matrix is obviously degenerate for all x since its first and second columns are proportional . On the other hand, for $i_1 = 1$ the second row is $(0, 1, x)$ which implies that to get rank of the whole matrix equal to 2 one has to set $x = 0$, i.e. the whole solution will be trivial. It remains to prove that the local multiplicity of each axis equals $(a-1)(b-1) + \gcd(b-a, c-b)$.

Further we calculate the number of recurrences with exactly one characteristic root equal to zero, i.e. the number of lines in $Vd_{3;I}$ lying in the coordinate planes but different from the coordinate axes.

Conjecture 20. *The sum of the multiplicities of all lines in $Vd^{\mathcal{A}}_{3;I}$ lying on coordinates planes but different from the coordinate axes equals $3(\gcd(i_2-i_1, i_3-i_2)-1)i_1$.*

Using the same notation as in the prevoous conjecture we will first show that in each coordinate plane with the coordinate axes removed there lie exactly $\gcd(b-$

$a, c-b) - 1$ lines belonging to $Vd^{\mathcal{A}}_{3;I}$. Setting $z = 0$ in (8) we get the condition

$$H_{3;I}(x,y,0) = \begin{pmatrix} h_{a-2}(x,y) & h_{a-1}(x,y) \\ h_{b-2}(x,y) & h_{b-1}(x,y)) \\ h_{c-2}(x,y) & h_{c-1}(x,y) \end{pmatrix} \tag{9}$$

has rank 1. Using homogenuity and condition $y \neq 0$ we set $y = 1$ and obtain

$$\Delta_{1,2} = \begin{vmatrix} h_{a-2}(x,1) & h_{a-1}(x,1) \\ h_{b-2}(x,1) & h_{b-1}(x,1) \end{vmatrix} = \frac{x^a(x^{b-a}-1)}{1-x}.$$

The roots of the equation $\Delta_{1,2}(x) = 0$ different from 0 are exactly all roots of unity of order $b - a$ except 1. Analogously, $\Delta_{1,3}(x) = 0$ has as solutions all roots of unity of order $c - a$ except 1. Finally, $\Delta_{2,3}(x) = 0$ has as solutions all roots of unity of order $c-a$ except 1. The solutions of the overdetermined system $\Delta_{1,2}(x) = \Delta_{1,3}(x) = \Delta_{2,3}(x) = 0$ are exactly the common roots of individual equations. There are exactly $\gcd(b-a, c-b) - 1$ since $\gcd(b-a, c-b, c-a) = \gcd(b-a, c-b)$ and we are excluding the root 1.It remains to show that the multiplicity of each root is a.

Finally, let us present the information about arithmetic sequences among our solutions which is most relevant in connection with the Skolem-Mahler-Lech theorem. Namely, some recurrences vanishing at a given I are degenerate (see Introduction), i.e. their solution vanishing at a given 4-tuple I must necessarily have infinitely many integer zeros. Using Macaulay we determined when this occur for all 4-tuples with $i_3 \leq 13$ restricting ourselves to consideration of the arithmetic progressions with difference ≤ 10, see Table 1. It turned out that we didn't get any solutions when the difference $d > 5$. For the case $d \leq 5$ our results are listed in the Appendix and below. (Since one gets the same result for $(0, i_1, i_2, i_3)$ as for $(0, i_3-i_2, i_3-i_1, i_3)$, we present only one of these.)

Explanations to Table 1. Its first column contains the degree of $Vd^{\mathcal{A}}_{3;I}$, the remaining columns contain the rest of the degree obtained after the removal of all solutions on the coordinate axes, the coordinate planes, and degenerate solutions containing the arithmetic progressions with difference 2,3,4,5, respectively.

We obtained solutions with arithmetic progressions with difference 2 when (i_1, i_2, i_3) equals: $(1,4,9), (1,4,10)$, $(1,4,12), (1,4,13), (1,5,8)$,(1,5,12),(1,5,13),(1,6,9), (1,6,13),(1,7,12),(1,8,12),(1,8,13),(1,9,12), (1,9,13),(3,4,12),(3,7,12), $(3,8,12)$, $(4,5,12), (4,5,13)$, $(4,6,12), (4,7,11), (4,7,12)$.

We obtained solutions with arithmetic progressions with difference 3 when (i_1, i_2, i_3) equals: $(1,3,7), (1,3,9), (1,3,10), (1,3,10), (1,3,12), (1,3,13), (1,4,6), (1,4,9), (1,4,12), (1,4,13), (1,5,8), (1,5,12), (1,5,13), (1,6,9), (1,6,13), (1,7,9), (1,7,10)$, $(1,7,12), (1,9,12), (1,9,13), (1,10,12), (1,10,13), (2,3,8), (2,3,9), (2,3,11), (2,3,12), (2,5,6), (2,5,9), (2,5,11), (2,5,12), (2,6,11), (2,6,13), (2,8,11), (2,9,12), (3,4,9), (3,4,10), (3,4,11), (3,4,12), (3,4,13), (3,5,9), (3,5,11), (3,5,12), (3,7,12), (3,7,13), (3,8,12), (3,9,13), (4,6,13), (4,7,11), (4,7,12), (5,6,13)$.

We obtained solutions with arithmetic progressions with difference 4 when (i_1, i_2, i_3) equals: $(1,4,12), (1,5,13), (1,8,12), (1,9,13), (3,4,12), (3,8,12), (4,5,13), (4,7,12)$.

Finally, we obtained solutions with arithmetic progressions with difference 5 when (i_1, i_2, i_3) equals: $(1,5,11), (1,6,10), (2,5,12), (2,7,10), (3,5,13)$.

Conjecture 21. *Non-degenerate solutions always form complete orbits of S_3. In particular, the total number is divisible by 6. Moreover, there are exactly 6 solutions which were roots of a real equation, i.e. the S_3-orbit of the unique linear recurrence with real coefficients.*

Here are some explanations for the tables in the appendix. In Tables 3 and 4 $CI(a,b,c)$ means that we have a complete intersection with generators of degrees a,b,c, and similarly for $CM(a,b,c,d)$.

3. Final Remarks

1. Results of [3] show that if a 4-tuple $(0,i_1,i_2,i_3)$ contains either a triple forming three consecutive terms of an arithmetic progression or 2 pairs with the same difference then any exponential polynomial of order 3 vanishing at these 4-tuple must necessarily contain an arithmetic progression of integer zeros. In our experiments we found only two other case of 4-tuples (up to the reversal of time and shift) such that the same fact holds but they are not covered by Beukers' theorem, namely, $(0,1,3,7)$ and $(0,1,3,9)$, see Table 1. We suspect that our exceptions are the only possible.

Problem 5. Is it true that if an $(k+1)$-tuple I consists of two pieces of arithmetic progression with the same difference then any exponential polynomial vanishing at I contains an arithmetic progression of integer zeros?

Problem 6. If the answer to the previous question is positive is it true that there are only finitely many exceptions from this rule leading to only arithmetic progressions?

2. A problem similar to that of J. H. Loxton and A. J. van der Poorten can be formulated for real zeros of exponential polynomials instead of integer. Namely, the following simple lemma is true.

Lemma 22. *Let $\lambda_1,...,\lambda_n$ be a arbitrary finite set of (complex) exponents having all distinct real parts then an arbitrary exponential polynomial of the form $c_1e^{\lambda_1 z} + c_2e^{\lambda_2 z} + .. + c_ne^{\lambda_n z}$, $c_i \in \mathbb{C}$ has at most finitely many real zeros.*

Problem 7. Does there exist an upper bound on the maximal number real for the set of exponential polynomials given in the latter lemma in terms of n only?

3. What about non-regular cases? Describe their relation to the existence of additional integer zeros and arithmetic progressions as well as additional Schur polynomials in the ideals.

4. Appendix

4.1. Table 1. 3×4**-case for** $\mathcal{A}$**.**

$4-tuple\ I$	$Degree$	$Axes$	$Planes$	$d=2$	$d=3$	$d=4$	$d=5$
$0,1,3,7$	5		2		0		
$0,1,3,8$	6						
$0,1,3,9$	7		4		0		
$0,1,3,10$	8				6		
$0,1,3,11$	9		6				
$0,1,3,12$	10				6		
$0,1,3,13$	11		8		6		
$0,1,4,6$	8				6		
$0,1,4,9$	14			8	6		
$0,1,4,10$	16		10	6			
$0,1,4,11$	18						
$0,1,4,12$	20			14	12	6	
$0,1,4,13$	22		16	10	6		
$0,1,5,7$	15		12				
$0,1,5,8$	18			12			
$0,1,5,11$	27		24				12
$0,1,5,12$	30			24			
$0,1,5,13$	33		24	18		12	
$0,1,6,8$	24						
$0,1,6,9$	28			24			
$0,1,6,10$	32				30		18
$0,1,6,13$	44			32	24		
$0,1,7,9$	35		32		30		
$0,1,7,10$	40		34		30		
$0,1,7,11$	45		42				
$0,1,7,12$	50			38	30		
$0,1,8,10$	48						
$0,1,8,11$	54						
$0,1,8,12$	60			54		48	
$0,1,8,13$	66			60			
$0,1,9,11$	63		60				
$0,1,9,12$	70			64	60		
$0,1,9,13$	77		68	62	60	54	
$0,1,10,12$	80				78		
$0,1,10,13$	88		82		78		
$0,1,11,13$	99		96				
$0,2,3,7$	12	6					
$0,2,3,8$	14	8			6		
$0,2,3,9$	10				6		
$0,2,3,10$	18	12					
$0,2,3,11$	20	14			12		
$0,2,3,12$	22	16			12		
$0,2,3,13$	24	18					
$0,2,5,6$	20	8			6		
$0,2,5,9$	32	20			18		
$0,2,5,11$	40	28	16		12		
$0,2,5,12$	44	32			30		18
$0,2,5,13$	48	36					
$0,2,6,7$	12						
$0,2,6,9$	40	16			12		
$0,2,6,11$	50	20			18		
$0,2,6,13$	60	24					
$0,2,7,8$	42	24					
$0,2,7,10$	54	36				24	

$4-tuple\ I$	*Degree*	*Axes*	*Planes*	$d=2$	$d=3$	$d=4$	$d=5$
$0,2,7,11$	60	42					
$0,2,7,13$	72	54					
$0,2,8,9$	56	32			30		
$0,2,8,11$	70	40	28		24		
$0,2,8,13$	84	48					
$0,2,9,10$	72	48					
$0,2,9,12$	88	64			60		
$0,2,9,13$	96	72					
$0,2,10,11$	90	60					
$0,2,10,13$	108	72					
$0,2,11,12$	110	80			78		
$0,2,12,13$	96						
$0,3,4,9$	34	16			12		
$0,3,4,10$	38	20			18		
$0,3,4,11$	38	20			18		
$0,3,4,12$	46	28		22	18	12	
$0,3,4,13$	50	32			30		
$0,3,5,9$	43	19	10		6		
$0,3,5,11$	53	29	20		18		
$0,3,5,12$	58	34			30		
$0,3,5,13$	63	39	30				18
$0,3,7,9$	61	25	16		12		
$0,3,7,12$	82	46		40	36		
$0,3,7,13$	89	53	44		42		
$0,3,8,9$	70	28			24		
$0,3,8,10$	78	36					24
$0,3,8,12$	94	52		46	42	36	
$0,3,9,10$	88	34			30		
$0,3,9,11$	97	37	28		24		
$0,3,9,13$	115	43	34		30		
$0,3,10,11$	108	54					
$0,3,10,12$	118	64			60		
$0,3,11,12$	130	70		64	60		
$0,3,11,13$	141	81	72				
$0,3,12,13$	154	82			78		
$0,4,5,7$	43	6					
$0,4,5,11$	66	30					
$0,4,5,12$	72	36		30		24	
$0,4,5,13$	78	42		36			
$0,4,6,7$	50	2			0		
$0,4,6,9$	64	10			6		
$0,4,6,11$	78	18					
$0,4,6,13$	92	26		24			
$0,4,7,9$	74	20			18		
$0,4,7,11$	90	36		30			
$0,4,7,12$	98	44		38	36	30	
$0,4,7,13$	106	52	28		24		
$0,4,9,10$	104	32			30		18
$0,4,9,11$	114	42					
$0,4,9,12$	124	52		46	42	36	
$0,4,10,11$	126	42					
$0,4,10,13$	148	58	34		30		
$0,4,11,12$	150	60		54		48	
$0,4,11,13$	162	72					
$0,4,12,13$	176	68		62	60	54	

$4-tuple\ I$	$Degree$	$Axes$	$Planes$	$d=2$	$d=3$	$d=4$	$d=5$
$0,5,6,8$	68	8			6		
$0,5,6,9$	76	16			12		
$0,5,6,13$	108	48					
$0,5,7,8$	78	6					
$0,5,7,11$	105	33	18				
$0,5,7,13$	123	51	36				
$0,5,8,9$	98	14		8	6		
$0,5,8,12$	128	44		38	36	30	
$0,5,9,11$	131	35	20		18		
$0,5,9,12$	142	46		40	36		
$0,5,11,12$	170	50		38	30		
$0,5,11,13$	183	63	48				
$0,5,12,13$	198	66		60			
$0,6,7,9$	100	10			6		
$0,6,7,10$	110	20			18		
$0,6,7,11$	120	30					
$0,6,8,9$	112	4			0		
$0,6,8,11$	134	20			18		
$0,6,8,13$	156	36					
$0,6,9,10$	136	10			6		
$0,6,9,11$	148	16			12		
$0,6,9,13$	172	28			24		
$0,6,10,11$	162	24					12
$0,6,10,13$	188	44			42		
$0,6,11,13$	204	54					
$0,7,8,10$	138	12					
$0,7,8,11$	150	24		18			
$0,7,8,12$	162	36		30		24	
$0,7,8,13$	174	48					
$0,7,9,10$	152	8			6		
$0,7,9,12$	178	34			30		
$0,7,9,13$	191	47	26		24		
$0,7,10,11$	180	18					
$0,7,10,12$	194	32			30		18
$0,7,11,12$	210	30		24			
$0,7,11,13$	225	45	24				
$0,7,12,13$	242	44		32	24		
$0,8,9,11$	182	14			12		
$0,8,9,12$	196	28		22	18	12	
$0,8,9,13$	210	42		36			
$0,8,10,11$	198	6					
$0,8,10,13$	228	30					18
$0,8,11,12$	230	20		14	12	6	
$0,8,11,13$	246	36					
$0,8,12,13$	264	24		18		12	
$0,9,10,12$	232	16			12		
$0,9,10,13$	248	32			30		
$0,9,11,12$	250	10			6		
$0,9,12,13$	286	16		10	6		
$0,10,11,13$	288	18					
$0,10,12,13$	308	8			6		

4.2. **Table 2.** 3×5**-case for** $\mathcal{A}$**.**

$5-tuple\ I$	$Dimension$	$Degree$	CM
$0,1,3,4,5$	0	6	y
$0,1,3,4,6$	1	2	y
$0,1,3,4,7$	1	2	y
$0,1,3,4,8$	0	12	y
$0,1,3,4,9$	1	2	y
$0,1,3,5,6$	0	12	y
$0,1,3,5,7$	1	3	y
$0,1,3,5,8$	0	18	y
$0,1,3,5,9$	1	3	y
$0,1,3,6,7$	1	2	y
$0,1,3,6,8$	0	24	y
$0,1,3,6,9$	1	4	y
$0,1,3,7,8$	0	30	y
$0,1,3,7,9$	1	5	y
$0,1,3,8,9$	0	42	y
$0,1,4,5,6$	0	24	y
$0,1,4,5,7$	0	30	y
$0,1,4,5,8$	1	6	y
$0,1,4,5,9$	1	6	y
$0,1,4,6,7$	1	2	n
$0,1,4,6,8$	0	48	y
$0,1,4,6,9$	1	2	n
$0,1,4,7,8$	0	60	y
$0,1,4,7,9$	1	2	n
$0,1,4,8,9$	1	6	n
$0,1,5,6,7$	0	60	y
$0,1,5,6,8$	0	72	y
$0,1,5,6,9$	0	84	y
$0,1,5,7,8$	0	90	y
$0,1,5,7,9$	1	3	n
$0,1,5,8,9$	1	6	n
$0,1,6,7,8$	0	120	y
$0,1,6,7,9$	1	2	n
$0,1,6,8,9$	0	168	y
$0,1,7,8,9$	0	210	y
$0,2,3,4,5$	1	6	y
$0,2,3,4,6$	1	6	y
$0,2,3,4,7$	1	6	y
$0,2,3,4,8$	1	6	y
$0,2,3,4,9$	1	6	y
$0,2,3,5,6$	1	8	y
$0,2,3,5,7$	1	6	n
$0,2,3,5,8$	1	8	y
$0,2,3,5,9$	1	8	y
$0,2,3,6,7$	1	6	n
$0,2,3,6,8$	1	8	n
$0,2,3,6,9$	1	10	y
$0,2,3,7,8$	1	6	n
$0,2,3,7,9$	1	6	n
$0,2,3,8,9$	1	8	y
$0,2,4,5,6$	1	12	y
$0,2,4,5,7$	1	12	y
$0,2,4,5,8$	1	12	y
$0,2,4,5,9$	1	12	y
$0,2,4,6,7$	1	18	y
$0,2,4,6,8$	2	3	y
$0,2,4,6,9$	1	24	y
$0,2,4,7,8$	1	18	y
$0,2,4,7,9$	1	18	y
$0,2,4,8,9$	1	24	y
$0,2,5,6,7$	1	12	n
$0,2,5,6,8$	1	14	n
$0,2,5,6,9$	1	14	n
$0,2,5,7,8$	1	12	n
$0,2,5,7,9$	1	12	y
$0,2,5,8,9$	1	14	n
$0,2,6,7,8$	1	18	n
$0,2,6,7,9$	1	18	n
$0,2,6,8,9$	1	26	n
$0,2,7,8,9$	1	18	n
$0,3,4,5,6$	1	18	y
$0,3,4,5,7$	1	18	y
$0,3,4,5,8$	1	18	y
$0,3,4,5,9$	1	18	y
$0,3,4,6,7$	1	20	n
$0,3,4,6,8$	1	18	n
$0,3,4,6,9$	1	22	y
$0,3,4,7,8$	1	24	n
$0,3,4,7,9$	1	20	n
$0,3,4,8,9$	1	18	n
$0,3,5,6,7$	1	24	n
$0,3,5,6,8$	1	26	n
$0,3,5,6,9$	1	28	y
$0,3,5,7,8$	1	24	n
$0,3,5,7,9$	1	33	y
$0,3,5,8,9$	1	26	n
$0,3,6,7,8$	1	36	n
$0,3,6,7,9$	1	40	y
$0,3,6,8,9$	1	46	y
$0,3,7,8,9$	1	36	n
$0,4,5,6,7$	1	36	y
$0,4,5,6,8$	1	36	y
$0,4,5,6,9$	1	36	y
$0,4,5,7,8$	1	36	n
$0,4,5,7,9$	1	36	n
$0,4,5,8,9$	1	42	n
$0,4,6,7,8$	1	42	n
$0,4,6,7,9$	1	44	n
$0,4,6,8,9$	1	48	n
$0,4,7,8,9$	1	54	n
$0,5,6,7,8$	1	60	y
$0,5,6,7,9$	1	60	y
$0,5,6,8,9$	1	66	n
$0,5,7,8,9$	1	66	n
$0,6,7,8,9$	1	90	y

4.3. **Table 3.** 3×6**-case for** $\mathcal{A}$**.**

$6-tuple\ I$	*Dimension*	*Degree*	*CM*	$6-tuple\ I$	*Dimension*	*Degree*	*CM*
$0,1,3,4,5,6$	0	6	$CI(1,2,3)$	$0,2,4,5,9,10*$	1	12	$CI(3,4)$
$0,1,3,4,5,7$	0	6	$CI(1,2,3)$	$0,2,4,6,7,8*$	1	18	$CI(3,6)$
$0,1,3,4,5,8$	0	6	$CI(1,2,3)$	$0,2,4,6,8,9*$	1	3	$CI(3,8)$
$0,1,3,4,5,9$	0	6	$CI(1,2,3)$	$0,2,4,6,9,10*$	1	24	$CI(3,8)$
$0,1,3,4,9,10$	1	2	$CI(1,2)$	$0,2,4,7,8,9*$	1	18	$CI(3,6)$
$0,1,3,5,6,7$	0	12	$CI(1,3,4)$	$0,2,4,7,9,10*$	1	18	$CI(3,6)$
$0,1,3,5,7,8$	0	18	$CI(1,3,6)$	$0,2,4,8,9,10*$	1	24	$CI(3,8)$
$0,1,3,5,8,9$	0	18	$CI(1,3,6)$	$0,2,5,6,7,8*$	1	12	n
$0,1,3,5,9,10$	0	24	$CI(1,3,8)$	$0,2,5,6,8,9$	1	14	n
$0,1,3,6,7,8$	0	18	$CM(1,4,5,6)$	$0,2,5,6,9,10*$	1	14	n
$0,1,3,6,8,9$	0	24	$CI(1,4,6)$	$0,2,5,7,8,9*$	1	12	n
$0,1,3,6,9,10$	1	2	n	$0,2,5,7,9,10*$	1	12	n
$0,1,3,7,8,9$	0	30	$CI(1,5,6)$	$0,2,5,8,9,10*$	1	14	n
$0,1,3,7,9,10$	1	2	n	$0,2,6,7,8,9*$	1	18	n
$0,1,3,8,9,10$	0	36	$CM(1,6,7,8)$	$0,2,6,7,9,10*$	1	18	n
$0,1,4,5,6,7$	0	24	$CI(2,3,4)$	$0,2,6,8,9,10*$	1	26	n
$0,1,4,5,7,8$	0	30	$CI(2,3,5)$	$0,2,7,8,9,10*$	1	18	n
$0,1,4,5,8,9$	1	6	$CI(2,3)$	$0,3,4,5,6,7*$	1	18	n
$0,1,4,5,9,10$	0	48	$CI(2,3,8)$	$0,3,4,5,7,8*$	1	18	n
$0,1,4,6,7,8$	0	36	$CM(2,4,5,6)$	$0,3,4,5,8,9*$	1	18	n
$0,1,4,6,8,9$	0	48	$CI(2,4,6)$	$0,3,4,5,9,10*$	1	18	n
$0,1,4,6,9,10$	1	2	n	$0,3,4,6,7,8*$	1	20	n
$0,1,4,7,8,9$	0	60	$CM(2,5,6,7)$	$0,3,4,6,8,9*$	1	18	n
$0,1,4,7,9,10$	1	2	n	$0,3,4,6,9,10$	1	22	
$0,1,4,8,9,10$	0	72	$CM(2,6,7,8)$	$0,3,4,7,8,9*$	1	24	n
$0,1,5,6,7,8$	0	60	$CI(3,4,5)$	$0,3,4,7,9,10$	1	20	n
$0,1,5,6,8,9$	0	72	$CM(3,4,6,7)$	$0,3,4,8,9,10*$	1	18	n
$0,1,5,6,9,10$	0	84	$CI(3,4,7)$	$0,3,5,6,7,8*$	1	24	n
$0,1,5,7,8,9$	0	90	$CM(3,5,6,7)$	$0,3,5,6,8,9$	1	26	n
$0,1,5,7,9,10$	0	96	$CM(3,5,7,8)$	$0,3,5,6,9,10*$	1	28	n
$0,1,5,8,9,10$	0	108	$CM(3,6,7,8)$	$0,3,5,7,8,9*$	1	24	n
$0,1,6,7,8,9$	0	120	$CI(4,5,6)$	$0,3,5,7,9,10*$	1	33	n
$0,1,6,7,9,10$	0	2	n	$0,3,5,8,9,10*$	1	26	n
$0,1,6,8,9,10$	0	168	$CM(4,6,7,8)$	$0,3,6,7,8,9*$	1	36	n
$0,1,7,8,9,10$	0	210	$CI(5,6,7)$	$0,3,6,7,9,10$	1	40	n
$0,2,3,4,5,6*$	1	6	$CI(2,3)$	$0,3,6,8,9,10*$	1	46	n
$0,2,3,4,6,7*$	1	6	$CI(2,3)$	$0,3,7,8,9,10*$	1	36	n
$0,2,3,4,7,8*$	1	6	$CI(2,3)$	$0,4,5,6,7,8*$	1	36	n
$0,2,3,4,8,9*$	1	6	$CI(2,3)$	$0,4,5,6,8,9*$	1	36	n
$0,2,3,4,9,10*$	1	6	$CI(2,3)$	$0,4,5,6,9,10*$	1	36	n
$0,2,3,5,6,7*$	1	8	n	$0,4,5,7,8,9*$	1	36	n
$0,2,3,5,7,8*$	1	6	n	$0,4,5,7,9,10*$	1	36	n
$0,2,3,5,8,9$	1	8	$CI(2,4)$	$0,4,5,8,9,10*$	1	42	n
$0,2,3,5,9,10*$	1	8	n	$0,4,6,7,8,9*$	1	42	n
$0,2,3,6,7,8*$	1	6	n	$0,4,6,7,9,10$	1	44	n
$0,2,3,6,8,9$	1	8	n	$0,4,6,8,9,10*$	1	48	n
$0,2,3,6,9,10*$	1	10	n	$0,4,7,8,9,10*$	1	54	n
$0,2,3,7,8,9*$	1	6	n	$0,5,6,7,8,9*$	1	60	n
$0,2,3,7,9,10*$	1	6	n	$0,5,6,7,9,10*$	1	60	n
$0,2,3,8,9,10*$	1	8	n	$0,5,6,8,9,10*$	1	66	n
$0,2,4,5,6,7*$	1	12	$CI(3,4)$	$0,5,7,8,9,10*$	1	66	n
$0,2,4,5,7,8*$	1	12	$CI(3,4)$	$0,6,7,8,9,10*$	1	90	n
$0,2,4,5,8,9*$	1	12	$CI(3,4)$				

4.4. **Table 4.** 4×6**- and** 5×8**-cases for** $\mathcal{A}$**.**

$6-tuple\ I$	$Dimension$	$Degree$	CM	$8-tuple\ I$	$Dimension$	$Degree$	CM
$0,1,2,4,5,6$	1	6	$CI(1,2,3)$	$0,1,2,3,5,6,7,8$	1	24	$CI(1,2,3,4)$
$0,1,2,4,5,7$	1	8	$CI(1,2,4)$	$0,1,2,3,5,6,7,9$	1	30	$CI(1,2,3,5)$
$0,1,2,4,5,8$	2	2	$CI(1,2)$	$0,1,2,3,5,6,7,10$	2	y	
$0,1,2,4,6,7$	1	12	$CI(1,3,4)$	$0,1,2,3,5,6,8,9$	1	40	$CI(1,2,4,5)$
$0,1,2,4,6,8$	2		n	$0,1,2,3,5,6,8,10$	1	48	$CI(1,2,4,6)$
$0,1,2,4,7,8$	1	20	$CI(1,4,5)$	$0,1,2,3,5,6,9,10$	1	60	$CI(1,2,5,6)$
$0,1,2,5,6,7$	1	24	$CI(2,3,4)$	$0,1,2,3,5,7,8,9$	1	60	$CI(1,3,4,5)$
$0,1,2,5,6,8$	1	30	$CI(2,3,5)$	$0,1,2,3,5,7,8,10$	1	72	$CI(1,3,4,6)$
$0,1,2,5,7,8$	1	40	$CI(2,4,5)$	$0,1,2,3,5,7,9,10$	1	90	$CI(1,3,5,6)$
$0,1,3,4,5,6$	1	24	$CI(2,3,4)$	$0,1,2,3,6,7,8,9$	1	144	$CI(2,3,4,5)$
$0,1,3,4,5,7$	1	30	$CI(2,3,5)$	$0,1,2,3,6,7,8,10$	1	180	$CI(2,3,4,6)$
$0,1,3,4,5,8$	1	36	$CI(2,3,6)$	$0,1,2,3,6,7,9,10$	1	180	$CI(2,3,5,6)$
$0,1,3,4,6,7$	2	8	$CI(2,4)$	$0,1,2,3,6,8,9,10$	1	240	$CI(2,4,5,6)$
$0,1,3,4,6,8$	1	48	$CI(2,4,6)$	$0,1,2,3,7,8,9,10$	1	360	$CI(3,4,5,6)$
$0,1,3,4,7,8$	1	60	$CI(2,5,6)$	$0,1,2,4,6,7,8,9$	1	360	$CI(3,4,5,6)$
$0,1,3,5,6,7$	1	60	$CI(3,4,5)$	$0,1,2,4,6,7,8,10$	2		y
$0,1,3,5,6,8$	1	72	$CI(3,4,6)$	$0,1,2,4,6,7,9,10$	1	504	$CI(3,4,6,7)$
$0,1,3,5,7,8$	1	90	$CI(3,5,6)$	$0,1,2,4,6,8,9,10$	2		y
$0,1,4,5,6,7$	1	96	y	$0,1,2,4,7,8,9,10$	1	840	$CI(4,5,6,7)$
$0,1,4,5,6,8$	1	114	y	$0,1,2,5,6,7,8,9$	1		n
$0,1,4,5,7,8$	1	140	y	$0,1,3,5,6,7,8,9$	2		n
$0,1,4,6,7,8$	1	180	y	$0,1,4,5,6,7,8,9$	1		y
$0,1,4,6,7,9$	2		n				
$0,1,4,6,8,9$	1	246	y				
$0,2,3,4,5,6$	2	12	$CI(3,4)$				
$0,2,3,4,5,7$	2	12	$CI(3,4)$				
$0,2,3,4,6,7$	2	12	n				
$0,2,3,5,6,7$	2	12	n				
$0,2,4,5,6,7$	2	24	n				
$0,3,4,5,6,7$	2	36	y				

4.5. **Table 5.** 3×5**-case for** $\mathcal{BC}$**.**

$5-tuple$	$Dimension$	$Degree$	$5-tuple$	$Dimension$	$Degree$
$0,1,3,4,6$	1		$0,2,3,5,9$		
$0,1,3,4,7$	1		$0,2,4,5,7$	0	6
$0,1,3,4,8$	0	6	$0,2,4,5,8$	0	12
$0,1,3,4,9$	1		$0,2,4,5,9$	0	18
$0,1,3,5,6$	0	6	$0,2,5,6,8$	1	
$0,1,3,5,7$	1		$0,2,5,6,9$	1	
$0,1,3,5,8$	0	12	$0,2,5,7,8$	1	
$0,1,3,5,9$	1		$0,2,5,7,9$	1	
$0,1,3,6,7$	1		$0,2,5,8,9$	1	
$0,1,3,6,8$	0	18	$0,2,6,7,9$	0	24
$0,1,3,6,9$	1		$0,2,6,8,9$	1	
$0,1,3,7,8$	0	24	$03,4,6,7$	1	
$0,1,3,7,9$	1		$0,3,4,6,8$	0	12
$0,1,3,8,9$	0	36	$0,3,4,6,9$	1	
$0,1,4,5,7$	0	6	$0,3,4,7,8$	1	
$0,1,4,5,8$	1		$0,3,4,7,9$	1	
$0,1,4,5,9$	1		$0,3,4,8,9$	0	48
$0,1,4,6,7$	1		$0,3,5,6,8$	1	
$0,1,4,6,8$	0	24	$0,3,5,6,9$	1	
$0,1,4,6,9$	1		$0,3,5,7,8$	1	
$0,1,4,7,8$	0	36	$0,3,5,7,9$	1	
$0,1,4,7,9$	1		$0,3,5,8,9$	1	
$0,1,4,8,9$	1		$0,3,6,7,9$	1	
$0,1,5,6,8$	0	12	$0,3,6,8,9$	1	
$0,1,5,6,9$	0	24	$0,4,5,7,8$	0	12
$0,1,5,7,8$	0	30	$0,4,5,7,9$	0	18
$0,1,5,7,9$	1		$0,4,5,8,9$	1	
$0,1,5,8,9$	1		$0,4,6,8,9$	1	
$0,1,6,7,9$	1		$0,5,6,8,9$	1	
$0,1,6,8,9$	0	48	$0,2,4,6,7$	1	
$0,2,3,5,6$	1		$0,2,4,6,9$	1	
$0,2,3,5,7$	0	6	$0,2,4,7,8$	1	
$0,2,3,5,8$	1		$0,2,4,7,9$	1	

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Department of Mathematics, Stockholm University, SE-106 91, Stockholm, Sweden
E-mail address: `ralff@math.su.se`

Department of Mathematics, Stockholm University, SE-106 91, Stockholm, Sweden
E-mail address: `shapiro@math.su.se`