

Algebraic relations between harmonic and anti-harmonic moments of plane polygons

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In this paper we describe the algebraic relations satisfied by the harmonic and anti-harmonic moments of simply connected, but not necessarily convex planar polygons with a given number of vertices.

To Isaac Schoenberg and Theodore Motzkin whose insights laid the foundations of this topic.

1 Introduction and Main Results

1.1 Basic notions and background

Let μ be a finite compactly supported Borel measure in the plane $\mathbb{R}^2 = \mathbb{C}$. For $j = 0, 1, \dots$, its j -th *harmonic moment* is a complex number given by:

$$m_j(\mu) \stackrel{\text{def}}{=} \int_{\mathbb{C}} z^j \mu(z).$$

Analogously, its j -th *anti-harmonic moment* is given by:

$$\bar{m}_j(\mu) \stackrel{\text{def}}{=} \int_{\mathbb{C}} \bar{z}^j \mu(z).$$

The function

$$u_\mu(z) \stackrel{\text{def}}{=} \int_{\mathbb{C}} \ln |z - \xi| \mu(\xi)$$

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is called the *logarithmic potential* of μ . It is harmonic outside the support of μ and well-defined almost everywhere in \mathbb{C} . The germ of $u_\mu(z)$ at ∞ is determined by the sequence of harmonic moments $\{m_j(\mu)\}_{j=0}^\infty$: the Taylor expansion at ∞ of the *Cauchy transform* of μ defined as

$$\mathfrak{C}_\mu(z) \stackrel{\text{def}}{=} \int_{\mathbb{C}} \frac{\mu(\xi)}{z - \xi} = 2 \frac{\partial u_\mu(z)}{\partial z},$$

and can be computed via the formula

$$\mathfrak{C}_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \dots \quad (1)$$

The problem of the recovery of a measure from its logarithmic potential at ∞ (alias “the inverse problem in logarithmic potential theory”) is a classical area of potential analysis, going back to the early 1920s and still quite active. One of its milestones is the fundamental paper [19], in which P.S. Novikov proved that Lebesgue measures of two different domains that are star-shaped with respect to the origin cannot have the same logarithmic potential near ∞ or, equivalently, the same sequences of harmonic moments.

For non-star-shaped domains a similar statement is false: see e.g. [4, p. 333] for examples of pairs of polygons having the same logarithmic potentials near ∞ .

In this century, the problem reappeared in mathematical physics in connection with integrable systems and the Hele-Shaw flow, see e.g. [15, 18, 17]. In [13] I.K. Kostov et. al. explicitly expressed, for domains with analytic boundary containing the origin, a conformal homeomorphism sending this domain to the unit disk via the moments of the domain. This result means, informally speaking, that harmonic moments may be used as a sort of coordinates on the set of such domains.

The picture becomes somewhat different if one considers moments of Lebesgue measures of polygons with a fixed number n of vertices. The harmonic (and anti-harmonic) moments of such polygons are algebraically dependent. The main goal of the present article is to describe these dependencies.

1.2 The main object: polygonal measures, an explicit formula, and complexification

A plane polygon $P \subset \mathbb{C}$ is determined by its sequence of vertices $z_1, \dots, z_n \in \mathbb{C}$, ordered counterclockwise. However, not every sequence of n points in \mathbb{C} is a sequence of vertices for some simply connected polygon. It is natural to generalize the notion of the Lebesgue measure of a polygon as follows.

Let $\mathbf{a} = (a_1, \dots, a_n)$, $a_j \in \mathbb{C}^2$ be a sequence of points with $a_j = (x_j, y_j)$, $j = 1, \dots, n$. Instead of x_j and y_j , we are going to use more convenient coordinates $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$. For brevity, denote $\mathbf{z} \stackrel{\text{def}}{=} (z_1, \dots, z_n)$ and $\bar{\mathbf{z}} \stackrel{\text{def}}{=} (\bar{z}_1, \dots, \bar{z}_n)$.

If all $a_j \in \mathbb{R}^2 \subset \mathbb{C}^2$ (that is, all x_j and y_j are real), then every \bar{z}_j is indeed the complex conjugate of z_j ; if we identify \mathbb{R}^2 with \mathbb{C} via $(x, y) \leftrightarrow x + iy$, then a_j becomes z_j . We will call such situation “the case of real vertices”. In general, though, \mathbf{z} and $\bar{\mathbf{z}}$ are n -tuples of independent complex variables.

Define an *oriented closed polygonal curve* $\Gamma_{\mathbf{a}}$ by

$$\Gamma_{\mathbf{a}} \stackrel{\text{def}}{=} [a_1, a_2] \xrightarrow{\quad} \cup \cdots \cup [a_{n-1}, a_n] \xrightarrow{\quad} \cup [a_n, a_1] \xrightarrow{\quad}.$$

Fix an auxiliary convex polygon $P_* \subset \mathbb{R}^2$ with the vertices w_1, \dots, w_n , ordered counterclockwise. Let \mathcal{T} be its *triangulation*, i.e. a set of diagonals of P_* having no common internal points and cutting P_* into triangles. Let $F_{\mathbf{a}, \mathcal{T}} : P_* \rightarrow \mathbb{C}^2$ be the map sending every w_j to a_j and affine on every triangle of the triangulation. (It is easy to show that $F_{\mathbf{a}, \mathcal{T}}$ exists and is unique and continuous.)

The image $\Delta_{\mathbf{a}, \mathcal{T}} = F_{\mathbf{a}, \mathcal{T}}(P_*) \subset \mathbb{C}^2$ is a polygonal disk in \mathbb{C}^2 bounded by $\Gamma_{\mathbf{a}}$. The disk $\Delta_{\mathbf{a}, \mathcal{T}} \subset \mathbb{C}^2$ is piecewise immersed (though $F_{\mathbf{a}, \mathcal{T}}$ is not always an immersion). For a generic \mathbf{a} , this disk is embedded.

The disk $\Delta_{\mathbf{a}, \mathcal{T}}$ supports the measure $\mu_{\mathbf{a}, \mathcal{T}} = (F_{\mathbf{a}, \mathcal{T}})_* dx dy$ which is the direct image of the Lebesgue measure on P under $F_{\mathbf{a}, \mathcal{T}}$; we call $\mu_{\mathbf{a}, \mathcal{T}}$ a *polygonal measure*.

The disk $\Delta_{\mathbf{a}, \mathcal{T}}$ itself depends on the triangulation \mathcal{T} , but certain integrals with respect to $\mu_{\mathbf{a}, \mathcal{T}}$ do not.

Theorem 1.1. (i) Let $h : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a holomorphic function of two variables. Then the integral $\int_{\Delta_{\mathbf{a}, \mathcal{T}}} h \mu_{\mathbf{a}, \mathcal{T}}$ does not depend on the triangulation \mathcal{T} and is equal to $\int_{\Gamma_{\mathbf{a}}} \omega$, where ω is any 1-form such that $d\omega = \frac{i}{2} h dz \wedge d\bar{z}$.

(ii) If the vertices a_1, \dots, a_n are real, then $\mu_{\mathbf{a}, \mathcal{T}}$ is independent of \mathcal{T} . It is supported on a compact subset of \mathbb{R}^2 and its density at a point $q \in \mathbb{R}^2 \setminus \Gamma_{\mathbf{a}}$ equals the linking number of the 1-cycle $\Gamma_{\mathbf{a}} \subset \mathbb{R}^2$ with the 0-cycle $q - \infty$. In particular, if $a_1, \dots, a_n \in \mathbb{R}^2$ are the vertices of a simply connected polygon, listed counterclockwise, then $\mu_{\mathbf{a}, \mathcal{T}}$ is the Lebesgue measure of this polygon. □

The theorem follows easily from the Stokes' theorem; see Section 2 for a detailed proof. The last claim in assertion (ii) explains the term “polygonal measure”.

Corollary 1.2 (of assertion (i)). The harmonic moment $m_j(\mu_{\mathbf{a}, \mathcal{T}})$ of the polygonal measure does not depend on the triangulation \mathcal{T} . □

Take again $\mathbf{a} = (a_1, \dots, a_n)$, $a_j = (x_j, y_j) \in \mathbb{C}^2$; $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$. Obviously, z_j and \bar{z}_j determine a_j since $x_j = (z_j + \bar{z}_j)/2$ and $y_j = (z_j - \bar{z}_j)/2i$. Set

$$\nu_k(\mathbf{z}, \bar{\mathbf{z}}) \stackrel{\text{def}}{=} \binom{k}{2} m_{k-2}(\mu_{\mathbf{a}, \mathcal{T}}) \quad \text{and} \quad \bar{\nu}_k(\mathbf{z}, \bar{\mathbf{z}}) = \binom{k}{2} \bar{m}_{k-2}(\mu_{\mathbf{a}, \mathcal{T}}).$$

In particular, $\nu_0 = \nu_1 = \bar{\nu}_0 = \bar{\nu}_1 = 0$ for all $(\mathbf{z}, \bar{\mathbf{z}})$. By Corollary 1.2, both sides of the equalities are independent of \mathcal{T} .

Remark 1.3. The index shift $(k-2) \mapsto k$, used above, is convenient since the normalized moments ν_k and $\bar{\nu}_k$ are homogeneous of degree k with respect to the dilatations on \mathbb{C}^2 . In other words, $\nu_k(t\mathbf{z}, t\bar{\mathbf{z}}) = t^k \nu_k(\mathbf{z}, \bar{\mathbf{z}})$ where $t\mathbf{z} \stackrel{\text{def}}{=} (tz_1, \dots, tz_n)$. □

The following theorem provides explicit formulas for $\nu_k(\mathbf{z}, \bar{\mathbf{z}})$ and $\bar{\nu}_k(\mathbf{z}, \bar{\mathbf{z}})$.

Theorem 1.4. For any positive integer $k \geq 2$, one has

$$\nu_k(\mathbf{z}, \bar{\mathbf{z}}) = \frac{i}{4} \sum_{j=1}^n (\bar{z}_j - \bar{z}_{j+1}) \frac{z_j^k - z_{j+1}^k}{z_j - z_{j+1}} = \frac{i}{4} \sum_{j=1}^n (\bar{z}_j - \bar{z}_{j+1}) (z_j^{k-1} + z_j^{k-2} z_{j+1} + \cdots + z_{j+1}^{k-1}). \quad (2)$$

The anti-harmonic moment $\bar{\nu}_k(\mathbf{z}, \bar{\mathbf{z}})$ is given by the same formula (2) with \mathbf{z} and $\bar{\mathbf{z}}$ interchanged. \square

Observe that, up to the factor $i/4$, each normalized harmonic and anti-harmonic moment of a polygonal measure is a polynomial with integer coefficients in the variables z_1, \dots, z_n and $\bar{z}_1, \dots, \bar{z}_n$. Notice additionally that $\bar{\nu}_k(\mathbf{z}, \bar{\mathbf{z}}) = \nu_k(\bar{\mathbf{z}}, \mathbf{z})$ for all k and $\bar{\nu}_2(\mathbf{z}, \bar{\mathbf{z}}) = -\nu_2(\mathbf{z}, \bar{\mathbf{z}})$.

1.3 Main results about the relations between the moments

In this paper, the problem of describing the algebraic relations among the moments of polygons will be understood in two different ways. We call them the *algebraic* and the *geometric* approaches, respectively. The algebraic approach amounts to finding the algebraic relations between the polynomials $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$, $j = 2, 3, \dots$, while the geometric approach deals with finding the algebraic relations that include both $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$ and $\bar{\nu}_j(\mathbf{z}, \bar{\mathbf{z}})$, $j = 2, 3, \dots$.

In case of the algebraic approach, our main result is relatively simple. Namely, all harmonic moments can be expressed as rational functions of the first $2n - 2$ moments $\nu_2(\mathbf{z}, \bar{\mathbf{z}}), \dots, \nu_{2n-1}(\mathbf{z}, \bar{\mathbf{z}})$, and these $2n - 2$ moments are algebraically independent. More precisely, denote by \mathfrak{F}_n the field extension of \mathbb{C} , generated by the sequence of polynomials $\{\nu_j(\mathbf{z}, \bar{\mathbf{z}})\}_{j=2}^{\infty}$.

Theorem 1.5. (i) $\mathfrak{F}_n = \mathbb{C}(\nu_2, \dots, \nu_{2n-1})$ and is isomorphic to the field of rational functions in $2n - 2$ independent complex variables.

(ii) $\mathfrak{F}_n \supset \mathbb{C}(\mathbf{z})^{S_n}$, where $\mathbb{C}(\mathbf{z})^{S_n}$ is the field of symmetric rational functions of z_1, \dots, z_n .

\square

Explicit formulas expressing harmonic moments $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$ with $j \geq 2n$ via the first $2n - 2$ ones are given by rational functions. However, all their denominators are powers of one fixed polynomial \mathfrak{D}_n , the determinant of the matrix (8) below. In fact, if one considers the *ring extension* \mathcal{R}_n of \mathbb{C} (as opposed to field), generated by the sequence of polynomials $\{\nu_j(\mathbf{z}, \bar{\mathbf{z}})\}_{j=2}^{\infty}$, the situation may be described in the following way.

Theorem 1.6. (i) The ring $\mathcal{R}_n = \mathbb{C}[\nu_2, \nu_3, \dots]$ is not generated by any finite collection of harmonic moments

$$\nu_2, \dots, \nu_N.$$

(ii) For the polynomial $\mathfrak{D}_n \in \mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]$, given by the determinant of (8), the localization $\mathcal{R}_n|_{\mathfrak{D}_n}$ is isomorphic to $\mathbb{C}[\nu_2, \dots, \nu_{2n-1}] \left[\frac{1}{\mathfrak{D}_n} \right]$.

\square

Notice that \mathcal{R}_n does not contain the ring $\mathbb{C}[\mathbf{z}]^{S_n}$ of symmetric polynomials in the variables z_1, \dots, z_n as a subring, since the expression of the basic (e.g. elementary) symmetric polynomials via ν_2, \dots, ν_{2n-1} involves division by some powers of \mathfrak{D}_n .

Further, in the geometric approach, we consider the field extension $\tilde{\mathfrak{F}}_n$ of \mathbb{C} , generated by both sequences $\{\nu_j\}_{j=2}^\infty$ and $\{\bar{\nu}_j\}_{j=2}^\infty$. (Recall that $\nu_2 = -\bar{\nu}_2$). Here the situation is more complicated.

Theorem 1.7. (i) The field $\tilde{\mathfrak{F}}_n$ is generated by the first $4n - 5$ harmonic and anti-harmonic moments

$$\nu_2, \nu_3, \bar{\nu}_3, \dots, \nu_{2n-1}, \bar{\nu}_{2n-1}.$$

(ii) The field $\tilde{\mathfrak{F}}_n$ contains a subfield $H = \mathbb{C}(\mathbf{z}, \bar{\mathbf{z}})^{S_n \times S_n}$ of rational functions, symmetric with respect to two groups of variables z_1, \dots, z_n and $\bar{z}_1, \dots, \bar{z}_n$ separately.

(iii) $\tilde{\mathfrak{F}}_n$ is an algebraic extension of H , generated by the single element ν_2 . The degree d_n of this extension equals $n!(n-1)!$ if n is odd and equals $2((n-1)!)^2$ if n is even. □

Remark 1.8. Notice that any algebraic extension of a field of characteristics zero is generated by a single element. So the essence of assertion (iii) of Theorem 1.7 is that a specific element ν_2 is a generator. We describe the Galois closure of this extension and its Galois group later in Section 4. □

Algebraic relations between the usual moments of polygons and polytopes in several special situations were discussed in a recent (joint with K. Kohn and B. Sturmfels) paper [12] of the third author. For algebraic domains, the relations between the moments were studied in e.g. [14, Section 3].

The structure of the paper is as follows. Section 2 contains detailed proofs of Theorems 1.1, 1.4, 1.5, and 1.6, as well as some formulas related to the logarithmic potential. In Section 3, we describe the action of the group $S_n \times S_n$ on the field $\tilde{\mathfrak{F}}_n$ and prove Theorem 1.7. In Section 4, we describe the Galois group of the (Galois closure of the) extension $\mathbb{C} \subset \tilde{\mathfrak{F}}_n$. Section 5 contains explicit description of this extension and its Galois group in the simplest case $n = 3$ (i.e. for triangles). We finish the paper with some questions expressing our outlook at the further development of the subject, see Section 6.

2 Harmonic Moments

2.1 Formulas for the moments.

In this section we are going to prove Theorems 1.1 and 1.4). We start with the following lemma:

Lemma 2.1. Assume that the triangulation \mathcal{T} contains the diagonal $(1, m)$ which divides \mathcal{T} into two parts, \mathcal{T}' and \mathcal{T}'' . If $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{a}' = (a_1, \dots, a_m)$ and $\mathbf{a}'' = (a_1, a_m, a_{m+1}, \dots, a_n)$, then $\Delta_{\mathbf{a}, \mathcal{T}} = \Delta_{\mathbf{a}', \mathcal{T}'} \cup \Delta_{\mathbf{a}'', \mathcal{T}''}$, and $\mu_{\mathbf{a}, \mathcal{T}} = \mu_{\mathbf{a}', \mathcal{T}'} + \mu_{\mathbf{a}'', \mathcal{T}''}$. □

The proof is straightforward.

Corollary 2.2. If \mathcal{T} consists of the edges $(1, 3), (1, 4), \dots, (1, n-1)$, then

$$\mu_{\mathbf{a}, \mathcal{T}} = \mu_{a_1 a_2 a_3} + \mu_{a_1 a_3 a_4} + \dots + \mu_{a_1 a_{n-1} a_n}.$$

□

(The right-hand side of the latter formula contains measures supported on triangles, so there is no need to specify a triangulation.) Notice that if the vertices are real, then, up to a sign, $\mu_{a_1 a_k a_{k+1}}$ is the Lebesgue measure of a triangle with the vertices a_1, a_k and a_{k+1} . If this triangle is degenerate then $\mu_{a_1 a_k a_{k+1}}$ is zero, otherwise the sign is taken to be $+$ if the triangle is oriented counterclockwise, and $-$ if it is oriented clockwise.

Proof of Theorem 1.1. To settle assertion (i), notice first that the 2-form $h(z, \bar{z}) dz \wedge d\bar{z}$ is closed and therefore exact since $h(z, \bar{z})$ is holomorphic in both variables. So $h(z, \bar{z}) dz \wedge d\bar{z} = 2id\omega$ for some 1-form ω . If $n = 3$, then there exists only one triangulation; the equality $-\frac{i}{2} \int_{\Delta_{\mathbf{a}}} h dz \wedge d\bar{z} = \int_{\Gamma_{\mathbf{a}}} \omega$ follows from the Stokes' theorem.

Next apply the formulas for $n = 3$ to each triangle of the triangulation \mathcal{T} and add them. Observe that the integrals over the internal diagonals of the triangulation contribute two equal terms with the opposite signs and therefore cancel, while the integrals over the “sides of the polygon” (i.e., segments of $\Gamma_{\mathbf{a}}$) appear once and survive in the total sum. On the other hand, all the maps $F_{\mathbf{a}, \mathcal{T}}$ are the same on the sides of the polygon P and map them to the same line $\Gamma_{\mathbf{a}}$; thus, the integrals over $\Gamma_{\mathbf{a}}$ are the same for all possible triangulations. This proves assertion (i) for any integer $n \geq 3$.

Assertion (ii) is evident for $n = 3$. Indeed, for q lying outside the triangle $\Gamma_{\mathbf{a}}$ the linking number of $q - \infty$ with $\Gamma_{\mathbf{a}}$ is 0, and for q lying inside the triangle, it equals ± 1 ; the choice of the sign depends on the orientation of the triangle. Notice now that the map sending $\Gamma_{\mathbf{a}}$ to the polygonal measure is additive, i.e., if $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{a}' = (a_1, \dots, a_m)$, $\mathbf{a}'' = (a_1, a_m, a_{m+1}, \dots, a_n)$, then $\Gamma_{\mathbf{a}} = \Gamma_{\mathbf{a}'} + \Gamma_{\mathbf{a}''}$ as 1-cycles. Therefore the linking number of $q - \infty$ with $\Gamma_{\mathbf{a}}$ is equal to the sum of its linking numbers with $\Gamma_{\mathbf{a}'}$ and $\Gamma_{\mathbf{a}''}$. This observation, together with Lemma 2.1, allows us to finish the proof by induction on n . ■

Now we are ready to derive an explicit formula for the moments of polygonal measures.

Proof of Theorem 1.4. For a harmonic moment ν_k , we are in the situation of Theorem 1.1 with $h(z, \bar{z}) = \frac{k(k-1)}{2} z^{k-2}$; so one can take $\omega = \frac{ik}{4} z^{k-1} d\bar{z}$.

If we parametrize a segment $[p, q] \subset \mathbb{C}$ as $z = p(1-t) + tq$, $t \in [0, 1]$, then

$$\frac{ik}{4} \int_{[p, q]} z^{k-1} d\bar{z} = \frac{ik}{4} \int_0^1 (p(1-t) + tq)^{k-1} (\bar{q} - \bar{p}) dt = \frac{i}{4} \int_0^1 \frac{\bar{q} - \bar{p}}{q - p} d((p(1-t) + tq)^k) = \frac{i}{4} (\bar{q} - \bar{p}) \frac{q^k - p^k}{q - p},$$

Equation (2) follows now from Theorem 1.1. ■

2.2 Generating functions and relations between harmonic moments.

Define now, following [20], the *normalized generating function* $\Psi_\mu(w)$ for harmonic moments of a measure μ as

$$\Psi_\mu(w) \stackrel{\text{def}}{=} \sum_{j=2}^{\infty} \nu_j(\mu) w^{j-2} = \sum_{j=0}^{\infty} \binom{j+2}{2} m_j(\mu) w^j. \quad (3)$$

$\Psi_\mu(w)$ is closely related to the Cauchy transform $\mathfrak{C}_\mu(z)$ at ∞ . Namely,

$$\Psi_\mu(w) = \frac{1}{2} \frac{d^2}{dw^2} \left(\sum_{j=0}^{\infty} m_j(\mu) w^{j+2} \right).$$

At the same time, (1) for a compactly supported measure μ and sufficiently large $|z|$ implies $w\mathfrak{C}_\mu(z) = \sum_{j=0}^{\infty} m_j(\mu)/z^j$. Hence, if $|w|$ is sufficiently small, then

$$\Psi_\mu(w) = \frac{1}{2} \frac{d^2}{dw^2} \left(w\mathfrak{C}_\mu \left(\frac{1}{w} \right) \right).$$

Similar multivariate generating functions were recently considered in [9].

For $n = 3$, sometimes we will use the notation

$$D_{i,j,k} \stackrel{\text{def}}{=} \nu_2(z_i, z_j, z_k, \bar{z}_i, \bar{z}_j, \bar{z}_k),$$

where the subscripts i, j, k may vary appropriately. Note that, for any n , if the vertices are real (that is, \mathbf{z} and $\bar{\mathbf{z}}$ are complex conjugate) and the points z_1, \dots, z_n are vertices of a simply connected polygon, then the moment $\nu_2(\mathbf{z}, \bar{\mathbf{z}})$ is equal to the (signed) area of the polygon; so $D_{i,j,k}$ is the area of a triangle with the vertices z_i, z_j, z_k .

The following observation, which can be found in [20], is important in our considerations:

Proposition 2.3.

$$\Psi_{\mu_{\mathbf{z}}}(w) = \frac{D_{123}}{(1 - z_1 w)(1 - z_2 w)(1 - z_3 w)}. \quad (4)$$

□

Proof. Substitution of (2) into (3) gives

$$\begin{aligned} \Psi_{\mu_{\mathbf{z}}}(w) &= \frac{i}{4} \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \sum_{k=2}^{\infty} (z_1^k - z_2^k) w^{k-2} + \text{cyclic} = \frac{i}{4} \left(\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \left(\frac{z_1^2}{1 - z_1 w} - \frac{z_2^2}{1 - z_2 w} \right) + \text{cyclic} \right) \\ &= \frac{i}{2} \frac{z_1 \bar{z}_2 - z_2 \bar{z}_1 + z_2 \bar{z}_3 - z_3 \bar{z}_2 + z_3 \bar{z}_1 - z_1 \bar{z}_3}{(1 - z_1 w)(1 - z_2 w)(1 - z_3 w)} = \frac{D_{123}}{(1 - z_1 w)(1 - z_2 w)(1 - z_3 w)}. \end{aligned}$$

(“+cyclic” here means the sum of two extra summands, cyclically shifting $z_1 \mapsto z_2 \mapsto z_3 \mapsto z_1$). ■

Proposition 2.3 and Corollary 2.2 imply the following.

Corollary 2.4. For any $\mathbf{z} = (z_1, \dots, z_n)$, one has

$$\Psi_{\mu_{\mathbf{z}}}(w) = \sum_{j=2}^{n-1} \frac{D_{1,j,j+1}}{(1-z_1w)(1-z_jw)(1-z_{j+1}w)}.$$

□

Corollary 2.5 (of Corollary 2.4). For every $\mathbf{z} = (z_1, \dots, z_n)$, there exists a unique polynomial $\mathcal{AD}_{\mathbf{z}}(w)$ of degree at most $n-3$ such that

$$\Psi_{\mu_{\mathbf{z}}}(w) = \frac{\mathcal{AD}_{\mathbf{z}}(w)}{\prod_{j=1}^n (1-z_jw)}, \quad (5)$$

□

Remark 2.6. Properties of the polynomial $\mathcal{AD}_{\mathbf{z}}(w)$ were studied in detail in [22] (see also [23]). For a generic \mathbf{z} , define, following [22], the *adjoint polynomial* $A_{\mathbf{z}}(u, v)$ of \mathbf{z} as the minimal degree polynomial vanishing at all points $\ell \in (\mathbb{CP}^2)^*$ that correspond to lines joining z_k and z_ℓ with $|k-\ell| \geq 2$. The adjoint polynomial is unique up to a multiplicative constant. It also has a multi-dimensional generalization studied in [11].

In particular, the following was proved:

Proposition ([22]). (i) The coefficient at w^{n-3} in $\mathcal{AD}_{\mathbf{z}}$ equals $\nu_2(\mathbf{z}, \bar{\mathbf{z}})$.

(ii) One has $\mathcal{AD}_{\mathbf{z}}(w) = A_{\mathbf{z}}(w, iw) \times \text{const.}$ where $A_{\mathbf{z}}$ is the adjoint polynomial, and the constant is determined by assertion (i).

□

□

Next, we settle Theorem 1.5.

Proof of Theorem 1.5. First, prove assertion (ii). The field $\mathbb{C}(\mathbf{z})^{S_n}$ of invariant rational functions is generated by the elementary symmetric polynomials $e_1(\mathbf{z}), \dots, e_n(\mathbf{z})$. Thus it is enough to show that they belong to \mathfrak{F}_n .

It is a well-known fact, see, e.g., [21, Th. 4.1.1], that if

$$\sum_{j=0}^{\infty} f(j)t^j = \frac{P(t)}{Q(t)}$$

where $\deg P < \deg Q$ and $Q(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \dots + \alpha_d t^d$ with $\alpha_d \neq 0$, then for all $k \geq 0$, one has the recurrence relation:

$$f(k+d) + \alpha_1 f(k+d-1) + \alpha_2 f(k+d-2) + \dots + \alpha_d f(k) = 0.$$

It follows from (5) that

$$\mathcal{AD}_Z(w) = \prod_{j=1}^n (1 - z_j w) \cdot \sum_{j=0}^{\infty} \nu_{j+2}(\mathbf{z}) w^j = (1 - e_1(\mathbf{z})w + \cdots + (-1)^n e_n(\mathbf{z})w^n) \cdot \sum_{j=0}^{\infty} \nu_{j+2}(\mathbf{z}) w^j.$$

Since $\deg \mathcal{AD}_z \leq n - 3$, for every $k \geq -2$, one has

$$\nu_{k+2+n}(\mathbf{z}) - e_1(\mathbf{z})\nu_{k+1+n}(\mathbf{z}) + e_2(\mathbf{z})\nu_{k+n}(\mathbf{z}) - \cdots + (-1)^n e_n(\mathbf{z})\nu_{k+2}(\mathbf{z}) = 0. \quad (6)$$

Consider the first n equations of the recurrence (6). This linear system has the form:

$$U \cdot E = V, \quad (7)$$

where U is the Toeplitz $n \times n$ -matrix, given by

$$U = \begin{pmatrix} \nu_{n-1}(\mathbf{z}, \bar{\mathbf{z}}) & \nu_{n-2}(\mathbf{z}, \bar{\mathbf{z}}) & \cdots & \nu_1(\mathbf{z}, \bar{\mathbf{z}}) & \nu_0(\mathbf{z}, \bar{\mathbf{z}}) \\ \nu_n(\mathbf{z}, \bar{\mathbf{z}}) & \nu_{n-1}(\mathbf{z}, \bar{\mathbf{z}}) & \cdots & \nu_2(\mathbf{z}, \bar{\mathbf{z}}) & \nu_1(\mathbf{z}, \bar{\mathbf{z}}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu_{2n-2}(\mathbf{z}, \bar{\mathbf{z}}) & \nu_{2n-3}(\mathbf{z}, \bar{\mathbf{z}}) & \cdots & \nu_n(\mathbf{z}, \bar{\mathbf{z}}) & \nu_{n-1}(\mathbf{z}, \bar{\mathbf{z}}) \end{pmatrix}, \quad (8)$$

and E and V are column vectors of length n , given by

$$E = (e_1(\mathbf{z}), -e_2(\mathbf{z}), \dots, (-1)^{n+1} e_n(\mathbf{z}))^T,$$

and

$$V = (\nu_n(\mathbf{z}, \bar{\mathbf{z}}), \nu_{n+1}(\mathbf{z}, \bar{\mathbf{z}}), \dots, \nu_{2n-1}(\mathbf{z}, \bar{\mathbf{z}}))^T.$$

(Recall that $\nu_0 = \nu_1 = 0$). Assuming that U is invertible, we get $E = U^{-1}V$. This means that every $e_j(\mathbf{z})$ is expressed as a rational function of the normalized moments $\nu_2(\mathbf{z}, \bar{\mathbf{z}}), \dots, \nu_{2n-1}(\mathbf{z}, \bar{\mathbf{z}})$ with the fixed denominator, equal to the determinant of U . Thus assertion (ii) of Theorem 1.5 is proved.

To settle assertion (i), we argue as follows. Using the recurrence relation (6), one can express every $\nu_{k+2+n}(\mathbf{z}, \bar{\mathbf{z}})$, where $k \geq n - 2$, as a rational function of the first $2n - 2$ normalized harmonic moments $\nu_2(\mathbf{z}, \bar{\mathbf{z}}), \dots, \nu_{2n-1}(\mathbf{z}, \bar{\mathbf{z}})$.

Now for $s = 1, 2, \dots$, denote by $h_s(\mathbf{z})$ the s -th complete symmetric polynomial of z_1, \dots, z_n . The identity

$$1 = \prod_{\ell=1}^n (1 - wz_\ell) \times \frac{1}{\prod_{\ell=1}^n (1 - wz_\ell)} = \sum_{m=0}^n (-1)^m e_m(\mathbf{z}) w^m \sum_{k=0}^{\infty} h_k(\mathbf{z}) w^k, \quad (9)$$

implies the standard relation $\sum_{m=0}^n (-1)^m e_m(\mathbf{z}) h_{j-m}(\mathbf{z}) = 0$ for all $j = 1, 2, \dots$. Corollary 2.4 implies that

$$\begin{aligned} \sum_{j=0}^{\infty} \nu_{j+2}(\mathbf{z}, \bar{\mathbf{z}}) w^j &= \left(\sum_{s=0}^{\infty} h_s(\mathbf{z}) w^s \right) \sum_{\ell=1}^{n-1} D_{1,\ell,\ell+1}(\mathbf{z}, \bar{\mathbf{z}}) (1 - wz_2) \dots (1 - \widehat{wz_\ell}) (1 - \widehat{wz_{\ell+1}}) \dots (1 - wz_n) \\ &= \left(\sum_{s=0}^{\infty} h_s(\mathbf{z}) w^s \right) \sum_{\ell=1}^{n-1} D_{1,\ell,\ell+1}(\mathbf{z}, \bar{\mathbf{z}}) \times \sum_{m=0}^{n-3} (-1)^m w^m e_m(z_2, \dots, \widehat{z_\ell}, \widehat{z_{\ell+1}}, \dots, z_n) \\ &= \left(\sum_{s=0}^{\infty} h_s(\mathbf{z}) w^s \right) \sum_{m=0}^{n-3} w^m Q_m(\mathbf{z}, \bar{\mathbf{z}}), \end{aligned} \quad (10)$$

where

$$Q_m(\mathbf{z}, \bar{\mathbf{z}}) \stackrel{\text{def}}{=} \sum_{\ell=1}^{n-1} (-1)^\ell D_{1,\ell,\ell+1}(\mathbf{z}, \bar{\mathbf{z}}) e_m(z_2, \dots, \widehat{z_\ell}, \widehat{z_{\ell+1}}, \dots, z_n). \quad (11)$$

This equation gives

$$\nu_j(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{m=0}^{n-3} (-1)^m Q_m(\mathbf{z}, \bar{\mathbf{z}}) h_{j-m-2}(\mathbf{z}), \quad (12)$$

where $j = 2, 3, \dots$, and $h_k(\mathbf{z}) \stackrel{\text{def}}{=} 0$ for $k < 0$.

Multiplying (10) by $\prod_{k=1}^n (1 - wz_k) = \sum_{\ell=0}^n (-1)^\ell e_\ell(\mathbf{z}) w^\ell$, we get

$$\sum_{\ell=0}^n (-1)^\ell e_\ell(\mathbf{z}) w^\ell \sum_{j=2}^{\infty} \nu_j(\mathbf{z}, \bar{\mathbf{z}}) w^{j-1} = \sum_{m=0}^{n-3} (-1)^m w^m Q_m(\mathbf{z}, \bar{\mathbf{z}}). \quad (13)$$

Relation (10) leads to

$$\mathfrak{F}_n = \mathbb{C}(e_1, \dots, e_n, Q_0, \dots, Q_{n-3}).$$

We are going to show that Q_0, \dots, Q_{n-3} are algebraically independent over $\mathbb{C}(e_1, \dots, e_n)$. Indeed, the functions z_1, \dots, z_n are the roots of

$$t^n - e_1(\mathbf{z}) t^{n-1} + \dots + (-1)^n e_n(\mathbf{z}) = 0$$

and, therefore, belong to an algebraic extension AE_n of $\mathbb{C}(e_1, \dots, e_n)$. The same field AE_n contains the functions $e_m(z_2, \dots, \widehat{z_k}, \widehat{z_{k+1}}, \dots, z_n)$, mentioned in (11), which implies that $D_{123}, \dots, D_{1,n-1,n} \in AE_n$ as well. Thus, the field $\mathbb{C}(e_1, \dots, e_n, D_{123}, \dots, D_{1,(n-1),n})$ is an algebraic extension of \mathfrak{F}_n .

The polynomial D_{123} depends on the variables z_2 and \bar{z}_2 , while the polynomials $\{D_{1,k,k+1}\}$ for $k = 3, \dots, (n-1)$, do not. Take any $u \in \mathbb{C}$ and substitute $\bar{z}_2 \mapsto \bar{z}_2 + u$, leaving all other \bar{z}_j and all z_j unchanged. This operation preserves the values of $e_1(\mathbf{z}), \dots, e_n(\mathbf{z})$, as well as the values of $D_{134}, \dots, D_{1,n-1,n}$. On the other hand, as u varies, D_{123} takes infinitely many values, and therefore it is not a root of any algebraic equation with the coefficients that depend only on $e_1(\mathbf{z}), \dots, e_n(\mathbf{z})$ and $D_{134}, \dots, D_{1,(n-1),n}$. Further take z_3, \bar{z}_3 and conclude in a similar way that D_{134} is algebraically independent of $D_{145}, \dots, D_{1,(n-1),n}$, etc.

In this way, we prove that $D_{123}, \dots, D_{1,n-1,n}$ are algebraically independent elements over the field $\mathbb{C}(e_1, \dots, e_n)$. Hence, in the tower of extensions

$$\mathbb{C}(e_1, \dots, e_n) \subset \mathfrak{F}_n = \mathbb{C}(e_1, \dots, e_n, Q_0, \dots, Q_{n-3}) \subset \mathbb{C}(e_1, \dots, e_n, D_{123}, \dots, D_{1,n-1,n}), \quad (14)$$

the transcendence degree of the last field over the first one equals $n - 2$. The second extension is algebraic, and therefore, the first extension has the transcendence degree $n - 2$ as well. Consequently, Q_0, \dots, Q_{n-3} are algebraically independent over $\mathbb{C}(e_1, \dots, e_n)$. Equation (12) implies that ν_2, \dots, ν_n are algebraically independent as well. Since $e_1(\mathbf{z}), \dots, e_n(\mathbf{z})$ are also algebraically independent, the field $\mathfrak{F}_n = \mathbb{C}(e_1, \dots, e_n, \nu_2, \dots, \nu_n)$ is isomorphic to the field of rational fractions in $2n - 2$ independent variables.

On the other hand, $\mathbb{C}(e_1, \dots, e_n) \subset \mathbb{C}(\nu_2, \dots, \nu_{2n-1}) \subset \mathfrak{F}_n$, which gives

$$\mathfrak{F}_n = \mathbb{C}(\nu_2, \dots, \nu_{2n-1}).$$

This completes the proof of Theorem 1.5. ■

Remark 2.7. Equation (7) coincides with [7, equation (2.8)], if one reads the coefficient vector E backwards and reflects U with respect to its vertical midline. □

Let us now settle our main result about the ring extension $\mathcal{R}_n = \mathbb{C}[\nu_2, \nu_3, \dots]$:

Proof of Theorem 1.6. By (2) each polynomial $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$ is a linear function in the variables $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$, and thus cannot be a polynomial function of the other $\nu_k(\mathbf{z}, \bar{\mathbf{z}})$'s of degree exceeding one. On the other hand, ν_j is homogeneous of degree $j - 1$ with respect to $\mathbf{z} = (z_1, \dots, z_n)$, so such linear dependence is impossible as well. This proves assertion (i) of the theorem.

Further, observe that, for $j > 2n - 1$, the recurrence relation (6) allows us to express each ν_j as a rational function of ν_2, \dots, ν_{2n-1} ; its denominator is a power of $\mathfrak{D}_n \stackrel{\text{def}}{=} \det U$, where U is defined by the equation (8). This fact is equivalent to assertion (ii). ■

3 Action of the Group $S_n \times S_n$ and Anti-harmonic Moments

3.1 Action of the group $S_n \times S_n$.

The group $S_n \times S_n$, where S_n is the symmetric group on n elements, acts on the field of rational functions $\mathbb{C}(\mathbf{z}, \bar{\mathbf{z}})$ by permuting the variables: the first copy of S_n acts on z_1, \dots, z_n while the second copy acts on $\bar{z}_1, \dots, \bar{z}_n$. We denote the action of a pair $(\sigma, \tau) \in S_n \times S_n$ on a rational function R by the subscript:

$$R(\mathbf{z}, \bar{\mathbf{z}})_{(\sigma\tau)} \stackrel{\text{def}}{=} R(z_{\sigma(1)}, \dots, z_{\sigma(n)}, \bar{z}_{\tau(1)}, \dots, \bar{z}_{\tau(n)}).$$

In particular, $\nu_{j(\sigma,\tau)}$ is the result of the permutation of variables in the j -th normalized moment $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$. Note that $\nu_{j(\sigma,\tau)}(\mathbf{z}, \bar{\mathbf{z}})$ is also the j -th normalized moment of $\mathbf{a} = (a_1, \dots, a_n)$, where $a_j = (x_j, y_j)$ with $z_{\sigma(j)} = x_j + iy_j$, $\bar{z}_{\tau(j)} = x_j - iy_j$ (in other words, $a_j = ((z_{\sigma(j)} + \bar{z}_{\tau(j)})/2, (z_{\sigma(j)} - \bar{z}_{\tau(j)})/(2i))$). If $\sigma = \tau$, then $a_1, \dots, a_n \in \mathbb{C}^2$ are actually the vertices of the polygonal line for $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$, but ordered differently. In particular, if all the vertices for $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$ are real, then the vertices for $\nu_{j(\sigma,\sigma)}(\mathbf{z}, \bar{\mathbf{z}})$ are real, too.

Let us now compute the stabilizer of the above $S_n \times S_n$ -action. To do this, we will especially need to describe the $S_n \times S_n$ -orbit of the lowest degree moment $\nu_2(\mathbf{z}, \bar{\mathbf{z}})$.

Observe that every $\nu_{2(\sigma,\tau)}(\mathbf{z}, \bar{\mathbf{z}})$ is a bilinear form in the variables $(\mathbf{z}, \bar{\mathbf{z}})$. We denote by $M_{(\sigma,\tau)}$ its matrix in the standard basis of \mathbb{C}^n .

Theorem 1.4 provides that

$$M_{(\text{id}, \text{id})} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}. \quad (15)$$

For an arbitrary pair (σ, τ) , the matrix $M_{(\sigma,\tau)}$ is obtained from (15) by permuting the rows of $M_{(\text{id}, \text{id})}$ according to the permutation σ and permuting its columns, according to τ . (These two permutation actions commute.)

Expression for $M_{(\sigma,\tau)}$ can be written in other terms as well. Namely, let $\mathfrak{P} : \mathbb{C}[S_n] \rightarrow \text{Mat}(n, \mathbb{C})$ be the standard permutation representation of the symmetric group, i.e., for any $\sigma \in S_n$, the (i, j) -th entry of the $n \times n$ -matrix $\mathfrak{P}[\sigma]$ is 1 for $j = \sigma(i)$ and 0 otherwise. Then $M_{(\text{id}, \text{id})} = \mathfrak{P}[C - C^{-1}]$, where $C = (1, 2, \dots, n)$ is the long cycle, sending j to $j + 1$ for $j = 1, \dots, n - 1$ and sending n to 1. Therefore $M_{(\sigma,\tau)} = \mathfrak{P}[\sigma]^* \mathfrak{P}[C - C^{-1}] \mathfrak{P}[\tau]$. The group representation \mathfrak{P} is orthogonal, that is, $\mathfrak{P}[\sigma]^* = \mathfrak{P}[\sigma^{-1}]$ for any $\sigma \in S_n$, implying

$$M_{(\sigma,\tau)} = \mathfrak{P}[\sigma^{-1}(C - C^{-1})\tau]. \quad (16)$$

Denote by $\mathcal{M}(n)$ the set of all $n \times n$ -matrices such that every row and every column of a matrix contain one entry equal to 1, another entry equal to -1 , and all the remaining entries vanish. The group $S_n \times S_n$ acts on $\mathcal{M}(n)$ by permutation of the rows and the columns. The set of all matrices $M_{(\sigma,\tau)} \in \mathcal{M}(n)$ for $(\sigma, \tau) \in S_n \times S_n$ is an orbit of this action, which we denote by \mathcal{O}_n .

The stabilizer of ν_2 under the $S_n \times S_n$ -action is the stabilizer of \mathcal{O}_n , which we denote by \mathcal{G} . We want to describe \mathcal{G} explicitly. Recall that $C \stackrel{\text{def}}{=} (1, 2, \dots, n)$ denotes the long cycle. For even $n = 2\ell$ set $C_1 \stackrel{\text{def}}{=} (1, 3, \dots, 2\ell - 1)$, $C_2 \stackrel{\text{def}}{=} (2, 4, \dots, 2\ell)$, $\delta_1 \stackrel{\text{def}}{=} (1, 2)(3, 4) \dots (2\ell - 1, 2\ell)$, and $\delta_2 \stackrel{\text{def}}{=} (2, 3)(4, 5) \dots (2\ell, 1)$.

The next lemma is straightforward:

Lemma 3.1. For n even, the following relations hold:

- (i) $C^2 = C_1 \cdot C_2 = C_2 \cdot C_1$,
- (ii) $C \cdot C_1 = C_2 \cdot C$ and $C \cdot C_2 = C_1 \cdot C$,
- (iii) $\delta_1 \cdot C_1 = C_2 \cdot \delta_1$, $\delta_1 \cdot C_2 = C_1 \cdot \delta_1$, $\delta_2 \cdot C_1 = C_2 \cdot \delta_2$ and $\delta_2 \cdot C_2 = C_1 \cdot \delta_2$,
- (iv) $\delta_2 \cdot C = C \cdot \delta_1$ and $\delta_1 \cdot C = C \cdot \delta_2$.

□

Proposition 3.2. (i) For n odd, the stabilizer $\mathcal{G} \stackrel{\text{def}}{=} \text{St}(\mathcal{O}_n) \subset S_n \times S_n$ coincides with the cyclic group \mathbb{Z}_n , generated by $(C, C) \in S_n \times S_n$.

(ii) For $n = 2\ell$ even, the stabilizer $\mathcal{G} \subset S_n \times S_n$ consists of all the elements of the form $(C_1^u \cdot C_2^v, C_1^v \cdot C_2^u) \in S_n \times S_n$ and of all the elements $(\delta_1 \cdot C_1^u \cdot C_2^v, \delta_2 \cdot C_1^v \cdot C_2^u) \in S_n \times S_n$, where $u, v = 0, \dots, \ell - 1$. As an abstract group, \mathcal{G} is non-commutative, but it contains an index 2 subgroup isomorphic to $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$.

□

Proof. An immediate check, using (16), shows that $M_{(\sigma, \tau)} = M_{(C \cdot \sigma, C \cdot \tau)}$ for all n . (It also follows from the obvious fact that the polygonal line Γ_a does not change under the cyclic shift of the points $a_1 \mapsto a_2 \mapsto \dots \mapsto a_n \mapsto a_1$.) For even $n = 2\ell$, the same formula and Lemma 3.1 additionally imply that $M_{(\sigma, \tau)} = M_{(C_1 \cdot \sigma, C_2 \cdot \tau)}$, $M_{(\sigma, \tau)} = M_{(C_2 \cdot \sigma, C_1 \cdot \tau)}$ and $M_{(\sigma, \tau)} = M_{(\delta_1 \cdot \sigma, \delta_2 \cdot \tau)}$.

Assume now that

$$M_{(\sigma, \tau)} = M_{(\sigma', \tau')}. \quad (17)$$

Observe that relabelling the variables one can, without loss of generality, choose $\sigma = \tau = \text{id}$, where id is the identity permutation. In the representation-theoretical notation, formula (17) is equivalent to $\mathfrak{P}[\sigma^{-1}(C - C^{-1})\tau] = \mathfrak{P}[C - C^{-1}]$, that is, $\mathfrak{P}[(C - C^{-1})\tau] = \mathfrak{P}[\sigma(C - C^{-1})]$.

If $u_1, \dots, u_n \in \mathbb{C}^n$ is the standard basis, then the latter equation means that

$$\mathfrak{P}[(C - C^{-1})\tau](u_i) = u_{\tau(i)+1} - u_{\tau(i)-1} = \mathfrak{P}[\sigma(C - C^{-1})](u_i) = u_{\sigma(i+1)} - u_{\sigma(i-1)}$$

for all $i = 1, \dots, n$. In other words, $\tau(i) + 1 = \sigma(i + 1)$ for all i , that is, $\tau \cdot C = C \cdot \sigma$ and $C \cdot \tau = \sigma \cdot C$. These relations imply

$$C^2 \cdot \tau = C \cdot \sigma \cdot C = \tau \cdot C^2. \quad (18)$$

For $n = 2\ell + 1$, one has

$$(C^2)^{\ell+1} \cdot \tau = \tau \cdot (C^2)^{\ell+1} \Leftrightarrow C \cdot \tau = \tau \cdot C.$$

Since C only commutes with its own powers, one obtains $\tau = C^k$ for some $k = 0, \dots, n - 1$, implying that $\sigma = C^k = \tau$.

Consider now the case $n = 2\ell$. Set $\mathcal{E}_1 = \{1, 3, \dots, 2\ell - 1\}$ and $\mathcal{E}_2 = \{2, 4, \dots, 2\ell\}$. Then $C^2 = C_1 \cdot C_2$ (a product of two independent cycles). The permutation τ commutes with $C_1 \cdot C_2$, and the subgroups of S_n

generated by C_1 and C_2 act transitively on \mathcal{E}_1 and \mathcal{E}_2 , respectively. Therefore either $\tau(\mathcal{E}_1) = \mathcal{E}_1$ and $\tau(\mathcal{E}_2) = \mathcal{E}_2$, or $\tau(\mathcal{E}_1) = \mathcal{E}_2$ and $\tau(\mathcal{E}_2) = \mathcal{E}_1$.

In the first case, the restrictions of τ to \mathcal{E}_1 and \mathcal{E}_2 commute with the cycles C_1 and C_2 , respectively, and therefore $\tau = C_1^u \cdot C_2^v$ and $\sigma = C \cdot \tau \cdot C^{-1} = C_1^v \cdot C_2^u$. In the second case, the same reasoning holds for the permutations $\tilde{\tau} \stackrel{\text{def}}{=} \delta_2 \cdot \tau$. So $\tau = \delta_2 \cdot C_1^u \cdot C_2^v$ and $\sigma = C \cdot \tau \cdot C^{-1} = \delta_1 \cdot C_1^v \cdot C_2^u$. \blacksquare

3.2 Relations between harmonic and anti-harmonic moments

Let us now settle Theorem 1.7.

Proof. Assertions (i) and (ii) are proved similarly to the corresponding statements in Theorem 1.5 about the field \mathfrak{F}_n .

To prove assertion (iii), set

$$P(t) \stackrel{\text{def}}{=} \prod_{(\sigma, \tau) \in (S_n \times S_n)/\mathcal{G}} (t - \nu_{2(\sigma, \tau)}). \quad (19)$$

Here the index (σ, τ) runs over a system of representatives of the right cosets of $S_n \times S_n$ with respect to the stabilizer subgroup \mathcal{G} . Thus $\deg P$ is equal to the number of these right cosets, that is, to $(n!)^2/|\mathcal{G}|$. By Proposition 3.2, one has $\deg P = n!(n-1)!$ for n odd, and $\deg P = 2((n-1)!)^2$ for n even.

Lemma 3.3. $P(t)$ is the minimal polynomial defining ν_2 over the field $\mathbb{C}(\mathbf{z}, \bar{\mathbf{z}})^{S_n \times S_n}$. \square

Proof. It is enough to show that P is irreducible.

Assume that $P(t)$ is reducible and $Q(t) = \prod_{(\sigma, \tau) \in U} (t - \nu_{2(\sigma, \tau)})$ is its irreducible factor, where U is some proper subset of $(S_n \times S_n)/\mathcal{G}$. Thus $\deg Q = \#U < \#(S_n \times S_n)/\mathcal{G}$. The coefficients of the polynomial Q are $S_n \times S_n$ -invariant implying that for any $(\sigma, \tau) \in U$ the element $\nu_{2(\sigma, \tau)}$ must be a root of Q . By Proposition 3.2, this forces U intersect every right coset in $(S_n \times S_n)/\mathcal{G}$. Thus $\deg Q \geq \#(S_n \times S_n)/\mathcal{G}$, a contradiction. \blacksquare

Now let us show that, for generic $(\mathbf{z}, \bar{\mathbf{z}})$, all the roots of the polynomial P defined by (19) are simple.

Lemma 3.4. For generic $(\mathbf{z}, \bar{\mathbf{z}})$, the values of all bilinear forms $\nu_{2(\sigma, \tau)}(\mathbf{z}, \bar{\mathbf{z}})$ are pairwise distinct, where (σ, τ) runs over all right cosets $(S_n \times S_n)/\mathcal{G}$ with respect to the stabilizer group \mathcal{G} . \square

Proof. Indeed, if it is not the case, then $\mathbb{C}^n \times \mathbb{C}^n$ is a union of finitely many sets $L_{\sigma, \tau, \sigma', \tau'} \stackrel{\text{def}}{=} \{(\mathbf{z}, \bar{\mathbf{z}}) \mid \nu_{2(\sigma, \tau)}(\mathbf{z}, \bar{\mathbf{z}}) = \nu_{2(\sigma', \tau')}(\mathbf{z}, \bar{\mathbf{z}})\}$. The functions $\nu_{2(\sigma, \tau)}$ are bilinear forms, so $L_{\sigma, \tau, \sigma', \tau'}$ are quadrics. A vector space over \mathbb{C} cannot be a union of finitely many nontrivial quadrics, so $L_{\sigma, \tau, \sigma', \tau'} = \mathbb{C}^n \times \mathbb{C}^n$ for some $\sigma, \tau, \sigma', \tau'$. But then $(\sigma, \tau) = (\sigma', \tau') \bmod \mathcal{G}$, which contradicts to the choice of (σ, τ) and (σ', τ') (one element from every right coset). The lemma follows. \blacksquare

Fix some generic c_1, \dots, c_n and d_1, \dots, d_n . Then the set

$$\{(\mathbf{z}, \bar{\mathbf{z}}) \mid e_j(\mathbf{z}) = c_j, e_j(\bar{\mathbf{z}}) = d_j, j = 1, \dots, n\}$$

is a generic $S_n \times S_n$ -orbit in $\mathbb{C}^m \times \mathbb{C}^n$, where e_j denotes the j -th elementary symmetric function. By Lemma 3.4, the values of ν_2 at distinct points of the orbit are distinct. So the values $e_1(\mathbf{z}), \dots, e_n(\mathbf{z}), e_1(\bar{\mathbf{z}}), \dots, e_n(\bar{\mathbf{z}})$ and $\nu_2(\mathbf{z}, \bar{\mathbf{z}})$ determine the point $(\mathbf{z}, \bar{\mathbf{z}})$ uniquely, and therefore, determine the values $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$ for all $j = 3, 4, \dots$

Now fix some j , and let $Y \subset \mathbb{C}^{2n+1}$ be the closure of the set

$$\{(e_1(\mathbf{z}), \dots, e_n(\mathbf{z}), e_1(\bar{\mathbf{z}}), \dots, e_n(\bar{\mathbf{z}}), \nu_2(\mathbf{z}, \bar{\mathbf{z}})) \in \mathbb{C}^{2n+1} \mid \mathbf{z}, \bar{\mathbf{z}} \in \mathbb{C}^m\}.$$

For every positive integer j we introduce the algebraic variety

$$\mathfrak{G}_j \stackrel{\text{def}}{=} \{(y, c) \in Y \times \mathbb{C} \mid \exists \mathbf{z}, \bar{\mathbf{z}} \in \mathbb{C}^m : y = (e_1(\mathbf{z}), \dots, e_n(\bar{\mathbf{z}}), \nu_2(\mathbf{z}, \bar{\mathbf{z}})), c = \nu_j(\mathbf{z}, \bar{\mathbf{z}})\} \subset \mathbb{C}^{2n+2}.$$

By Lemma 3.4, the projection map $p : \mathfrak{G}_j \rightarrow \mathbb{C}^{2n+1}$, given by $p(y, c) \stackrel{\text{def}}{=} y$, is generically one-to-one onto its image. Hence there exists a rational map $\tilde{R} : \mathbb{C}^{2n+1} \rightarrow \mathfrak{G}_j$ such that $\tilde{R}|_{p(\mathfrak{G})} = p^{-1}$, see [16]. If $R \stackrel{\text{def}}{=} q \circ \tilde{R}$, where $q : \mathbb{C}^{2n+1} \times \mathbb{C} \rightarrow \mathbb{C}$ is the standard projection, then $\nu_j(\mathbf{z}, \bar{\mathbf{z}}) = R(e_1(\mathbf{z}), \dots, e_n(\mathbf{z}), e_1(\bar{\mathbf{z}}), \dots, e_n(\bar{\mathbf{z}}), \nu_2(\mathbf{z}, \bar{\mathbf{z}}))$.

Thus we have shown that

$$\nu_j \in \mathbb{C}(e_1(\mathbf{z}), \dots, e_n(\mathbf{z}), e_1(\bar{\mathbf{z}}), \dots, e_n(\bar{\mathbf{z}}))(\nu_2) = \mathbb{C}(\mathbf{z}, \bar{\mathbf{z}})^{S_n \times S_n}(\nu_2).$$

On the other hand, it follows from assertion (i) that

$$\tilde{\mathfrak{F}}_n = \mathbb{C}(e_1(\mathbf{z}), \dots, e_n(\bar{\mathbf{z}}), \nu_2, \nu_3, \bar{\nu}_3, \dots, \nu_{2n-1}, \bar{\nu}_{2n-1}),$$

which implies $\tilde{\mathfrak{F}}_n = \mathbb{C}(e_1(\mathbf{z}), \dots, e_n(\bar{\mathbf{z}}), \nu_2) = \mathbb{C}(\mathbf{z}, \bar{\mathbf{z}})^{S_n \times S_n}(\nu_2)$. ■

Proposition 3.2 and assertion (iii) of Theorem 1.7 imply the following claim:

Corollary 3.5. $\tilde{\mathfrak{F}}_n \subset \mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]^{\mathcal{G}}$, where $\mathcal{G} \subset S_n \times S_n$ is the stabilizer group of ν_2 , described in Proposition 3.2. □

For any nonnegative integer k , denote by $Y_k \subset \mathbb{C}^{2n+2k+1}$ the closure of the set

$$\{(e_1(\mathbf{z}), \dots, e_n(\mathbf{z}), e_1(\bar{\mathbf{z}}), \dots, e_n(\bar{\mathbf{z}}), \nu_2(\mathbf{z}, \bar{\mathbf{z}}), \nu_3(\mathbf{z}, \bar{\mathbf{z}}), \bar{\nu}_3(\mathbf{z}, \bar{\mathbf{z}}), \dots, \nu_{k+2}(\mathbf{z}, \bar{\mathbf{z}}), \bar{\nu}_{k+2}(\mathbf{z}, \bar{\mathbf{z}})) \mid \mathbf{z}, \bar{\mathbf{z}} \in \mathbb{C}^m\} \subset \mathbb{C}^{2n+2k+1}.$$

Using this notation, assertion (iii) of Theorem 1.7 can be reformulated as follows.

Corollary 3.6. For every positive integer k , the variety Y_k is birationally equivalent to Y_0 . □

Remark 3.7. Denote by $\tilde{\mathcal{R}}_n \stackrel{\text{def}}{=} \mathbb{C}[\nu_2, \nu_3, \bar{\nu}_3, \dots]$ the ring extension, generated by all harmonic and anti-harmonic moments. Although $\mathbb{C}(\mathbf{z}, \bar{\mathbf{z}})^{S_n \times S_n} \subset \tilde{\mathfrak{F}}_n$, it is not true that $\mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]^{S_n \times S_n} \subset \tilde{\mathcal{R}}_n$ because the elementary symmetric

functions are only expressed as rational functions of the moments. On the other hand, the inclusion $\tilde{\mathcal{R}}_n \subset \mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]^G$ obviously holds.

Similarly to Theorem 1.7, the same circumstance (i.e. the presence of a denominator in formulas) does not allow us to conclude that $\nu_2, \nu_3, \bar{\nu}_3, \dots, \nu_{2n-1}, \bar{\nu}_{2n-1}$ generate $\tilde{\mathcal{R}}_n$. Probably (though we have not yet proved this), the situation is similar to assertion (i) of Theorem 1.6: the ring cannot be generated by any proper subset of $\nu_j, \bar{\nu}_j, j = 2, 3, \dots$. Also we can conjecture that an analog of assertion (ii) of the same theorem holds: all the denominators in question are powers of a single polynomial $\tilde{\mathcal{D}}_n$. \square

Remark 3.8. Formulas (2) additionally show that $\nu_{j(\xi, \xi)} = -\nu_j$ for all j , where ξ is the involution reading the sequence $(12\dots n)$ in the opposite direction, i.e. $\xi = (1, n)(2, n-1)\dots$. Together with the cyclic group $\mathbb{Z}_n \subset S_n \times S_n$ generated by (C, C) , the involution ξ generates the dihedral group. \square

4 Relations between the Roots and the Galois Group of the Equation, Satisfied by ν_2

Assertion (iii) of Theorem 1.7 claims that the minimal polynomial $P(t)$ for the element ν_2 generates the algebraic extension of the field $\mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]^{S_n \times S_n}$ of degree

$$d_n \stackrel{\text{def}}{=} \begin{cases} n!(n-1)! & \text{if } n \text{ is odd,} \\ 2((n-1)!)^2 & \text{if } n \text{ is even.} \end{cases}$$

This extension is not Galois; its Galois closure is the field, generated by all the roots of P , that is, by $\nu_{2(\sigma, \tau)}$ for all $\sigma, \tau \in S_n$. In this section, we calculate the Galois group of the closure, or, equivalently, the Galois group of the polynomial $P(t)$. To do this, we need to describe the algebraic dependencies between the polynomials $\nu_{2(\sigma, \tau)}$; by definition, the Galois group of $P(t)$ is the subgroup of $S_n \times S_n$ preserving all these dependencies.

4.1 Linear relations and dimension

Denote by \mathfrak{M}_n the linear span of the set of all $n \times n$ -matrices $M_{(\sigma, \tau)}$, where $(\sigma, \tau) \in S_n \times S_n$ (see equation (15) above and the text following it).

Lemma 4.1. For any $n \geq 3$, the space $\mathfrak{M}_n \subset \text{Mat}_n$ consists of all $n \times n$ -matrices with vanishing row and column sums. \square

Proof. Recall that \mathfrak{P} is the standard n -dimensional representation of the group S_n . By equation (16), the matrix $M_{(\sigma, \tau)}$ belongs to the image of \mathfrak{P} . Representation \mathfrak{P} is reducible; it splits into the trivial 1-dimensional representation in the subspace $V_0 \subset \mathbb{C}^n$, spanned by the vector $v_0 = (1, 1, \dots, 1)$, and the irreducible representation of dimension $(n-1)$ in the space $V \stackrel{\text{def}}{=} \{(z_1, \dots, z_n) \mid z_1 + \dots + z_n = 0\}$. So both V_0 and V are invariant subspaces of all the $M_{(\sigma, \tau)}$.

The representation of S_n on V_0 is trivial. Thus $C|_{V_0} = \text{id}$, so that $M_{(\sigma,\tau)} = 0$ on the space V_0 . Therefore the sum of matrix elements of $M_{(\sigma,\tau)}$ in every row and column vanishes.

According to (16), Lemma 4.1 is equivalent to the following statement. For every linear operator $X : V \rightarrow V$ there exist constants $a_{\sigma,\tau} \in \mathbb{C}$ such that $\sum_{\sigma,\tau \in S_n} a_{\sigma,\tau} \mathfrak{P}[\sigma^{-1}(C - C^{-1})\tau] = X$. A standard result in representation theory says that the image of an irreducible representation of the group algebra of any finite group is the full matrix algebra of the representation space. So, for any linear operator $Y : V \rightarrow V$, there exist constants $a_{\tau,\sigma} \in \mathbb{C}$ such that $\sum_{\tau \in S_n} a_{\tau,\sigma} \mathfrak{P}[\tau] = Y$ on V , see e.g. [24]. Now Lemma 4.1 is equivalent to the following statement: for any linear operator $X : V \rightarrow V$, there exist operators $Y_\sigma, \sigma \in S_n$, such that

$$\sum_{\sigma \in S_n} \mathfrak{P}[\sigma^{-1}(C - C^{-1})]Y_\sigma = X. \quad (20)$$

To prove this claim, we observe that the operator $M_0 \stackrel{\text{def}}{=} \mathfrak{P}[C - C^{-1}] : V \rightarrow V$ is nonzero. So, the linear hull $W \subset V$ of the spaces $\mathfrak{P}[\sigma^{-1}(C - C^{-1})](V) \subset V$, $\sigma \in S_n$, is S_n -invariant and nonzero. The representation V is irreducible which implies that $W = V$. In other words, there exist (not necessarily distinct) permutations $\sigma_1, \dots, \sigma_{n-1} \in S_n$ and vectors $w_1, \dots, w_{n-1} \in V$ such that the vectors $v_i = \mathfrak{P}[\sigma_i^{-1}(C - C^{-1})](w_i)$, $i = 1, \dots, n-1$, form a basis in V .

For $i, j = 1, \dots, n-1$ let u_{ij} be the constants such that $X(v_j) = \sum_{i=1}^{n-1} u_{ij} w_i$ for all j . Define the operators $Y_\sigma, \sigma \in S_n$, by the formula:

$$Y_\sigma(e_j) = \sum_{i:\sigma_i=\sigma} u_{ij} w_i.$$

(If no σ_i equals σ , then $Y_\sigma = 0$ and it does not enter (20).) An immediate check shows that (20) holds, and the claim follows. Lemma 4.1 is settled. \blacksquare

To move further, for $1 \leq i, j \leq n-1$, denote by $\phi_{ij} \in \mathfrak{M}_n$ the matrix whose entries are equal to 1 at positions (i, j) and (n, n) , and to -1 at positions (i, n) and (n, j) , and vanish elsewhere.

Corollary 4.2. (i) For any $n \geq 3$, $\dim \mathfrak{M}_n = (n-1)^2$.

(ii) For any $n \geq 3$, matrices $\phi_{i,j}$ with $1 \leq i \leq n-1$; $1 \leq j \leq n-1$, form a basis in \mathfrak{M}_n . \square

4.2 Quadratic relations

Denote by $\mathcal{A}_n \subset \mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]$ the subalgebra generated by the bilinear forms $\nu_{2(\sigma,\tau)}$ with $(\sigma, \tau) \in S_n \times S_n$.

Proposition 4.3. For any $n \geq 3$, one has

$$\mathcal{A}_n \simeq \frac{\mathbb{C}[D]}{\langle I_2 \rangle}, \quad (21)$$

where $D = (d_{ij})$ is a $(n-1) \times (n-1)$ matrix with variable entries d_{ij} , and $\langle I_2 \rangle$ is the ideal, generated by all 2×2 -minors of D . \square

Proof. Observe that \mathcal{A}_n is generated by the linear space of bilinear forms whose matrices (written in the basis $(\mathbf{z}, \bar{\mathbf{z}})$) belong to \mathfrak{M}_n . For simplicity, we will identify bilinear forms with their matrices and denote this space by \mathfrak{M}_n as well. By Lemma 4.2 the $(n-1)^2$ forms ϕ_{ij} , $1 \leq i, j \leq n-1$, constitute a basis for \mathfrak{M}_n . More explicitly,

$$\phi_{i,j} = z_i \bar{z}_j - z_i \bar{z}_n - z_n \bar{z}_j + z_n \bar{z}_n = (z_i - z_n)(\bar{z}_j - \bar{z}_n).$$

One can easily check the equalities

$$\phi_{i_1, j_1} \phi_{i_2, j_2} - \phi_{i_1, j_2} \phi_{i_2, j_1} = 0, \quad (22)$$

coming from 2×2 -minors.

So, \mathcal{A}_n is isomorphic to the sub-algebra of $\mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]$, generated by ϕ_{ij} , $1 \leq i, j \leq n-1$. The substitution $u_i = z_i - z_n$, $i = 1 \dots n-1$, $u_n = z_n$ and $v_i = \bar{z}_i - \bar{z}_n$, $i = 1, \dots, n-1$, $v_n = \bar{z}_n$ shows that \mathcal{A}_n is isomorphic to $\mathbb{C}[u_i v_j, 1 \leq i, j \leq n-1]$.

Now observe that $\mathbb{C}[u_i v_j, 1 \leq i, j \leq n-1]$ is the coordinate ring of the Segre embedding $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2} \hookrightarrow \mathbb{P}^{(n-1)^2-1}$, where

$$([u_1 : \dots : u_{n-1}], [v_1 : \dots : v_{n-1}]) \mapsto [u_1 v_1 : u_1 v_2 : \dots : u_{n-1} v_{n-1}].$$

It is well-known (see e.g. [5, p. 14]) that the coordinate ring of $\mathbb{P}^{(n-1)^2-1}$ coincides with $S = \mathbb{C}[d_{ij}]/I_2$, where I_2 is the ideal generated by all 2×2 -minors $d_{ij} d_{pq} - d_{iq} d_{pj}$, $1 \leq i, j, p, q \leq n-1$. The image of (u_i, v_j) under the Segre embedding is d_{ij} . This observation completes the proof. \blacksquare

The Hilbert series of \mathcal{A}_n is given in [2, p. 53] and is equal to

$$\mathcal{H} = \sum_{d=0}^{\infty} \binom{d+n-1}{n-1}^2 t^d. \quad (23)$$

It is also known that \mathcal{A}_n is both Gorenstein and Koszul. The Gorenstein property was first proved in [8]; the Koszul property was first settled in [1]; see also [2].

Consider now the map $\Theta_n : \mathbb{C}[x_{(\sigma, \tau)}, \sigma, \tau \in S_n] \rightarrow \mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]$ which sends each variable $x_{(\sigma, \tau)}$ to $\nu_{2(\sigma, \tau)}$. This map is graded and doubles the degree.

Proposition 4.4. Let $n \geq 4$, where $1 \leq i, j \leq n-1$, and set

$$\rho_{ij} \stackrel{\text{def}}{=} \frac{1}{2} (x_{(1i)(1n), (1j)(2, n-1)(2n)} - x_{(1i), (1j)(2, n-1)(2n)} + x_{(1i)(1n), (1j)(2n)} - x_{(1i), (1j)(2n)}). \quad (24)$$

Then $\Theta_n(\rho_{ij}) = \phi_{ij} \in M_n$. \square

Proof. For $i = j = 1$, the proof is an immediate check. For any other i and j , one has

$$\rho_{ij} = \mathfrak{R}[(1i), (1j)] \rho_{11},$$

where \mathfrak{R} is a regular representation of $S_n \times S_n$ in $\mathbb{C}[x_{(\sigma,\tau)}, \sigma, \tau \in S_n]$, given by

$$\mathfrak{R}[(\sigma', \tau')](x_{\sigma,\tau}) = x_{\sigma'\sigma, \tau'\tau}.$$

Recall that we denote by \mathfrak{P} a standard n -dimensional representation of S_n . Then one has

$$\Theta_n(\rho_{ij}) = \mathfrak{P}[(1i)]\Theta_n(\rho_{11})\mathfrak{P}[(1j)] = \mathfrak{P}[(1i)]\phi_{11}\mathfrak{P}[(1j)] = \phi_{ij}.$$

■

The kernel $J_n \stackrel{\text{def}}{=} \text{Ker}(\Theta_n) \subset \mathbb{C}[S_n \times S_n]$ is the ideal of relations. Obviously, J_n is a homogeneous ideal: $J_n = \bigoplus_k J_{nk}$, where $J_{nk} \stackrel{\text{def}}{=} \text{Ker} \Theta_n|_k$ is the kernel of Θ_n , restricted to the degree k component of the polynomial ring $\mathbb{C}[x_{(\sigma,\tau)}, (\sigma, \tau) \in S_n \times S_n]$.

The condition $x = \sum_{\sigma,\tau \in S_n} u_{\sigma,\tau} x_{\sigma,\tau} \in J_{n1}$ means that, for any $i = 1, \dots, n$, one has

$$\begin{aligned} 0 &= \Theta(x)(e_i) = \sum_{\sigma,\tau \in S_n} u_{\sigma,\tau} \mathfrak{P}[\sigma^{-1}(C - C^{-1})(e_{\tau(i)})] \\ &= \sum_{\sigma,\tau \in S_n} u_{\sigma,\tau} \mathfrak{P}[\sigma^{-1}](e_{\tau(i)+1} - e_{\tau(i)-1}) \\ &\quad (\text{addition and subtraction are modulo } n) \\ &= \sum_{\sigma,\tau \in S_n} u_{\sigma,\tau} u_{\sigma,\tau} (e_{\sigma^{-1}(\tau(i)+1)} - e_{\sigma^{-1}(\tau(i)-1)}). \end{aligned}$$

In other words, this equality means that, for all $i, j = 1, \dots, n$, one has

$$\sum_{(\sigma,\tau): \sigma(j)=\tau(i)+1} u_{\sigma,\tau} = \sum_{(\sigma,\tau): \sigma(j)=\tau(i)-1} u_{\sigma,\tau}. \quad (25)$$

Propositions 4.3 and 4.4 imply the following.

Corollary 4.5. The ideal of relations is generated by all linear elements $x = \sum_{\sigma,\tau \in S_n} u_{\sigma,\tau} x_{\sigma,\tau}$, where the coefficients $u_{\sigma,\tau}$ satisfy equations (25), and by the quadratic elements $\rho_{i_1, j_1} \rho_{i_2, j_2} - \rho_{i_1, j_2} \rho_{i_2, j_1}$, where the elements ρ_{ij} are defined by equation (24). □

Corollary 4.6 (of Corollary 4.5). The Galois group of the Galois closure of the field extension $\mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]^{S_n \times S_n}(\nu_2) : \mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]^{S_n \times S_n}$ consists of all maps $\gamma : S_n \times S_n \rightarrow S_n \times S_n$ such that the linear transformation sending $x_{\sigma,\tau} \mapsto x_{\gamma(\sigma,\tau)}$ for all $\sigma, \tau \in S_n$, preserves all the relations, described in Corollary 4.5. □

5 Examples and Illustrations: Triangle

In this section, we illustrate our general results in the simplest nontrivial case $n = 3$, i.e. when the considered polygons are triangles.

5.1 S_3 -action and the Galois group

First of all, for $n = 3$, the numerator of equation (5) is a constant equal to $\nu_2 = \nu_2(\mathbf{z}, \bar{\mathbf{z}})$. Therefore, the harmonic moments of a triangle satisfy the relation

$$\frac{\nu_{j+2}}{\nu_2} = \binom{j+2}{2} h_j(z_1, z_2, z_3), \quad (26)$$

where $h_j(z_1, z_2, z_3)$ denotes the complete symmetric function of degree j in three variables, that is, the sum of all monomials of degree j in (z_1, z_2, z_3) . Thus ν_2 will be playing a crucial role in the following considerations; denote $M \stackrel{\text{def}}{=} \nu_2$ for short (cf. $M_{\text{id}, \text{id}}$ in Section 3; again, we do not distinguish bilinear forms from their matrices.)

Theorem 5.1. The generators $M = \nu_2, \nu_3, \bar{\nu}_3, \nu_4, \bar{\nu}_4, \nu_5, \bar{\nu}_5$ of the field $\tilde{\mathfrak{F}}_3$ satisfy a sole relation $L(M, e_1, \dots, \bar{e}_3) = 0$, where $L \stackrel{\text{def}}{=} \text{Res}_S(R, Q)$. Here $R = 16M^2 + \det \Omega(S)$, where

$$\Omega(S) = \begin{pmatrix} 3 & e_1 & \bar{e}_1 \\ e_1 & e_1^2 - 2e_2 & S \\ \bar{e}_1 & S & \bar{e}_1^2 - 2\bar{e}_2 \end{pmatrix}$$

and

$$Q = \prod_{\sigma \in S_3} (S - z_1 \bar{z}_{\sigma(1)} - z_2 \bar{z}_{\sigma(2)} - z_3 \bar{z}_{\sigma(3)}).$$

□

(Here $\text{Res}_S(R, Q)$ denotes the resultant of polynomials R and Q with respect to the variable S).

Remark 5.2. Explicitly, one has

$$R = -3S^2 + 2e_1 \bar{e}_1 S + 16M^2 + e_1^2 \bar{e}_1^2 - 4e_1^2 \bar{e}_2 - 4\bar{e}_1^2 e_2 + 12e_2 \bar{e}_2.$$

Q is a polynomial of degree 6 with respect to S ; it is symmetric in the $\{z_i\}$ and the $\{\bar{z}_i\}$ separately. Hence Q can be regarded as a polynomial of degree 6 in S with the coefficients being polynomials in the variables $\{e_k\}$ and $\{\bar{e}_k\}$, $k = 1, 2, 3$. The total degree of Q is 20; it contains 66 terms. □

Proof of Theorem 5.1. By Theorem 1.4, $M = -\frac{i}{2} \det \omega$, where

$$\omega = \begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{pmatrix}. \quad (27)$$

We will follow the argument suggested by R. Bryant in [3]. One has $\omega \cdot \omega^* = \Omega(S)$, where $S = z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3$. Thus $16M^2 = -\det \Omega(S)$ for this value of S . The same value of S is a root of the polynomial Q , so $L = \text{Res}_S(R, Q) = 0$. An explicit formula for the resultant shows that $\text{Res}_S(R, Q)$ has degree 12 with respect to M . Theorem 1.7 implies that L is the minimal polynomial for M . \blacksquare

Remark 5.3. Combining Theorem 5.1 with equations (26), one obtains a relation among $\nu_2, \nu_3, \nu_4, \nu_5, \bar{\nu}_3, \bar{\nu}_4, \bar{\nu}_5$. We calculated it explicitly using *Macaulay* computer algebra system. The result is a very long polynomial with integer coefficients (of the order of several millions) which is weighted homogeneous of degree 64 with ν_k and $\bar{\nu}_k$ having weight k for $k = 2, 3, 4, 5$. \square

Let us now present the relations between $M_{(\sigma, \tau)}$. By equation (19), one should take one pair (σ, τ) for every right coset of $S_3 \times S_3$ with respect to the cyclic group generated by (C, C) , where C is the cyclic shift (123). The number of these cosets is $3!2! = 12$, and a convenient system of representatives is $\{(\sigma, \tau) \mid \sigma \in \{\text{id}, (12)\}, \tau \in S_3\}$.

The vector space, spanned by $M_{(\sigma, \tau)}$, has dimension $(3-1)^2 = 4$. So, there exist $12 - 4 = 8$ independent linear relations between $M_{(\sigma, \tau)}$. Of them, 6 are two-term:

$$M_{(12), (12)\tau} + M_{\text{id}, \tau} = 0, \quad \tau \in S_3, \quad (28)$$

and the additional two are three-term:

$$\begin{aligned} M_{\text{id}, \text{id}} + M_{\text{id}, (123)} + M_{\text{id}, (132)} &= 0, \\ M_{(12), \text{id}} + M_{(12), (123)} + M_{(12), (132)} &= 0. \end{aligned} \quad (29)$$

The basis in the image of the map Θ_3 is formed by 4 vectors, ϕ_{11} , ϕ_{12} , ϕ_{21} , and ϕ_{22} . For $n = 3$, all quadratic relations (22) reduce to the single relation

$$\phi_{12}\phi_{21} = \phi_{11}\phi_{22}. \quad (30)$$

Direct computation shows that for $n = 3$, the forms ϕ_{ij} can be expressed via $M_{\sigma,\tau}$ as follows (recall that the general formulas (24) work only for $n \geq 4$):

$$\begin{aligned}\phi_{11} &= \frac{1}{3}(M_{\text{id},\text{id}} + 2M_{\text{id},(123)} - M_{\text{id},(12)} - M_{\text{id},(23)}), \\ \phi_{12} &= \frac{1}{3}(2M_{\text{id},\text{id}} + M_{\text{id},(123)} - M_{\text{id},(12)} - 2M_{\text{id},(23)}), \\ \phi_{21} &= \frac{1}{3}(-M_{\text{id},\text{id}} + M_{\text{id},(123)} - M_{\text{id},(12)} - 2M_{\text{id},(23)}), \\ \phi_{22} &= \frac{1}{3}(M_{\text{id},\text{id}} + 2M_{\text{id},(123)} - 2M_{\text{id},(12)} - M_{\text{id},(23)}).\end{aligned}$$

Substitution of these formulas into the quadratic relation (30) gives

$$M_{\text{id},\text{id}}^2 + M_{\text{id},\text{id}}M_{\text{id},(123)} + M_{\text{id},(123)}^2 = M_{\text{id},(12)}^2 + M_{\text{id},(12)}M_{\text{id},(23)} + M_{\text{id},(23)}^2. \quad (31)$$

The Galois group G_3 of the equation (19) permutes its 12 roots $M_{\text{id},\tau}, M_{(12),\tau}, \tau \in S_3$, while preserving the linear relations (28) and (29), together with the quadratic relation (31). Thus $G_3 \subset S_{12}$.

For $\gamma \in G_3$, relations (28) imply that there exists a bijection $\tilde{\gamma} : S_3 \rightarrow S_3$ and a map $\epsilon : S_3 \rightarrow \{1, -1\}$ such that $\gamma(M_{\text{id},\tau}) = \epsilon[\tau]M_{\text{id},\tilde{\gamma}[\tau]}$ for all $\tau \in S_3$. Then it follows from (28) that $\gamma(M_{(12),\tau}) = -\epsilon[\tau]M_{\text{id},(12)\tilde{\gamma}[\tau]} = \epsilon[\tau]M_{(12),\tilde{\gamma}[\tau]}$, which means that the bijection $\tilde{\gamma} \in S_6$ and the map ϵ determine γ uniquely. In other words, G_3 is a subgroup of the Coxeter group B_6 of signed permutations (which is naturally embedded into S_{12} , as described above).

Further, to preserve relations (29), $\tilde{\gamma}$ should either map the subsets $A_3 \stackrel{\text{def}}{=} \{\text{id}, (123), (132)\}$ and $S_3 \setminus A_3 = \{(12), (13), (23)\}$ of S_3 to themselves or to each other. In both cases, the numbers $\epsilon[\tau]$ should remain the same, while τ is changing within a set. The pairs $(\tilde{\gamma}, \epsilon) \in G_3$, where $\tilde{\gamma}$ preserves the sets, form a subgroup $G_3^+ \subset G_3$ of index 2.

Notice now that the quadratic form $Q(u) = u_1^2 + u_1u_2 + u_2^2$ is S_3 -invariant on the subspace $V_3 \stackrel{\text{def}}{=} \{u_1e_1 + u_2e_2 + u_3e_3 \mid u_1 + u_2 + u_3 = 0\} \subset \mathbb{C}^3 = \langle e_1, e_2, e_3 \rangle$ with the permutation action of S_3 . (This can be checked by an easy computation; actually, up to a factor, this form is equal to the restriction of the form $u_1^2 + u_2^2 + u_3^2$, defined in \mathbb{C}^3 , to V_3). So, any mapping γ described above automatically preserves the relation (31). Hence, the subgroup G_3^+ consists of *all* pairs $(\tilde{\gamma}, \epsilon)$, where $\tilde{\gamma}$ preserves the sets A_3 and $S_3 \setminus A_3$ and ϵ is constant within either set; thus, G_3^+ is isomorphic to the group $S_3 \times S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and contains 144 elements. The whole group G_3 contains 288 elements and is a semi-direct product of G_3^+ and \mathbb{Z}_2 .

5.2 Graphic presentation of the moment $M = \nu_2$

It follows from (26) that, in order to analyze the moments for $n = 3$, it is enough to study the lowest moment ν_2 , defined by the equation (27).

To represent the points $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{C}^2$, let us draw two triples of complex numbers: $\mathbf{z} = (z_1, z_2, z_3)$ and $\bar{\mathbf{z}} = (\bar{z}_1, \bar{z}_2, \bar{z}_3)$, where $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$, $j = 1, 2, 3$. For generic choice of z_j, \bar{z}_j , there exist unique numbers $\alpha, \beta \in \mathbb{C}$ such that $\alpha z_1 + \beta = \bar{z}_1$ and $\alpha z_2 + \beta = \bar{z}_2$. Therefore

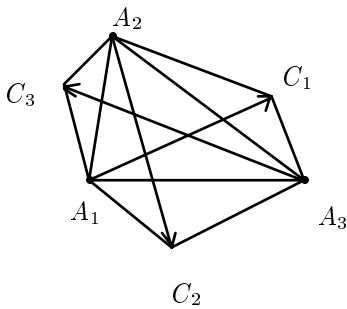
$$\nu_2 = \frac{1}{2i} \det \begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ 0 & 0 & \bar{z}_3 - (\alpha z_3 + \beta) \end{pmatrix} = \frac{1}{2i} (\bar{z}_3 - (\alpha z_3 + \beta))(z_2 - z_1). \quad (32)$$

So if $w = \alpha z_3 + \beta$, then the triangle $(\bar{z}_1 \bar{z}_2 w)$ is similar to the triangle $(z_1 z_2 z_3)$, where the similarity map sends $z_1 \mapsto \bar{z}_1, z_2 \mapsto \bar{z}_2, z_3 \mapsto w$. Obviously, this condition determines w uniquely. Then the vector connecting \bar{z}_3 and w represents the complex number $\bar{z}_3 - (\alpha z_3 + \beta)$. Thus it follows from (32) that the moment ν_2 is the product of this number by the complex number, represented by the vector that joins the vertices z_1 and z_2 , divided by $2i$. In particular, $|\nu_2|$ is one half of the product of the lengths of these two vectors. Thus, ν_2 can be thought as a measure of non-similarity of two triangles.

The action of $S_3 \times S_3$ preserves triples \mathbf{z} and $\bar{\mathbf{z}}$, but changes the numbering of these points. Identity (29) now involves moments ν_2 , calculated using (32) with the same z_1, z_2, z_3 in all three terms and $\bar{z}_1, \bar{z}_2, \bar{z}_3$, changing their labels in a cycle.

If the vertices of the triangle are real, then (29) translates into a statement from the elementary Euclidean geometry. Namely, denote by A_j the point $\bar{z}_j \in \mathbb{C} = \mathbb{R}^2$ and by C_j , the point $\alpha z_j + \beta$ from (32); here $j = 1, 2, 3$. Then (29) and (32) give:

Theorem 5.4. Let $A_1 A_2 A_3$ be a triangle in the plane \mathbb{R}^2 . Let $C_1, C_2, C_3 \in \mathbb{R}^2$ be points such that the triangles $A_1 C_3 A_2, A_2 C_1 A_3$ and $A_3 C_2 A_1$ are similar with the similarity maps sending vertices to vertices as written (e.g. $A_1 \mapsto A_2, A_2 \mapsto A_3, C_3 \mapsto C_1$ for the first two triangles, etc.). Then the sum of the vectors $\vec{A_1 C_1} + \vec{A_2 C_2} + \vec{A_3 C_3}$ vanishes. \square



$$\begin{aligned} A_1 C_3 A_2 &\sim A_2 C_1 A_3 \sim A_3 C_2 A_1 \\ \vec{A_1 C_1} + \vec{A_2 C_2} + \vec{A_3 C_3} &= 0 \end{aligned}$$

6 Further Outlook

1. According to assertion (iii) of Theorem 1.7, each moment $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$ is a rational function of $e_1(\mathbf{z}), \dots, e_n(\mathbf{z}), e_1(\bar{\mathbf{z}}), \dots, e_n(\bar{\mathbf{z}})$, and $\nu_2(\mathbf{z}, \bar{\mathbf{z}})$. Is it possible to find these rational functions explicitly?
2. The main motivation for the present paper comes from a recent article [12] by the third author (joint with K. Kohn and B. Sturmfels), where general (not harmonic) moments for convex polytopes were considered. In particular, [12] contains a complete description of relations between the axial moments of such polytopes.

A similar problem for the fields and the rings of general moments is still widely open and is apparently closely related to the complicated questions about the ring of diagonal harmonics, discussed in [10].

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