REAL POLYNOMIALS WITH CONSTRAINED REAL DIVISORS, II. (CO)HOMOLOGY AND STABILIZATION

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Abstract. In the late 80s, V. Arnold and V. Vassiliev initiated the study of the topology of the space of real univariate polynomials of a given degree and with no real roots of multiplicity exceeding a given positive integer. Expanding their studies, we consider the spaces $\mathcal{P}_d^{\mathbf{c}\Theta}$ of real monic univariate polynomials of degree d whose real divisors avoid sequences of root multiplicities taken from a given poset Θ of compositions, closed under a certain natural combinatorial operation. We reduce the computation of $H^*(\mathcal{P}_d^{\mathbf{c}\Theta})$ to the computation of the homology of a differential complex, defined purely combinatorially in terms of the poset Θ. Building upon the combinatorics of this complex, we determine the homotopy type of $\mathcal{P}_d^{\mathbf{c}\Theta}$ and calculate $H^*(\mathcal{P}_d^{\mathbf{c}\Theta})$ for several classes of Θ. We also consider the stabilization of $H^*(\mathcal{P}_d^{\mathbf{c}\Theta})$ when d goes to infinity. In particular, we prove stability results and state conjectures in case when Θ is generated by a finite set Δ of compositions.

Contents

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1. Introduction

In [Ar] V. Arnold proved the following statements which were later generalized by V. Vassiliev, see [Va]. In their formulation, we keep the original notation of [Ar], which we will abandon later on. For $1 \leq k \leq d$, let G_k^d be the space of real monic polynomials $x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{R}[x]$ with no real roots of multiplicity greater than k. (In what follows, theorems, conjectures etc. labelled by letters are borrowed from the existing literature while those labelled by numbers are hopefully new.)

Theorem A. *If $k < d < 2k+1$, then G_k^d is diffeomorphic to the product of a sphere S^{k-1} by an Euclidean space. In particular,*

$$\pi_i(G_k^d) \simeq \pi_i(S^{k-1}) \text{ for all } i.$$

An analogous result holds for the space of polynomials whose sum of roots vanishes, i.e., polynomials with vanishing coefficient a_{d-1}.

Theorem B. *The homology groups with integer coefficients of the space G_k^d are nonzero only for dimensions which are multiples of $k-1$. Namely, for $(k-1)r \leq d$,*

$$H^{r(k-1)}(G_k^d) \simeq \mathbb{Z}.$$

This paper being a sequel of [KSW1] is aimed to generalize Theorem A and Theorem B to the situation when the multiplicities of the real roots *avoid a given set of patterns.* To make this paper independent of [KSW1], we repeat below some basic definitions, notation and results of loc. cit.

Let $\mathcal{P}_d$ denote the space of real monic univariate polynomials of degree d. Given a polynomial $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$ with real coefficients, we define its *real divisor* $D_{\mathbb{R}}(P)$ as the multiset

$$\underbrace{x_1 = \cdots = x_{i_1}}_{\omega_1 = i_1} < \underbrace{x_{i_1+1} = \cdots = x_{i_1+i_2}}_{\omega_2 = i_2 - i_1} < \cdots < \underbrace{x_{i_{\ell-1}+1} = \cdots = x_{i_\ell}}_{\omega_\ell = i_\ell - i_{\ell-1}}$$

of the real roots of $P(x)$. The tuple $\omega = (\omega_1, \ldots, \omega_\ell)$ is called the (ordered) *real root multiplicity pattern* of $P(x)$. Let $\mathring{\mathsf{R}}_d^\omega$ be of the set of all polynomials with root multiplicity pattern ω, and let R_d^ω be closure in $\mathcal{P}_d$ of $\mathring{\mathsf{R}}_d^\omega$.

For a given collection Θ of root multiplicity patterns, we consider the union $\mathcal{P}_d^\Theta$ of the subspaces $\mathring{\mathsf{R}}_d^\omega$, taken over all $\omega \in \Theta$. We denote by $\mathcal{P}_d^{\mathbf{c}\Theta}$ its complement $\mathcal{P}_d \setminus \mathcal{P}_d^\Theta$.

One can easily observe that, in most of the cases, $\mathcal{P}_d^\Theta$ is contractible. Thus it makes more sense to consider its one-point compactification $\bar{\mathcal{P}}_d^\Theta$ (which is the union of the one point compactifications $\bar{\mathsf{R}}_d^\omega$ for $\omega \in \Theta$ with the points at infinity being identified). If the set $\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$, by the Alexander duality on $\bar{\mathcal{P}}_d \cong S^d$,

$$H^j(\mathcal{P}_d^{\mathbf{c}\Theta}; \mathbb{Z}) \approx H_{d-j-1}(\bar{\mathcal{P}}_d^\Theta; \mathbb{Z}),$$

which implies that the spaces $\mathcal{P}_d^{\mathbf{c}\Theta}$ and $\bar{\mathcal{P}}_d^\Theta$ carry the same (co)homological information.

Example 1.1. For Θ comprising all ω's with at least one component greater than or equal to k, we get $\mathcal{P}_d^{\mathbf{c}\Theta} = G_k^d$, in Arnold's notation.

Roughly speaking, we will be interested in the following type of problems.

Question 1.2. Given a collection Θ such that $\mathcal{P}_d^{\Theta}$ is closed in $\mathcal{P}_d$,

- ▷ how to describe $\pi_i(\mathcal{P}_d^{\mathbf{c}\Theta})$ and $\bar{H}^i(\mathcal{P}_d^{\mathbf{c}\Theta};\mathbb{Z})$ in terms of the combinatorics of Θ?
- ▷ does $\bar{H}^i(\mathcal{P}_d^{\mathbf{c}\Theta};\mathbb{Z})$ stabilize when $d \to \infty$?

We will make this question more precise in § 1.2, once more terminology will become available. One of our essential tools will be a description of a combinatorial differential complex $(\mathbb{Z}[\Theta], \partial)$ of free $\mathbb{Z}$-modules that calculates the homology of the spaces $\bar{\mathcal{P}}_d^{\Theta}$ (see Corollary 2.6). Then, applying discrete Morse theory to this complex and different combinatorial arguments, we will be able to study the cohomology of the spaces $\mathcal{P}_d^{\mathbf{c}\Theta}$. All in all, our methods provide an alternative approach to the study of various spaces of real univariate polynomials with restricted real divisors, as compared to the one used earlier by V. Arnold and V. Vassiliev.

1.1. **Cell structure on the space of real univariate polynomials.** Let us first introduce a well-known stratification of the space of real univariate polynomials of a given degree.

For any real polynomial $P(x)$, we have already defined its real divisor $D_{\mathbb{R}}(P)$, i.e. the ordered set of its real zeros, counted with their multiplicities. Denote by $D_{\mathbb{C}}(P)$ its complex conjugation-invariant *non-real divisor* in $\mathbb{C}$, i.e. the set of its non-real roots with their multiplicities. The standard *divisor* $D(P)$ of $P(x)$ is the multiset of all its complex roots, i.e. $D(P) = D_{\mathbb{R}}(P) \cup D_{\mathbb{C}}(P)$.

We have also associated to a polynomial $P(x) \in \mathbb{R}[x]$ its real root multiplicity pattern $(\omega_1, \dots, \omega_\ell)$. (Combinatorics of such multiplicity patterns will play the key role in our investigations). An arbitrary sequence $\omega = (\omega_1, \dots, \omega_\ell)$ of positive integers is called a *composition* of the number $|\omega| := \omega_1 + \cdots + \omega_\ell$. We also allow the *empty composition* $\omega = ()$ of the number $|()| = 0$.

Definition 1.3. For $\omega = (\omega_1, \dots, \omega_\ell)$, we call $|\omega|$ the *norm* and $|\omega|' := |\omega| - \ell$ the *reduced norm* of ω.

It is clear that the stratum $\mathcal{P}_d^{\omega}$ is empty if and only if either $|\omega| > d$, or $|\omega| \leq d$ and $|\omega| \not\equiv d \mod 2$.

Denote by $\mathbf{\Omega}$ the set of all compositions of natural numbers. Let us define two (sequences of) operations on $\mathbf{\Omega}$ that will govern our subsequent considerations, see also [Ka].

The *merge operations* $\mathsf{M}_j : \mathbf{\Omega} \to \mathbf{\Omega}$, sending $\omega = (\omega_1, \dots, \omega_\ell)$ to the partition

$$\mathsf{M}_j(\omega) = (M_j(\omega)_1, \dots, M_j(\omega)_{\ell-1}),$$

where for any $j \geq \ell$, one has $\mathsf{M}_j(\omega) = \omega$ and for $1 \leq j < \ell$, one has

$$\begin{aligned} \mathsf{M}_j(\omega)_i &= \omega_i \text{ if } i < j, \\ \mathsf{M}_j(\omega)_j &= \omega_j + \omega_{j+1}, \\ \mathsf{M}_j(\omega)_i &= \omega_{i+1} \text{ if } i + 1 < j \leq \ell - 1. \end{aligned} \tag{1.1}$$

.

Similarly, we define the *insertion operations* $\mathsf{I}_j : \mathbf{\Omega} \to \mathbf{\Omega}$, sending $\omega = (\omega_1, \dots, \omega_\ell)$ to the partition $\mathsf{I}_j(\omega) = (I_j(\omega)_1, \dots, I_j(\omega)_{\ell+1})$, where for any $j > \ell + 1$, one has $\mathsf{I}_j(\omega) = \omega$ and for $1 \leq j \leq \ell + 1$, one has

$$\begin{aligned} \mathsf{I}_j(\omega)_i &= \omega_i \text{ if } i < j, \\ \mathsf{I}_j(\omega)_j &= 2, \\ \mathsf{I}_j(\omega)_i &= \omega_{i-1} \text{ if } j \leq i \leq \ell + 1. \end{aligned} \tag{1.2}$$

The next proposition collects some basic properties of R_d^ω, see [Ka, Theorem 4.1] for details.

Proposition C. *Take $d \geq 1$ and $\omega = (\omega_1, \dots, \omega_\ell) \in \mathbf{\Omega}$ such that $|\omega| \leq d$ and $|\omega| \equiv d$* mod 2*. Then $\mathring{\mathsf{R}}_d^\omega \subset \mathcal{P}_d$ is an (open) cell of codimension $|\omega|'$. Moreover, R_d^ω is the union of the cells $\{\mathring{\mathsf{R}}_d^{\omega'}\}_{\omega'}$, taken over all ω' that are obtained from ω by a sequence of merging and insertion operations. In particular,*

(a) *The cell $\mathring{\mathsf{R}}_d^\omega$ has (maximal) dimension d if and only if $\omega = (\underbrace{1, 1, \dots, 1}_{\ell})$ for $0 \leq \ell \leq d$ and $\ell \equiv d \mod 2$.*

(b) *The cell $\mathring{\mathsf{R}}_d^\omega$ has dimension 1 if and only if $\omega = (d)$. In this case, $\mathring{\mathsf{R}}^{(d)} = \mathsf{R}^{(d)} = \{(x - a)^d \mid a \in \mathbb{R}\}$.*

Geometrically speaking, if a point moves in $\mathring{\mathsf{R}}_d^\omega$ and approaches the boundary $\mathsf{R}_d^\omega \setminus \mathring{\mathsf{R}}_d^\omega$, then either there is at least one value of j such that the distance between the j^{th} and $(j+1)^{\text{st}}$ distinct real roots goes to 0, or there is a value of j such that two complex-conjugate real roots converge to a real root which is then the j^{th} largest. The first situation corresponds to the application of the merge operation M_j to ω, and the second one to the application of the insertion I_j.

Note that the norm $|\omega| = \deg(D_{\mathbb{R}}(P))$ is preserved under the merge operations, while the insert operations increase $|\omega|$ by 2 and thus preserve its parity.

The merge and insert operations can be used to define a natural *partial order* "$\succ$" on the set $\mathbf{\Omega}$ of all compositions.

Definition 1.4. For $\omega, \omega' \in \mathbf{\Omega}$, we say that ω' *is smaller than* ω (notation "$\omega \succ \omega'$"), if ω' can be obtained from ω by a sequence of merge and insert operations $\{\mathsf{M}_j\}$, $j \geq 1$, and $\{\mathsf{I}_j\}$, $j \geq 0$. For a given $\omega \succ \omega'$, if there is no ω'' such that $\omega \succ \omega'' \succ \omega'$, then we say that $\omega \succ \omega'$ is a *cover relation*, or that ω' is *covered* by ω.

For a fixed d, by Proposition C, the above partial order reflects the adjacency of the non-empty cells $\{\mathcal{P}_d^\omega\}_\omega$. From now on, we will consider a subset $\Theta \subseteq \mathbf{\Omega}$ as a *poset*, ordered by $\succ$. As an immediate consequence of Proposition C, we get the following statement.

Corollary D. *For $\Theta \subseteq \mathbf{\Omega}$,*

(i) *$\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$ if and only if, for any $\omega \in \Theta$ and $\omega' \prec \omega$, we have $\omega' \in \Theta$;*

(ii) *if $\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$, then $\bar{\mathcal{P}}_d^\Theta$ carries the structure of a compact CW-complex with open cells $\{\mathring{\mathsf{R}}_d^\omega\}_{\omega\in\Theta}$, labeled by $\omega \in \Theta$, and the unique 0-cell, represented by the point $\bullet$ at infinity.*

The corollary motivates the following definition.

Definition 1.5. A subposet $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$ is called *closed* in $\mathbf{\Omega}_{\langle d]}$ if, for any $\omega' \prec \omega$ and $\omega \in \Theta$, we have $\omega' \in \Theta$.

Revisiting the beginning of § 1, we observe that the closed posets $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$ are exactly the posets for which we would like to study the spaces $\mathcal{P}_d^{\mathbf{c}\Theta}$ and $\bar{\mathcal{P}}_d^\Theta$.

Let us additionally define several natural subposets of $\mathbf{\Omega}$ which inherit the partial order $\succ$. Namely, let $\mathbf{\Omega}_{\langle d]} \subset \mathbf{\Omega}$ denote the set of all compositions ω such that $|\omega| \leq d$ and $|\omega| \equiv d$ mod 2. Further let $\mathbf{\Omega}_d \subset \mathbf{\Omega}_{\langle d]}$ denote the set of all $\omega \in \mathbf{\Omega}$ such that $|\omega| = d$. One has obvious inclusions

$$\mathbf{\Omega}_{\langle 1]} \subset \mathbf{\Omega}_{\langle 3]} \subset \mathbf{\Omega}_{\langle 5]} \subset \cdots \quad \text{and} \quad \mathbf{\Omega}_{\langle 2]} \subset \mathbf{\Omega}_{\langle 4]} \subset \mathbf{\Omega}_{\langle 6]} \subset \cdots .$$

Denote by $\mathbf{\Omega}_o$ (resp. $\mathbf{\Omega}_e$) the poset obtained as the colimit of the first (resp. the second) sequence. Clearly, $\mathbf{\Omega}_o$ contains all compositions with an odd norm and $\mathbf{\Omega}_e$ contains all compositions with an even norm. In particular, $\mathbf{\Omega} = \mathbf{\Omega}_o \cup \mathbf{\Omega}_e$. Clearly, $\mathbf{\Omega}_e$, $\mathbf{\Omega}_o$ are closed subposets of $\mathbf{\Omega}$, and $\mathbf{\Omega}_d$ is a closed subposet of $\mathbf{\Omega}_{\langle d]}$.

1.2. **Main questions and further background.** We are finally in position to precisely formulate our main quests:

Problem 1.6. For a given closed subposet $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$,

- ▷ calculate the homotopy groups $\pi_i(\bar{\mathcal{P}}_d^\Theta)$ and $\pi_i(\mathcal{P}_d^{\mathbf{c}\Theta})$ in terms of the combinatorics of Θ;
- ▷ calculate the integer homology of $\bar{\mathcal{P}}_d^\Theta$ or, equivalently, the integer cohomology of $\mathcal{P}_d^{\mathbf{c}\Theta}$ in terms of the combinatorics of Θ.

Below we will introduce a spectral sequence and a combinatorial differential complex which, *in principle*, enable us, to calculate the homology $\bar{H}_*(\bar{\mathcal{P}}_d^\Theta, \mathbb{Z})$ of the one-point compactification $\bar{\mathcal{P}}_d^\Theta$ for *any* closed poset $\Theta \subset \mathbf{\Omega}_{\langle d]}$, see § 2. However, for a general poset Θ, to obtain $\bar{H}_*(\bar{\mathcal{P}}_d^\Theta, \mathbb{Z})$ *in closed form* seems impossible!

To justify this claim, consider the closed subposet $\mathbf{\Omega}_d \subseteq \mathbf{\Omega}_{\langle d]}$ and the corresponding space $\mathcal{P}_d^{\mathbf{\Omega}_d}$, consisting of all polynomials of degree d having only real roots. It is easily seen that the poset $\mathbf{\Omega}_d$ is isomorphic to the powerset of $\{1, \dots, d-1\}$, ordered by inclusion. The latter can be identified with the face poset of the $(d-2)$-simplex. Through this identification, closed subposets $\Theta \subseteq \mathbf{\Omega}_d$ correspond to the simplicial complexes on $d-1$ vertices. In this case, it is known that the one-point compactification $\bar{\mathcal{P}}_d^{\Theta}$ is a *double suspension* the corresponding simplicial complex on $d-1$ vertices, see e.g. [SW]. In particular, it can carry *any homology* that a simplicial complex on $d-1$ vertices can carry! Since $\bar{\mathcal{P}}_d \backslash \bar{\mathcal{P}}_d^{\Theta} = \mathcal{P}_d \backslash \mathcal{P}_d^{\Theta}$, for d big enough, the cohomology of any space of finite homotopy type can occur in such a way. In particular, an arbitrary torsion may occur.

However, in this example, the dimension of the simplicial complex, corresponding to Θ, and the degree d of the polynomials under consideration are closely linked. If it is possible to loosen this link, i.e. increase d while "keeping" Θ, then one typically observes different types of (co)homological stabilization and rather "tame" answers for the limiting $\bar{H}_*(\bar{\mathcal{P}}_d^{\Theta}, \mathbb{Z})$, when $d \to \infty$.

Such stabilization is usually based on the standard affine inclusion $\underline{\text{inc}} : \mathcal{P}^d \hookrightarrow \mathcal{P}^{d+2}$, obtained by multiplication of each polynomial $P(x) \in \mathcal{P}^d$ by x^2+1. (This inclusion was already used by Arnold and Vassiliev and apparently long before them as well.) Note that the map $\underline{\text{inc}}$ preserves the combinatorial types of real divisors, i.e.

$$\omega_{D_{\mathbb{R}}(\underline{\text{inc}}(P))} = \omega_{D_{\mathbb{R}}(P)}. \tag{1.3}$$

Let us now describe a general stabilization set-up whose special cases we consider below. Let Δ be any antichain of $\mathbf{\Omega}_e$ (resp. $\mathbf{\Omega}_o$), i.e. a subset consisting of pairwise incomparable elements. We say that Δ is of *even* (resp. *odd*) parity if $\Delta \subseteq \mathbf{\Omega}_e$ (resp. $\Delta \subseteq \mathbf{\Omega}_o$). We denote by $\langle\Delta\rangle$ the closed subposet in $\mathbf{\Omega}$, generated by Δ. Note that, $\langle\Delta\rangle$ is always an infinite poset even in case when the original set Δ is finite and nonempty. But for any Δ, whose elements have the same parity of their norms as d, the restriction $\langle\Delta\rangle_d := \langle\Delta\rangle \cap \mathbf{\Omega}_{\langle d]}$ is always a finite closed subposet.

Assuming that the norms of all the elements from Δ and d are of the same parity, we have the obvious sequence of inclusions:

$$\langle\Delta\rangle_d \subseteq \langle\Delta\rangle_{d+2} \subseteq \langle\Delta\rangle_{d+4} \subseteq \cdots. \tag{1.4}$$

Additionally, set

$$\mathcal{P}_d^{\langle\Delta\rangle} := \mathcal{P}_d^{\langle\Delta\rangle_d} \quad \text{and} \quad \mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle} := \mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle_d}.$$

By (1.3), the sequence (1.4) of the poset inclusions combined with the multiplication inc by x^2+1, gives rise to the following two filtrations:

$$\begin{array}{ccccccc} \mathcal{P}_d & \xrightarrow{\text{inc}} & \mathcal{P}_{d+2} & \xrightarrow{\text{inc}} & \mathcal{P}_{d+4} & \xrightarrow{\text{inc}} & \cdots \\ \cup & & \cup & & \cup & & \\ \mathcal{P}_d^{\langle\Delta\rangle} & \xrightarrow{\text{inc}} & \mathcal{P}_{d+2}^{\langle\Delta\rangle} & \xrightarrow{\text{inc}} & \mathcal{P}_{d+4}^{\langle\Delta\rangle} & \xrightarrow{\text{inc}} & \cdots \end{array} \tag{1.5}$$

$$\begin{array}{ccccccc} \mathcal{P}_d & \xrightarrow{\text{inc}} & \mathcal{P}_{d+2} & \xrightarrow{\text{inc}} & \mathcal{P}_{d+4} & \xrightarrow{\text{inc}} & \cdots \\ \cup & & \cup & & \cup & & \\ \mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle} & \xrightarrow{\text{inc}} & \mathcal{P}_{d+2}^{\mathbf{c}\langle\Delta\rangle} & \xrightarrow{\text{inc}} & \mathcal{P}_{d+4}^{\mathbf{c}\langle\Delta\rangle} & \xrightarrow{\text{inc}} & \cdots \end{array} \tag{1.6}$$

Additionally, the inclusions in (1.5) extend to that of the one-point compactifications of all spaces in the diagram. For every non-negative integer n, the inclusions of the spaces in (1.5) and their one-point compactifications together with (1.6) induce the following sequence of homomorphisms of the homology and cohomology groups.

$$H_n(\bar{\mathcal{P}}_d^{\langle\Delta\rangle};\mathbb{Z}) \;\to\; H_n(\bar{\mathcal{P}}_{d+2}^{\langle\Delta\rangle};\mathbb{Z}) \to H_n(\bar{\mathcal{P}}_{d+4}^{\langle\Delta\rangle};\mathbb{Z}) \to \cdots, \tag{1.7}$$

$$H^n(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z}) \;\leftarrow\; H^n(\mathcal{P}_{d+2}^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z}) \leftarrow H^n(\mathcal{P}_{d+4}^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z}) \leftarrow \cdots, \tag{1.8}$$

Denote by $H_n(\bar{\mathcal{P}}_\infty^{\langle\Delta\rangle};\mathbb{Z})$ the colimit of (1.7) and by $H^n(\mathcal{P}_\infty^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ the limit of (1.8).

Problem 1.7. (i) For which Δ, does (1.7) or, equivalently, (1.8) stabilize? More precisely, for which Δ, there exists a number $\flat_n$ such the maps in (1.7) or, equivalently, in (1.8) are isomorphisms for all $d \geq \flat_n$?

(ii) If the answer to (i) is positive, calculate the limit homology $H_n(\bar{\mathcal{P}}_\infty^{\langle\Delta\rangle};\mathbb{Z})$ or cohomology $H^n(\mathcal{P}_\infty^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$.

(iii) For which Δ, (1.7) is a sequence of *injective* maps for all d sufficiently large? Equivalently, for which Δ, (1.8) is a sequence of *surjective* maps for all d sufficiently large?

(iv) If the answer to (iii) is positive, what can be said about the growth of the homology groups $H_n(\bar{\mathcal{P}}_\infty^{\langle\Delta\rangle};\mathbb{Z})$ or cohomology groups $H^n(\mathcal{P}_\infty^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$?

In what follows, we will study Problem 1.7 both in the cases when Δ could be finite and infinite.

Among infinite complexes Δ, we concentrate on a natural particular case, important for applications.

(1) $(d-k)$-skeleta of $\mathcal{P}_d$:

Consider

$$\Delta = \{\omega \in \mathbf{\Omega}_e \mid |\omega|' = k\} \text{ or } \Delta = \{\omega \in \mathbf{\Omega}_o \mid |\omega|' = k\},$$

where k is a given non-negative number. For such a Δ, whose elements ω are assumed to have the norms $|\omega| \equiv d \mod 2$, the set $\mathcal{P}_d^{\langle\Delta\rangle}$ is the union of all cells $\mathring{\mathsf{R}}_d^\omega$ of codimension $\geq k$, i.e. the $(d-k)$-skeleton of $\mathcal{P}_d$. § 4 contains our results, related to this choice of Δ. (This subproject can be thought as an extension of our study of the fundamental group of the $(d-2)$-skeleton carried out in [KSW1].)

For finite complexes Δ, we can consider a rather general situation as follows.

(2) finite Δ and the sequence $\{\langle\Delta\rangle_d\}$:

We allow Δ to be an arbitrary antichain, i.e. a finite collection of incompatible compositions ω with the same parity of their norms. Consideration of the sequence $\{\langle\Delta\rangle_d\}$ and the corresponding sequences (1.7) – (1.8) is called (the problem of) the *direct stabilization*. In § 5 we consider the special case when Δ contains a single composition ω, which is both non-trivial and allows us to obtain interesting explicit results for each individual d. At the same time, as $d \to \infty$, we obtain rather general stabilization results about $\bar{H}_*(\bar{\mathcal{P}}_d^{\langle\Delta\rangle};\mathbb{Z})$, see § 6 below.

Remark 1.8. In the concluding part [KSW2] of our study we consider Arnold–Vassiliev type situation which includes and generalizes the original problem considered by V. Arnold [Ar] and V. Vassiliev [Va]. Namely, for a given composition $\omega = (\omega_1, \dots, \omega_\ell) \in \mathbf{\Omega}$, we call a composition $\omega' = (\omega'_1, \dots, \omega'_m)$ an *extension* of ω if $|\omega|$ and $|\omega'|$ are of the same parity and there are numbers $1 \leq i_1 < \cdots < i_\ell \leq m$ such that $\omega_j = \omega'_{i_j}$ for $1 \leq j \leq \ell$ and $\omega'_j = 1$ for $j \in \{1, \dots, m\} \setminus \{i_1, \dots, i_\ell\}$. add a sentence that describes this informally In other words, to get such an ω' from ω, we may insert any number of 1's in-between the entries of ω, as well as any number of 1's prior or after the string ω. Let $\mathrm{ext}(\omega) \subset \mathbf{\Omega}$ denote the (infinite) set of all extensions of ω. By definition, the compositions in $\mathrm{ext}(\omega)$ are pairwise incomparable.

Given $\omega \in \mathbf{\Omega}_e$ and an $d \geq |\omega|$ of the same parity as $|\omega|$, the space $\mathcal{P}_d^{\langle\langle\omega\rangle\rangle}$ may be described as the set of real polynomials of the form

$$T(x) \prod_{j=1}^{q} (x - \alpha_j)^{\omega_j},$$

where $T(x) \in \mathcal{P}_{d-|\omega|}$, and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_q$ is a non-decreasing sequence of real numbers.

Let $\Delta = \{\omega^{(1)}, \dots, \omega^{(m)}\} \subset \mathbf{\Omega}$ be an antichain in the above partial order, i.e., a finite collection of pairwise incomparable compositions with $|\omega^{(1)}|, \dots, |\omega^{(m)}|$ of the same parity. Set $\mathrm{ext}(\Delta) := \mathrm{ext}(\omega^{(1)}) \cup \cdots \cup \mathrm{ext}(\omega^{(m)})$. Note that under our assumptions, $\mathrm{ext}(\Delta)$ is a set of pairwise incomparable composition with norms of equal parity. Finally, denote by $\langle\langle\Delta\rangle\rangle \subset \mathbf{\Omega}$ the closed subposet, generated by $\mathrm{ext}(\Delta)$. Similarly to the above we have the sequences

$$H_n(\bar{\mathcal{P}}_d^{\langle\langle\Delta\rangle\rangle}; \mathbb{Z}) \to H_n(\bar{\mathcal{P}}_{d+2}^{\langle\langle\Delta\rangle\rangle}; \mathbb{Z}) \to H_n(\bar{\mathcal{P}}_{d+4}^{\langle\langle\Delta\rangle\rangle}; \mathbb{Z}) \to \cdots, \tag{1.9}$$

$$H^n(\mathcal{P}_d^{\mathbf{c}\langle\langle\Delta\rangle\rangle}; \mathbb{Z}) \leftarrow H^n(\mathcal{P}_{d+2}^{\mathbf{c}\langle\langle\Delta\rangle\rangle}; \mathbb{Z}) \leftarrow H^n(\mathcal{P}_{d+4}^{\mathbf{c}\langle\langle\Delta\rangle\rangle}; \mathbb{Z}) \leftarrow \cdots, \tag{1.10}$$

The main goal of [KSW2] is to study the topology of the spaces $\bar{\mathcal{P}}_d^{\langle\langle\Delta\rangle\rangle}$ and $\mathcal{P}_d^{\mathbf{c}\langle\langle\Delta\rangle\rangle}$.

Let us outline the general structure of the paper. In § 2, we describe a combinatorial complex associated with any closed subposet $\Theta \subset \mathcal{P}_{\langle d]}$ which, in principle, allows us to compute $H_*(\bar{\mathcal{P}}_d^{\Theta}; \mathbb{Z})$ or, equivalently, $H^*(\mathcal{P}_d^{\mathbf{c}\Theta}; \mathbb{Z})$. A short § 3 provides a brief description of the basic results from the discrete Morse theory to be used in the rest of the paper. In § 4, using the discrete Morse theory, we provide explicit answers for the homotopy type of the $(d-k)$-skeleta of $\mathcal{P}_d$. In § 5 we describe the homotopy type of $\bar{\mathcal{P}}_d^{\Theta}$ when Θ is generated by a single composition ω. In § 6, we describe our stabilization results for $H_*(\bar{\mathcal{P}}_d^{\Theta}; \mathbb{Z})$ and $H^*(\mathcal{P}_d^{\mathbf{c}\Theta}; \mathbb{Z})$ for arbitrary poset Θ generated by a finite set of compositions. In § 7 we present examples of (co)homology groups of our spaces obtained with the help of computer program in GAP written by the third author whose algorithm is based on the results of § 2. These examples illustrate our stabilization results as well as show the existence of some unstable (co)homology which we at present can not explain or describe. We also formulate a number of natural open problems not covered by our results. Finally, in § 9 we present a motivating example of the cohomology class of one of the spaces under consideration which is relevant in the study of traversing vector fields on manifolds with boundary.

Remark 1.9. Besides the previous studies of V. Arnold [Ar] and V. Vassiliev [Va], our major motivation for this paper comes from the results of the first author, connecting the cohomology $H^*(\mathcal{P}_d^{\mathbf{c}\Theta};\mathbb{Z})$ with certain characteristic classes, arising in the theory of traversing flows, see [Ka], [Ka1]. For traversing vector flows on compact manifolds X with boundary ∂X and with an a priory forbidden tangency patterns Θ of their trajectories to ∂X, the spaces $\mathcal{P}_d^{\mathbf{c}\Theta}$ play an important role, similar to the role of Grassmanians in the category of vector bundles. (§ 9 gives some hints about this role).

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2. A combinatorial differential complex computing $H_*(\bar{\mathcal{P}}_d^{\Theta}, \mathbb{Z})$

In this section, we use the natural structure of CW-complex for $\mathcal{P}_d$, presented in Proposition C, to construct a combinatorial differential complex $(\mathbb{Z}[\Theta], \partial)$ that calculates the homology of the one-point compactification $\bar{\mathcal{P}}_d^{\Theta}$ for any closed subposet $\Theta \subset \mathbf{\Omega}_{\langle d]}$. Recall that the cells $\{e_\omega\}$ of the CW-complex $\mathcal{P}_d$ are indexed by the compositions $\omega \in \mathbf{\Omega}_{\langle d]}$. The dimension of the cell e_ω equals $d - |\omega|'$. This CW-complex has a single 1-dimensional cell, labelled by $\omega_\star = (d)$. In particular, the Euler formula for $\bar{\mathcal{P}}_d \approx S^d$ amounts to

$$\sum_{k=0}^{d-1} (-1)^{d-k} \big|\big\{\omega \in \mathbf{\Omega}_{\langle d]} : |\omega|' = k\big\}\big| = 1 + (-1)^d.$$

Our next goal is, for a given closed subposet $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$, to define a spectral sequence, converging to the reduced homology $\bar{H}_*(\bar{\mathcal{P}}_d^{\mathbf{\Theta}}, \mathbb{Z})$, and to describe its properties.

For a closed subposet $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$, consider a decreasing filtration of $\mathcal{P}_d^{\Theta}$ given by

$$\mathcal{P}_d^{\Theta} = F_0(\mathcal{P}_d^{\Theta}) \supset F_1(\mathcal{P}_d^{\Theta}) \supset \cdots \supset F_{d-1}(\mathcal{P}_d^{\Theta})$$

by the closed subsets

$$F_k(\mathcal{P}_d^{\Theta}) := \mathcal{P}_d^{\Theta_{|\sim|' \geq k}} = \bigcup_{\{\omega \in \Theta:\ |\omega|' \geq k\}} \mathcal{P}_d^{\omega}. \tag{2.1}$$

By Proposition C, each $\mathring{\mathsf{R}}_d^{\omega}$ is an open $(d - |\omega|')$-cell. The filtration (2.1) immediately extends to the one-point compactification by taking

$$\bar{\mathcal{P}}_d^{\Theta} = F_0^{\bullet}(\bar{\mathcal{P}}_d^{\Theta}) \supset F_1^{\bullet}(\bar{\mathcal{P}}_d^{\Theta}) \supset \cdots \supset F_{d-1}^{\bullet}(\bar{\mathcal{P}}_d^{\Theta}) \supset F_d^{\bullet}(\bar{\mathcal{P}}_d^{\Theta}) := \bullet. \tag{2.2}$$

Note that for any $k < d$, we have that

$$F_k(\mathcal{P}_d^\Theta) \setminus F_{k+1}(\mathcal{P}_d^\Theta) = F_k^\bullet(\bar{\mathcal{P}}_d^\Theta) \setminus F_{k+1}^\bullet(\bar{\mathcal{P}}_d^\Theta)$$

is a disjoint union of the open cells $\mathring{\mathcal{P}}_d^\omega$, where $\omega \in \Theta$ runs over all compositions with $|\omega|' = k$. By Proposition C, each such cell is homeomorphic to $\mathbb{R}^{d-k}$.

Consider the spectral sequence $\{E_{p,q}^s\}$ calculating reduced homology of the one-point compactification $\bar{\mathcal{P}}_d^\Theta$ of the space $\mathcal{P}_d^\Theta$, associated with the decreasing filtration (2.2). Its $E_{p,q}^2$-term consists of the groups

$$\begin{aligned} E_{p,q}^2 &\cong \bigoplus_{\{\omega \in \Theta,\ |\omega|' = p\}} H_{p+q}(\bar{\mathsf{R}}_d^\omega, \partial \bar{\mathsf{R}}_d^\omega;\ \mathbb{Z}) \\ &\cong \bigoplus_{\{\omega \in \Theta,\ |\omega|' = p\}} \bar{H}_{p+q}(S^{d-|\omega|'}; \mathbb{Z}) \\ &\cong \bigoplus_{\{\omega \in \Theta,\ |\omega|' = p\}} \bar{H}_{p+q}(S^{d-p}; \mathbb{Z}). \end{aligned}$$

It is clear that $E_{p,q}^2$ is non-trivial only when $p+q = d-p$ or, equivalently, when $2p+q = d$. Therefore,

$$E_{p,d-2p}^2 \approx \mathbb{Z}[\Theta_{|\sim|'=p}]$$

is the group generated by formal integral combinations of the elements taken from the set $\Theta_{|\sim|'=p}$, and vanishes otherwise. In particular, $E_{p,q}^2 = 0$ outside of the range $\{(p,q)|\ 0 \leq p \leq d-1;\ 0 \leq p+q \leq d\}$.

Remark 2.1. Notice that the coincidence of the differences of strata for the original space and for its one-point compactification does not imply that the E_2-terms of the corresponding spectral sequences are isomorphic.

Theorem 2.2. *The spectral sequence $\{E_{p,q}^s, d^s\}$ degenerates at the E^3-term.*

Proof. Indeed due to the constraint $2p+q = d$, all the differentials

$$d^s : E_{p,q}^s \to E_{p+s-1,\, q-s}^s,$$

with $s \geq 3$, have trivial targets. As a result, for all $i \in [0, d]$,

$$\bar{H}_i(\bar{\mathcal{P}}_d^\Theta; \mathbb{Z}) \approx \frac{\mathsf{ker}\big\{d_{d-i,\,2i-d}^2 : \mathbb{Z}[\Theta_{|\sim|'=d-i}] \longrightarrow \mathbb{Z}[\Theta_{|\sim|'=d-i+1}]\big\}}{\mathsf{im}\big\{d_{d-i-1,\,2i-d+2}^2 : \mathbb{Z}[\Theta_{|\sim|'=d-i-1}] \longrightarrow \mathbb{Z}[\Theta_{|\sim|'=d-i}]\big\}}.$$

Note that if $\Theta_{|\sim|'=d-i} = \{\emptyset\}$, then, by definition, $\mathbb{Z}[\emptyset] = 0$. □

Our next goal is to explicitly describe the complex $\{E^2, d^2\}$ purely combinatorially. Observe that the free $\mathbb{Z}$-module $\mathbb{Z}[\mathbf{\Omega}]$ comes with a *bigrading*, induced by the norm $|\omega|$ together with the reduced norm $|\omega|'$ where $\omega \in \mathbf{\Omega}$. By definition, the first grading of an element $\sum_\omega n_\omega \cdot \omega \in \mathbb{Z}[\mathbf{\Omega}]$, $n_\omega \neq 0$ is greater than or equal to k, if $|\omega|' \geq k$ for all ω in the sum. We call this grading *codimensional*. It gives rise to a decreasing filtration $\{\mathbb{Z}[\mathbf{\Omega}]_{'\geq k}\}_{k \in \mathbb{Z}_+}$ of $\mathbb{Z}[\mathbf{\Omega}]$. The same reasoning works verbatim for both $\mathbb{Z}[\mathbf{\Omega}_o]$ and $\mathbb{Z}[\mathbf{\Omega}_e]$.

By definition, the second grading of an element $\sum_\omega n_\omega \cdot \omega$ is greater than or equal to d, if $|\omega| \geq d$ for all ω in the sum. We call this grading *degree-based*. Thus if $\mathbb{Z}[\mathbf{\Omega}_{\langle d]}]$ denotes the $\mathbb{Z}$-submodule of $\mathbb{Z}[\mathbf{\Omega}_e]$ (resp. $\mathbb{Z}[\mathbf{\Omega}_o]$), generated by the elements whose degree-based grading is less than or equal to d, we get an increasing filtration

$$\ldots \subset \mathbb{Z}[\mathbf{\Omega}_{\langle d]}] \subset \mathbb{Z}[\mathbf{\Omega}_{\langle d+2]}] \subset \mathbb{Z}[\mathbf{\Omega}_{\langle d+4]}] \subset \ldots$$

of $\mathbb{Z}[\mathbf{\Omega}_e]$ (resp. $\mathbb{Z}[\mathbf{\Omega}_o]$).

Below we will concentrate on the case when d is even, i.e. $|\omega| \equiv 0 \mod 2$ and study the sequence

$$\mathbf{\Omega}_{\langle 2]} \subset \mathbf{\Omega}_{\langle 4]} \subset \mathbf{\Omega}_{\langle 6]} \subset \ldots.$$

In this case it will be convenient to use $|\omega|/2$ for the degree-based grading. (The case of odd d is completely parallel.) In fact, when d is fixed in order to introduce the degree-based grading we can instead of $|\omega|$ consider $\frac{d-|\omega|}{2}$ which is integer for both even and odd d.

Using the merge operator $\mathsf{M}_\bullet$ and the insert operator $\mathsf{I}_\bullet$ on $\mathbf{\Omega}_e$ (resp. $\mathbf{\Omega}_o$) from § 1.1, we define two homomorphisms on $\mathbb{Z}[\mathbf{\Omega}_e]$ given by

$$\partial_{\mathsf{M}}(\omega) := -\sum_{k=1}^{s_\omega - 1}(-1)^k \mathsf{M}_k(\omega) \ \text{ and } \ \partial_{\mathsf{I}}(\omega) := \sum_{k=0}^{s_\omega}(-1)^k \mathsf{I}_k(\omega),$$

where $s_\omega := |\omega| - |\omega|'$ is the cardinality of the support of ω.

Next, we define a homomorphism

$$\partial = \partial_{\mathsf{M}} + \partial_{\mathsf{I}} : \mathbb{Z}[\mathbf{\Omega}_{\langle d]}] \to \mathbb{Z}[\mathbf{\Omega}_{\langle d]}]$$

by the formula

$$\partial(\omega) := \begin{cases} -\sum_{k=1}^{s_\omega-1}(-1)^k\, \mathsf{M}_k(\omega) + \sum_{k=0}^{s_\omega}(-1)^k\, \mathsf{I}_k(\omega), & \text{for } |\omega| < d, \\ \\ -\sum_{k=1}^{s_\omega-1}(-1)^k\, \mathsf{M}_k(\omega), & \text{for } |\omega| = d. \end{cases} \tag{2.3}$$

Assuming that $\Theta \subset \mathbf{\Omega}_e$ is some closed subposet, set

$$\Theta_d := \Theta \cap \mathbf{\Omega}_d \text{ and } \Theta_{\langle d]} := \Theta \cap \mathbf{\Omega}_{\langle d]}.$$

(Obviously, $\Theta_{\langle d]}$ is a closed subposet of $\mathbf{\Omega}_{\langle d]}$).

Lemma 2.3. *In the above notation the following facts hold:*

1. *The homomorphisms $\partial_{\mathsf{M}}, \partial_{\mathsf{I}} : \mathbb{Z}[\Theta] \to \mathbb{Z}[\Theta]$ are anti-commuting differentials, i.e., $\partial_{\mathsf{I}}^2 = \partial_{\mathsf{M}}^2 = \partial_{\mathsf{I}}\partial_{\mathsf{M}} + \partial_{\mathsf{M}}\partial_{\mathsf{I}} = 0$ which implies that $\partial = \partial_{\mathsf{M}} + \partial_{\mathsf{I}}$ is a differential as well;*
2. *∂_{M} increases the codimensional grading by 1 and preserves the degree-based grading, while ∂_{I} increases both the degree-based and the codimensional gradings by 1.*

Remark 2.4. Again when d is fixed, we can introduce the quantities $\kappa(\omega) := \frac{d-|\omega|}{2}$ and $\nu(\omega) := \frac{d-|\omega|}{2} + s_\omega$ which we will use later in this text. Here as above $s(\omega)$ is the cardinality of the support of ω. In these notations, ∂_{M} preserves $\kappa(\omega)$ and decreases $\nu(\omega)$ by 1 while ∂_{I} preserves $\nu(\omega)$ and decreases $\kappa(\omega)$ by 1. The sum $\kappa(\omega) + \nu(\omega)$ equals the dimension of the stratum $\mathcal{P}_d^\omega$.

Proof. To settle (1), we need to show that $\partial_{\mathsf{I}}^2 = 0$, $\partial_{\mathsf{M}}^2 = 0$, and $\partial_{\mathsf{I}}\partial_{\mathsf{M}} + \partial_{\mathsf{M}}\partial_{\mathsf{I}} = 0$. Consider any composition $\omega = (\dots a, b, c, \dots)$, where a, b, c are arbitrary natural numbers. Then the expression for $\partial_{\mathsf{M}}(\omega)$ will include the terms

$$\pm(\dots a + b, c, \dots) \mp (\dots a, b + c, \dots).$$

Thus $\partial_{\mathsf{M}}^2(\omega)$ will consist of the terms

$$\pm(\dots a + b + c, \dots) \mp (\dots a + b + c, \dots),$$

which cancel each other. When the support of ω has cardinality ≤ 2 an even simpler argument applies.

To show that $\partial_{\mathsf{I}}^2 = 0$, we take $\omega = (\dots, a, b, \dots)$ and calculate

$$\partial_{\mathsf{I}}(\omega) = \dots \pm (\dots, a, 2, b, \dots) \mp \dots.$$

Then $\partial_{\mathsf{I}}^2(\omega)$ will only contain the terms

$$\pm(\dots, a, 2, 2, b, \dots) \mp (\dots, a, 2, 2, b, \dots),$$

which again cancel each other.

Finally, let us compute $(\partial_{\mathsf{I}}\partial_{\mathsf{M}} + \partial_{\mathsf{M}}\partial_{\mathsf{I}})(\omega)$ with $\omega = (\dots, a, b, \dots)$. Observe that $\partial_{\mathsf{I}}(\omega)$ consists of the terms

$$\pm(\dots, 2, a, b, \dots) \mp (\dots, a, 2, b, \dots) \pm (\dots, a, b, 2, \dots).$$

Computing $\partial_{\mathsf{M}}\partial_{\mathsf{I}}(\omega)$, after a cancellation, we will be left with the contribution

$$\mp(\dots, 2, a + b, \dots) \pm (\dots, a + b, 2, \dots).$$

In a similar computation of $\partial_{\mathsf{I}}\partial_{\mathsf{M}}(\omega)$, we will be left with the contribution

$$\pm(\dots, 2, a + b, \dots) \mp (\dots, a + b, 2, \dots),$$

which implies that $(\partial_{\mathsf{I}}\partial_{\mathsf{M}} + \partial_{\mathsf{M}}\partial_{\mathsf{I}})(\omega) = 0$.

The second claim of the lemma is straightforward. □

Proposition 2.5. *For any closed subposet $\Theta \in \mathbf{\Omega}_{\langle d]}$, the differential complex $(\mathbb{Z}[\Theta], \partial)$ coincides with the E^2-term of the spectral sequence $\{E^s_{p,q}, d^s\}$, associated with the filtration (2.1) and converging to the reduced homology $\bar{H}_*(\bar{\mathcal{P}}^\Theta_d; \mathbb{Z})$ of $\bar{\mathcal{P}}^\Theta_d$.*

Proof. By Proposition C, the (topological) boundary $\partial \mathsf{R}^\omega_d$ coincides with

$$\Big(\bigcup_{k=1}^{s_\omega - 1} \mathsf{R}_d^{\mathsf{M}_k(\omega)}\Big) \bigcup \Big(\bigcup_{k=0}^{s_\omega} \mathsf{R}_d^{\mathsf{I}_k(\omega)}\Big).$$

Therefore, the boundary $\partial[\bar{\mathsf{R}}^\omega_d]$ of the chain $[\bar{\mathsf{R}}^\omega_d]$ in the cellular chains complex $\bar{C}_{d-|\omega|'}(\bar{\mathcal{P}}^\Theta_d; \mathbb{Z})$ is equal to the sum

$$\sum_{k=1}^{s_\omega - 1} a_k [\bar{\mathsf{R}}_d^{\mathsf{M}_k(\omega)}] + \sum_{k=0}^{s_\omega} b_k [\bar{\mathsf{R}}_d^{\mathsf{I}_k(\omega)}] \tag{2.4}$$

for some choice of integral coefficients $\{a_k\}$ and $\{b_k\}$.

Note that the chain $\bar{\mathsf{R}}_d^\omega$ can be also thought of as a generator of $\bar{H}_{d-|\omega|'}(\bar{\mathsf{R}}_d^\omega, \partial\bar{\mathsf{R}}_d^\omega;\ \mathbb{Z})$, or even as an element of $\bar{H}_{d-|\omega|'}\big(\bar{\mathcal{P}}_d^{\Theta_{|\omega|'}}, \bar{\mathcal{P}}_d^{\Theta_{|\omega|'+1}}; \mathbb{Z}\big)$.

Recall how the preferred orientation of the chain $[\bar{\mathsf{R}}_d^\omega]$ is generated by the canonical orientation of the open cell $\mathring{\mathsf{R}}_d^\omega \approx Sym^{s_\omega}(\mathbb{R}) \times Sym^{m_\omega}(\mathsf{H})$, where H denotes the upper half-plane $\{z \in \mathbb{C} |\, \mathrm{Im}(z) > 0\}$ and $m_\omega := (d - |\omega|)/2$ (see [Ka]). Namely, the orientation of the open cell $Sym^{m_\omega}(\mathsf{H})$ is canonically induced by its complex structure, while the orientation of $Sym^{s_\omega}(\mathbb{R})$ is induced from its embedding, as a convex (open) polyhedron Π°_ω, in the space $\mathbb{R}^{s_\omega}$ with the coordinates $(x_1, x_2, \dots, x_{s_\omega})$. The polyhedron Π°_ω is given by the inequalities $x_1 < x_2 < \dots < x_{s_\omega}$. In other words, the orientation of $\mathring{\mathsf{R}}_d^\omega$ is given by the volume form

$$\theta_\omega := (dx_1 \wedge \dots \wedge dx_{s_\omega}) \wedge \Big(\frac{i}{2}\Big)^{m_\omega} (dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_{m_\omega} \wedge d\bar{z}_{m_\omega})$$

considered on the product $Sym^{s_\omega}(\mathbb{R}) \times Sym^{m_\omega}(\mathsf{H})$.

We claim that in the formula (2.4) for the boundary operator ∂, $a_k = (-1)^{k+1}$. Note that, if $\mathsf{M}_k(\omega) = \mathsf{M}_l(\omega)$, then $k = l$.[1] Therefore, for each polynomial $P \in \mathring{\mathsf{R}}_d^{\mathsf{M}_k(\omega)}$ and any path $P_t \subset \mathring{\mathsf{R}}_d^\omega$ such that $\lim_{t\to 0} P_t = P$, there exists for all sufficiently small t, a single pair of real roots $x_k(t), x_{k+1}(t)$ of P_t that merge, forming a root $x_k^\star$ of P. Moreover, for any two t-paths $P_t, Q_t \subset \mathring{\mathsf{R}}_d^\omega$ such that $\lim_{t\to 0} P_t = \lim_{t\to 0} Q_t = P$, their germs at P are isotopic in $\mathring{\mathsf{R}}_\circ^\omega$. Such isotopy is produced by the linear homotopy that connects each root $x_l(P_t)$ to the root $x_l(Q_t)$. Such an isotopy extends to the identity on $\mathsf{R}_d^{\mathsf{M}_k(\omega)}$. Therefore, the germ of $\mathring{\mathsf{R}}_d^\omega$ at P has a single connected component. As a result, the incidence index a_k of the cell $\mathring{\mathsf{R}}_d^\omega$ with the cell $\mathring{\mathsf{R}}_d^{\mathsf{M}_k(\omega)}$ equals ± 1.

Now consider a t-family of polynomials $P_t \subset \mathring{\mathsf{R}}_d^\omega$ such that:

(1) $\lim_{t\to 0} P(t) = P \in \mathring{\mathsf{R}}_d^{\mathsf{M}_k(\omega)}$;
(2) polynomials $P(t)$ share the all roots with P, except for the roots $x_k(t), x_{k+1}(t)$ and $x_k^\star$;
(3) $x_k(t) = x_k^\star - t$, $x_{k+1}(t) = x_k^\star + t$.

The tangent vector to the path $P(t)$ at $t = 0$ is given by $w = (0, \dots, 0, -1, 1, 0, \dots)$. This vector is the inward normal to the face $\{x_k = x_{k+1}\}$ in the polyhedron $\Pi_\omega \subset \mathbb{R}^{s_\omega}$ at the point $p^\star$ representing the ordered sequence of real numbers that coincide with the support of $D_{\mathbb{R}}(P) \subset \mathbb{R}$. The inner product "$\rfloor$" of w with the volume form in $\mathbb{R}^{s_\omega}$ is given by

$$w \,\rfloor (dx_1 \wedge \dots \wedge dx_k \wedge dx_{k+1} \wedge \dots \wedge dx_{s_\omega}) = (-1)^{k+1}(dx_1 \wedge \dots \wedge dx_k \wedge dx_{k+2} \wedge \dots \wedge dx_{s_\omega}).$$

Therefore, the orientation of the boundary $\partial\Pi^\circ_\omega$, induced by the orientation of Π°_ω, differs from the preferred orientation of $\Pi^\circ_{\mathsf{M}_k(\omega)}$ exactly by the factor $(-1)^{k+1}$.

[1] It is possible that $\mathsf{M}_k(\omega) = \mathsf{M}_l(\omega')$ for $k \neq l$ and an appropriate $\omega' \neq \omega$.

Next, let us determine the incidence coefficient b_k of the cell R_d^ω with the cell $\mathsf{R}_d^{\mathsf{I}_k(\omega)}$ in the sum (2.4). It is a more delicate task since, in the vicinity of $P \in \mathring{\mathsf{R}}_d^{\mathsf{I}_k(\omega)}$, multiple sheets (components) of $\mathring{\mathsf{R}}_d^\omega$ may appear.

The simplest example of such phenomenon is the case when P has two consecutive real roots, x_i and x_{i+1}, each of the multiplicity 2. Let us denote by P_1 and P_2 two polynomials obtained by resolving the first and the second roots into a pair of simple complex-conjugate roots. Then the combinatorial patterns of $D_{\mathbb{R}}(P_1)$ and $D_{\mathbb{R}}(P_2)$ are identical. Although P_1 and P_2 are close to each other in R_d^ω, there is no short path in $\mathring{\mathsf{R}}_d^\omega$ that connects them in a neighborhood of P, i.e. these polynomials P_1 and P_2 represent different connected components of $\mathring{\mathsf{R}}_d^\omega$.

Denote by $x_1^\star, \dots, x_{s_\omega+1}^*$ the distinct real roots of $P \in \mathring{\mathsf{R}}_d^{I_k(\omega)}$ ordered by their magnitude, and by $\{(z_l^\star, \bar{z}_l^\star)\}_l$ the unordered collection of non-real roots of P. Assume that $x_{k+1}^\star$ is a root of multiplicity 2. Let $P_t \subset \mathring{\mathsf{R}}_d^\omega$ be a path in $\mathring{\mathsf{R}}_d^\omega$. Denote by $\{x_j(t)\}$ all real and $\{(z_l(t), \bar{z}_l(t)\}$ all non-real roots of $P(t)$. We can choose P_t so that:

(1) $x_j(t) = x_j^\star$ for all $j \le k$ and $x_j(t) = x_{j+1}^\star$ for all $j > k+1$,
(2) the pair $(z_{k+1}(t), \bar{z}_{k+1}(t)) = (x_{k+1}^\star + \mathsf{i}\,t, x_{k+1}^\star - \mathsf{i}\,t)$ is a root pair for P_t,
(3) all the other non-real root pars for P_t and P coincide.

In the ordered "root space" $\mathbb{R}^{s_\omega} \times \mathbb{C}^{m_\omega}$, $w = (0, \dots, 0, \mathsf{i}, 0, \dots 0)$ is the tangent vector to the curve P_t at P.

The volume form θ_ω can be written as $\theta_\omega^{\mathbb{R}} \wedge \theta_\omega^{\mathbb{C}}$, where

$$\theta_\omega^{\mathbb{R}} := dx_1 \wedge \cdots \wedge dx_{s_\omega}$$

and

$$\theta_\omega^{\mathbb{C}} := \left(\frac{\mathsf{i}}{2}\right)^{m_\omega} (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_{m_\omega} \wedge d\bar{z}_{m_\omega}).$$

Since $\mathsf{i} \rfloor \left(\frac{\mathsf{i}}{2} dz \wedge d\bar{z}\right) = (0,1) \rfloor (dx \wedge dy) = -dx$, we get

$$w \rfloor (\theta_\omega^{\mathbb{R}} \wedge \theta_\omega^{\mathbb{C}}) = (-1)^{s_\omega} \theta_\omega^{\mathbb{R}} \wedge (w \rfloor \theta_\omega^{\mathbb{C}}) = (-1)^{s_\omega} \theta_\omega^{\mathbb{R}} \wedge dx_{k+1}^\star \wedge \theta_{\mathsf{I}_k(\omega)}^{\mathbb{C}}$$

$$= (-1)^{2s_\omega + k} \theta_{\mathsf{I}_k(\omega)}^{\mathbb{R}} \wedge \theta_{\mathsf{I}_k(\omega)}^{\mathbb{C}} = (-1)^k \theta_{\mathsf{I}_k(\omega)}.$$

Similar considerations apply to the case $k = 0$ when $x_0^\star < x_1(t)$, or to the case $k = s_\omega$ when $x_{s_\omega+1}^\star > x_{s_\omega}(t)$.

This means that the part $\mathring{\mathsf{R}}_d^{\mathsf{I}_k(\omega)}$ of the boundary $\partial(\mathsf{R}_\omega)$, being approached via the path $P_{t,k} := P_t \subset \mathsf{R}_d^\omega$ as above, acquires an orientation that differs from its $\theta_{\mathsf{I}_k(\omega)}$-induced orientation by the factor $b_k = (-1)^k$. However, in general, this factor is not the incidence coefficient of R_d^ω with $\mathsf{R}_d^{\mathsf{I}_k(\omega)}$! We already mentioned that several sheets of R_d^ω can join along $\mathring{\mathsf{R}}_d^{\mathsf{I}_k(\omega)}$ implying that topologically the germ of R_d^ω along $\mathring{\mathsf{R}}_d^{\mathsf{I}_k(\omega)}$ is an open book. This fact is consistent with formula (2.3), where $\mathsf{I}_k(\omega)$ could be equal to $\mathsf{I}_l(\omega)$ for some $l \neq k$.

Thus we have shown that, (E^2, d^2)-term of the $\bar{H}_*$-homology spectral sequence, associated to the filtration $\{F_k(\bar{\mathcal{P}}_d^\Theta)\}_k$ of the space $\bar{\mathcal{P}}_d^\Theta$, coincides with the differential complex complex $\partial : \mathbb{Z}[\Theta] \to \mathbb{Z}[\Theta]$, defined by the formula (2.3). □

As a first application of Proposition 2.5 we will compute $\mathcal{P}_d^{\mathbf{c}\Theta} = \mathcal{P}_d \setminus \mathcal{P}_d^\Theta = S^d \setminus \bar{\mathcal{P}}_d^\Theta$. This computation extends part (1) in Theorem 2.7 of [SW] which deals with closed subposets Θ whose cover relations only involve merging operations.

Corollary 2.6. *Let $\Theta \subset \mathbf{\Omega}_{\langle d]}$ be a closed subposet. Then, by the Alexander duality, for any $j \in \{0, 1, \dots, d-1\}$,*

$$H^j(\mathcal{P}_d^{\mathbf{c}\Theta}; \mathbb{Z}) \approx H_{d-j-1}(\partial : \mathbb{Z}[\Theta] \to \mathbb{Z}[\Theta]) \tag{2.5}$$
$$:= \frac{\mathsf{ker}\{\partial : \mathbb{Z}[\Theta_{|\sim|'=j+1}] \to \mathbb{Z}[\Theta_{|\sim|'=j+2}]\}}{\mathsf{im}\{\partial : \mathbb{Z}[\Theta_{|\sim|'=j}] \to \mathbb{Z}[\Theta_{|\sim|'=j+1}]\}}.$$

Observe that our choice of the lower index $d - j - 1$ in the formulation of Corollary 2.6 for the homology of the differential complex $(\mathbb{Z}[\Theta], \partial)$ differs from the standard indexing used in the homological algebra.

Proof. The result follows by a combination of the Alexander duality

$$H^j(\mathcal{P}_d^{\mathbf{c}\Theta}; \mathbb{Z}) \approx \bar{H}_{d-j-1}(\bar{\mathcal{P}}_d^\Theta; \mathbb{Z})$$

with Proposition 2.5. In particular, with the collapse of the spectral sequence of the filtration $\{F_k^\bullet(\bar{\mathcal{P}}_d^{\mathbf{\Theta}})\}_k$ at the E^3-term, validates formula (2.5). □

Given two polynomials, $\mathsf{P}(t)$ and $\mathsf{Q}(t)$, we write "$\mathsf{P}(t) \succeq \mathsf{Q}(t)$" if the formal t-power series $\big(\mathsf{P}(t) - \mathsf{Q}(t)\big)(1+t)^{-1}$ is a polynomial and all its coefficients are nonnegative.

Corollary 2.7. *Let $\Theta \subset \mathbf{\Omega}_{\langle d]}$ be a closed subposet. Consider the t-polynomial*

$$\mathsf{P}_{(\mathbb{Z}[\Theta], \partial)}(t) := \sum_{j=0}^{d-1} \big|\Theta_{|\sim|'=j+1}\big|\, t^j.$$

Then:

- $\mathrm{rk}\{H^j(\mathcal{P}_d^{\mathbf{c}\Theta}; \mathbb{Z})\} \leq \big|\Theta_{|\sim|'=j+1}\big|$;
- *the Poincaré polynomial $\mathsf{P}_{\mathcal{P}_d^{\mathbf{c}\Theta}}(t)$ of the space $\mathcal{P}_d^{\mathbf{c}\Theta}$ satisfies the relation*
$$\mathsf{P}_{\mathcal{P}_d^{\mathbf{c}\Theta}}(t) \;\preceq\; \mathsf{P}_{(\mathbb{Z}[\Theta], \partial)}(t);$$
- *the Euler characteristic of $\mathcal{P}_d^{\mathbf{c}\Theta}$ is given by the formula*
$$\chi(\mathcal{P}_d^{\mathbf{c}\Theta}) = \pm \sum_{j=0}^{d-1} (-1)^j \big|\Theta_{|\sim|'=j+1}\big|.$$

Proof. The claim follows from a well-known observation that the Poincaré polynomial of a finite graded differential complex is greater than or equal to ("$\succeq$") the Poincaré polynomial of its graded homology, see e.g. [F, Lemma 1.3]. □

3. Crash course in discrete Morse theory

The proofs of a number of our (co)homological results are based on discrete Morse theory. For that reason we introduce its basics in a short section. We use and describe discrete Morse theory for algebraic complexes as developed in [Sk, JW]. This theory is an adaption of discrete Morse theory for cell complexes by Forman [Fo] to an algebraic setting.

Let R be a ring and

$$\mathcal{F} := \{\cdots \xrightarrow{d_{+1}} F_i \xrightarrow{d_i} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} 0\}$$

be a chain complex of free R-modules of finite ranks. Choose an R-basis B_i for F_i for all i. We build a directed graph $G = (B, E)$ whose vertices are labeled by the elements of the set $B = \bigcup_{i \geq 0} B_i$. The edge set E of G consists of the edges $[b \to b']$, where $b \in B_i$, $b' \in B_{i-1}$ and b' appears in the support of the differential $\partial_i b$ with a coefficient which is invertible in the coefficient ring. In particular, the coefficient must be nonzero.

Example 3.1. In all our applications we have the following situation. Let $\Theta \subseteq \mathbf{\Omega}$ be a closed subposet and as above, given a positive integer $d \geq 0$, set $\Theta_{\langle d]} := \Theta \cap \mathbf{\Omega}_{\langle d]}$. Let $\mathcal{F}$ be the chain complex of $\mathbb{Z}$-modules F_i, freely generated by $\omega \in \Theta$ with $|\omega|' = i$. The differential acting on $\mathcal{F}$ will be the differential ∂ of $\mathbb{Z}[\Theta_{\langle d]}]$ from Proposition 2.5. (Additionally, by Proposition 2.5, we know that $\widetilde{H}_i(\bar{\mathcal{P}}_d^{\Theta}; \mathbb{Z}) \cong H_i(\mathcal{F})$).

We define the graph $G = (B, E)$ corresponding to this complex as follows. It vertex set B coincides with $\Theta \cap \mathbf{\Omega}_{\langle d]}$ where $B_i = \{\omega \in \Theta_{\langle d]} \mid |\omega|' = i\}$. The edge set E consists of edges $[\omega \to \omega']$ corresponding to the cover relations $\omega \succ \omega'$ in $\Theta_{\langle d]}$ for which the coefficient of ω in $\partial\omega'$ is ± 1. Note that if $\omega \succ \omega'$ is a cover relation, then the coefficient of ω in $\partial\omega'$ can only be zero if ω arises from ω' by insertion of a 2 in a block of an odd number of 2's. In all other cases it is ± 1.

To move further, we need some graph theoretic notation.

Definition 3.2. Given a graph $G = (B, E)$, we say that $M \subseteq E$ is a *matching*, if no vertex $b \in B$ appears in two different edges belonging to M.

Given the graph $G = (B, E)$, we say that $M \subseteq E$ is an *acyclic matching* if

- M is a matching;
- the graph
$$G^M = \big(B,\ (E \setminus M) \cup \{[b' \to b] \bigm| [b \to b'] \in M\}\big)$$
where the directions of the edges from M are reversed, contains no directed cycles.

Given an acyclic matching M, we call $b \in B$ *critical* with respect to M, if b is not contained in an edge from M.

The following remark is well known.

Remark 3.3. Let $G = (B, E)$ be the directed graph associated to a chain complex of free R-modules with basis B. If $M \subseteq E$ is a matching such that G^M contains a directed cycle, then for each cycle in G^M, there is an index i such that the cycle contains only vertices from B_i and B_{i+1}.

Proof. On a directed cycle there cannot be two consecutive edges that increase the homological dimensions since both edges would have to come from the matching M. Thus an edge increasing the homological dimension by 1 is always followed by an edge decreasing the homological dimension by 1. It follows that if i is the maximal homological dimension of a vertex $b \in B$ appearing in a cycle then there cannot be a vertex $b' \in B$ in the cycle of homological dimension $< i-1$. The assertion follows. $\square$

Finally we come to the result that makes discrete Morse theory useful for our purposes (see for example [JW, Theorem 2.2]).

Proposition 3.4. *Let $\mathcal{F} := \big\{ \cdots \to F_i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_1} F_0 \to 0 \big\}$ be a chain complex of finitely generated free R-modules. Let $G = (B, E)$ be the directed graph associated to $\mathcal{F}$ after choosing bases B_i for F_i and setting $B = \bigcup_i B_i$. For an acyclic matching M on G, there exists a chain complex*

$$\mathcal{F}^M := \big\{ \cdots \to F_i^M \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_1} F_0^M \to 0 \big\}$$

of free R-modules, where F_i^M is a free R-module, generated by the set

$$B_i' = \{ b \in B_i \mid b \text{ critical with respect to } M \}$$

such that $H_i(\mathcal{F}) \cong H_i(\mathcal{F}^M)$. If for any i and any $b \in B_i'$, there are no directed paths in G^M to an element of B_{i-1}' then for all i, we have that $H_i(\mathcal{F})$ is the free R-module generated by B_i'.

4. Homotopy types of $(d-k)$-skeleta of $\mathcal{P}_d$

Observe that for $\Theta = \mathbf{\Omega}_{\langle d]}$, the one-point compactification $\bar{\mathcal{P}}_d^\Theta = \bar{\mathcal{P}}_d$ is a d-dimensional sphere. Thus, by Corollary 2.6, all the homology of the complex $\partial : \mathbb{Z}[\mathbf{\Omega}_{\langle d]}] \to \mathbb{Z}[\mathbf{\Omega}_{\langle d]}]$ is trivial, except for the top dimension d, in which it equals $\mathbb{Z}$. In this section we show that this observation admits a generalization to the situation when Θ is the $(d-k)$-skeleton of our celluation of the space of real polynomials of degree d, i.e. $\Theta = \mathbf{\Omega}_{|\sim|' \geq k}$, $k \geq 1$.

We will actually consider a more general situation. For that, let $\mathbf{\Omega}^{(q)}_{|\sim|' \geq k}$ be the set of all $\omega \in \mathbf{\Omega}$ with $|\omega|' \geq k$ and $|\omega| \geq q$. Note that, for $\Theta = \mathbf{\Omega}^{(0)}_{|\sim|' \geq k} = \mathbf{\Omega}_{|\sim|' \geq k}$ we have that $\bar{\mathcal{P}}_d^\Theta$ is the full $(d-k)$-skeleton of $\mathcal{P}_d$.

To formulate our next result, we need the following invariant. For a composition $\omega = (\omega_1, \ldots, \omega_\ell)$, we set $t(\omega) := i$ if there is an $1 \leq i < \ell$ such that $\omega_1 = \cdots = \omega_{i-1} = 1, \omega_i = 2$ and $t(\omega) = +\infty$ otherwise.

Now, for $k \geq 1$ and $d \geq q \geq 0$, $q \equiv d \mod 2$, let us introduce the nonnegative integer $A(k, d, q)$ by

$$A(k,d,q) := \left| \left\{ \omega = (\omega_1, \ldots, \omega_\ell) \;\middle|\; \begin{array}{cc} |\omega|' = k, & |\omega| \equiv d \mod 2, \\ q \leq |\omega| \leq d, & t(\omega) = +\infty \end{array} \right\} \right|.$$

Proposition 4.1. *Fix* $1 \leq k < d$ *and* $q \geq 0$ *such that* $q \equiv d \mod 2$. *Then the one-point compactification* $\bar{\mathcal{P}}_d^{\mathbf{\Omega}^{(q)}_{|\sim|' \geq k}}$ *has the homotopy type of a wedge of* $(d-k)$*-dimensional spheres. The number of spheres in the wedge equals* $A(k,d,q)$*.*

The proof of the proposition is based on discrete Morse theory. (Consult § 3 for definitions and the basic setup). Since this is our first application of discrete Morse theory, we will carefully guide the reader and provide all the details.

Proof. Let $G = (B, E)$ be the directed graph associated to $B = \mathbf{\Omega}^{(q)}_{|\sim|' \geq k} \cap \mathbf{\Omega}_{\langle d]}$ in Example 3.1. Note that $\mathbf{\Omega}^{(q)}_{|\sim|' \geq k} \cap \mathbf{\Omega}_{\langle d]}$ is a closed subposet of $\mathbf{\Omega}_{\langle d]}$. Next we define an acyclic matching M on G. In M we assemble the edges $[\gamma \to \omega]$, where $\omega = (\omega_1, \ldots, \omega_\ell)$, $i = t(\gamma) \leq \ell < \infty$ and

$$\gamma = (1 = \omega_1, 1 = \ldots, 1 = \omega_{i-1}, 2, \omega_i - 2, \omega_{i+1}, \ldots, \omega_\ell).$$

In particular, we must have $\omega_i > 2$, $|\gamma|' = |\omega|' + 1 \geq k$, $|\gamma| = |\omega|$ and $\gamma \succ \omega$, Thus the coefficient of ω in the differential of γ in $\partial\omega$ is ± 1 (and, in particular, is non-vanishing). Thus we have $[\gamma \to \omega]$ in E.

We claim that M is an acyclic matching on the graph G. Indeed,

- (matching) This is obvious from the definition.
- (acyclic) Clearly, there can be an edge $[\gamma \to \omega]$ in M only if $t(\omega) = \infty$.

  We prove acyclicity by contradiction. Assume that we have a directed cycle in G^M. By Remark 3.3 it follows that along the directed cycles the edge from $E \setminus M$ and reversed edges from M alternate. Hence the cycle must contain an edge $[\omega \to \gamma]$, where $[\gamma \to \omega] \in M$. By definition of M we must have $t(\gamma) < t(\omega) = \infty$.

  Now consider a covering $\omega' \succ \gamma$ where $\omega' \neq \omega$. We need to treat the following cases:
  - if ω' arises from γ by inserting a 2, then $t(\omega') \leq t(\gamma) < \infty$. But then we cannot continue the directed cycle at ω' since there is no edge $[\gamma' \to \omega']$ in M.
  - if ω' arises from γ by merging two blocks, then we have to distinguish two subcases. If the 2 at position $t(\gamma)$ is merged with the 1 to its left then $t(\omega') = \infty$. But then the unique γ' with $[\gamma' \to \omega'] \in M$ arise from ω' by splitting the just merged 3 into a 2 and a 1 to its right. Thus $t(\gamma') = t(\gamma) - 1$. In all other cases either $\omega = \omega'$ or $t(\omega') < \infty$ and there is no edge $[\gamma' \to \omega']$ in M.

  Thus we can arrange the composition visited on a directed cycle in such a way that the t-values alternate between $+\infty$ and a strictly decreasing sequence of numbers. This implies that the cycle cannot be closed, a contradiction implying that the matching is acyclic.

Next we determine the critical cells of M. Assume that $\omega = (\omega_1, \ldots, \omega_\ell)$ is not matched by M. There are two possible subcases to consider:

- $i = t(\omega) < \infty$

Then we must have $\omega = (1,\ldots,1,2,\omega_{i+1},\ldots,\omega_\ell)$ for some $\ell > i$ and ω is matched with $\omega = (1,\ldots,1,2+\omega_{i+1},\omega_{i+2},\ldots,\omega_\ell)$. Thus there are no critical cells ω with $t(\omega) < \infty$.

- $t(\omega) = \infty$

  In this case, either $\omega = (1,\ldots,1)$, $(1,\ldots,1,2)$ or $\omega = (1,\ldots,1,w,\ldots)$ for some $w > 2$. In the first situation, we must have $k = 0$ which we have excluded. In the second situation, we have $k = 1$. In the third situation ω is matched with $(1,\ldots,1,2,w-2,\ldots)$ unless $|\omega|' = k$. By $w > 2$ it follows that this can only be the case when $k > 1$.

Considering the cases, it follows that the number of unmatched compositions is $A(d,k,q)$. In addition, all unmatched compositions have reduced norm equal to k. By Proposition 3.4 all of them contribute one copy of $\mathbb{Z}$ to homology in dimension $d-k$. The homological isomorphism in Proposition 4.1 now follows.

Using the fact that all the critical cells have reduced norm equal to k, it follows that they are maximal cells in the CW-complex $\bar{\mathcal{P}}_d^{\mathbf{\Omega}_{|\sim|'\geq k}}$. Hence we can remove them and retain the CW-structure on the remaining space X. By the above arguments the CW-complex X has an acyclic matching with no critical cells. Hence by Proposition 3.4 it is acyclic. It follows from the long exact sequence of the pair $(\bar{\mathcal{P}}_d^{\mathbf{\Omega}_{|\sim|'\geq k}}, X)$ that the projection $p : \bar{\mathcal{P}}_d^{\mathbf{\Omega}_{|\sim|'\geq k}} \to \bar{\mathcal{P}}_d^{\mathbf{\Omega}_{|\sim|'\geq k}}/X$ induces an isomorphism in homology.

For the maximal possible $k = d-1$, the poset $\mathbf{\Omega}_{|\sim|'\geq d-1}$ consists of a single critical cell $\omega = (d)$ and the stratum $\bar{\mathcal{P}}_d^{\mathbf{\Omega}_{|\sim|'\geq d-1}}$ is a circle S^1. So in this case, the map p reduces to a homeomorphism of two circles.

For $k \leq d-2$, it follows from Theorem 2.2. of [KSW1] that $\bar{\mathcal{P}}_d^{\mathbf{\Omega}_{|\sim|'\geq d-1}}$ is simply connected. By the Hurewicz theorem, p is a homotopy equivalence, which concludes the proof. $\square$

Note that for $q = d$ we have that $\bar{\mathcal{P}}_d^{\mathbf{\Omega}^{(q)}_{|\sim|'\geq k}}$ is the full $(d-k)$-skeleton of $\bar{\mathcal{P}}_d$.

Corollary 4.2. *Let $1 \leq k < d$, $0 \leq q \leq d$, $q = d \mod 2$, and set $\Theta = \mathbf{\Omega}^{(q)}_{|\sim|'\geq k}$. Then*

$$\bar{H}^j(\mathcal{P}_d^{\mathbf{c}\,\Theta};\mathbb{Z}) \;\cong\; \begin{cases} 0 & \textit{for } j \neq k-1, \\ \mathbb{Z}^{A(k,d,q)} & \textit{for } j = k-1. \end{cases}$$

Next we consider a subposet of $\mathbf{\Omega}^{(0)}_{|\sim|'\geq k}$. Let $\mathbf{\Omega}^{\neq 1}_{|\sim|'\geq k}$ be the set of all $\omega = (\omega_1,\ldots,\omega_r) \in \mathbf{\Omega}$ such that $|\omega|' \geq k$ and $\omega_i \neq 1$ for $i = 1,\ldots,r$. This poset will become crucial in the study of stabilizazion results in § 6. Of similar use as $t(\omega)$ for the analysis of $\mathbf{\Omega}^{(q)}_{|\sim|'\geq k}$ is the following invariant.

For a composition $\omega = (\omega_1,\ldots,\omega_\ell)$, we set $s(\omega) := i$ if there is an $1 \leq i < \ell$ such that $\omega_1 = \cdots = \omega_{i-1} = 3, \omega_i = 2$ and $t(\omega) = +\infty$ otherwise.

Proposition 4.3. *Fix $1 \leq k \leq d$. Then the one-point compactification $\bar{\mathcal{P}}_d^{\mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k}}$ has the homotopy type of a wedge of $(d-k)$-spheres. The spheres are indexed by the compositions $\omega \in \mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k}$ such that $|\omega|' = k$ and of the following types:*

(i) $\omega = (3, \ldots, 3, 2)$,

(ii) $\omega = (3, \ldots, 3, w, \ldots)$ *for some* $w > 3$.

Proof. Except for some technicalities, the proof proceeds along the lines of the proof of Proposition 4.1 where $s(\tilde{)}$ assumes the role of $t(\tilde{)}$ and the role of parts 1 is now played by parts 3. We again use discrete Morse theory. Let $G = (B, E)$ be the directed graph associated to $B = \mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k} \cap \mathbf{\Omega}_{\langle d]}$. Again $\mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k} \cap \mathbf{\Omega}_{\langle d]}$ is a closed subposet of $\mathbf{\Omega}_{\langle d]}$. Next we define an acyclic matching M on G. In M we assemble the edges $[\gamma \to \omega]$ of the following two types:

(Type I)

$$\omega = (\omega_1, \ldots, \omega_\ell) \in \mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k} \cap \mathbf{\Omega}_{\langle d]} \text{ and } \gamma = (\omega_1, \ldots, \omega_{i-1}, 2, \omega_i - 2, \omega_{i+1}, \ldots, \omega_\ell),$$

where $\omega_1, \ldots, \omega_{i-1} = 3$, $\omega_i > 3$, and $|\gamma|' = |\omega|' - 1 \geq k$.

(Type II) $\omega = (\underbrace{3, \ldots, 3}_{r}, 2)$ and $\gamma = (\underbrace{3, \ldots, 3}_{r})$.

Note that for both types, we have $\gamma \prec \omega$, $|\gamma|' = |\omega|' + 1 \geq k$ and the coefficient of ω in $\partial\gamma$ is ± 1 (and is non-vanishing). In particular, if $\gamma \in \mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k}$ then $[\gamma \to \omega]$ in E. In type I, we also have $|\gamma| = |\omega|$.

We again claim that M is an acyclic matching on the graph G. Indeed,

- (matching) This is obvious from the definition.
- (acyclic) Clearly, there is an edge $[\gamma \to \omega]$ in M then $s(\omega) = \infty$.

  We prove acyclicity by contradiction. Assume that we have a directed cycle in G^M. Any directed cycle must contain an edge $[\omega \to \gamma]$, where $[\gamma \to \omega] \in M$. If $[\gamma \to \omega]$ is of type II then the directed cycle must involve an edge $[\gamma' \to \omega] \in E$ for some $\gamma' \neq \gamma$. But then ω arises from γ' be merging or insertion of a 2. If it arises by inserting a 2 then we must have $\gamma' = \gamma$ and if it arises from a merging then one of the merged parts has to be a 1, which is excluded. It follows that a directed cycle cannot involve $[\omega \to \gamma]$ for some $[\gamma \to \omega] \in M$ of type II.

  Thus the directed cycle only involves edges from M of type I. By definition of type I we must have $s(\gamma) < s(\omega) = \infty$.

  Now consider a covering $\gamma \succ \omega'$ where $\omega' \neq \omega$. We need to treat the following cases:

  – if ω' arises from γ by inserting a 2, then $s(\omega') \leq s(\gamma) < \infty$. But then we cannot continue the directed cycle at ω' since there is no edge $[\gamma' \to \omega']$ in M.

  – if ω' arises from γ by merging two blocks, then we have to distinguish two subcases. If the 2 at position $t(\gamma)$ is merged with a 3 to its left then $t(\omega') = \infty$. But then the unique γ' with $[\gamma' \to \omega'] \in M$ arises from ω' by splitting the just merged 3 into a 2 and a 1 to its right. Thus $s(\gamma') = s(\gamma) - 1$. In all other cases either $\omega = \omega'$ or $s(\omega') < \infty$ and there is no edge $[\gamma' \to \omega']$ in M.

Again it follows that we can arrange the composition visited on a directed cycle in such a way that the s-values alternate between $+\infty$ and a strictly decreasing sequence of numbers. This implies that the cycle cannot be closed, a contradiction.

Next we determine the critical cells. Assume that $\omega = (\omega_1, \dots, \omega_\ell)$ is not matched by M.

- $s(\omega) < \infty$ Then we must have $\omega = (3, \dots, 3, 2, \omega_{i+1}, \dots, \omega_\ell)$ for some $\ell > i$ and ω is matched with $(3, \dots, 3, 2 + \omega_{i+1}, \omega_{i+2}, \dots, \omega_\ell)$. Thus there are no critical cells ω with $s(\omega) < \infty$.
- $s(\omega) = \infty$
  In this case, either $\omega = (3, \dots, 3)$, $(3, \dots, 3, 2)$ or $\omega = (3, \dots, 3, w, \dots)$ for some $w > 3$. $(\underbrace{3, \dots, 3}_{r})$ is matched with $(\underbrace{3, \dots, 3}_{r}, 2)$ and $(3, \dots, 3, 2)$ is unmatched if and only if $|(3, \dots, 3, 2)| = k$. In the third situation ω is matched with $(3, \dots, 3, 2, w - 2, \dots)$ unless $|\omega|' = k$.

Thus unmatched compositions are of codimension k and by Proposition 3.4 each contributes one copy of $\mathbb{Z}$ to homology in the respective dimension. The homological isomorphism in Proposition 4.3 now follows.

All critical cells are maximal in the CW-complex $\bar{\mathcal{P}}_d^{\mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k}}$. Hence we can remove them and retain the CW-structure on the remaining space X. By the arguments above the CW-complex X has an acyclic matching with no critical cells. Hence by Proposition 3.4 it is acyclic. It follows from the long exact sequence of the pair $(\bar{\mathcal{P}}_d^{\mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k}}, X)$ that the projection $p : \bar{\mathcal{P}}_d^{\mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k}} \to \bar{\mathcal{P}}_d^{\mathbf{\Omega}_{|\sim|' \geq k}}/X$ induces an isomorphism in homology.

For the maximal possible $k = d - 1$, the poset $\mathbf{\Omega}^{\neq 1}_{|\sim|' \geq d-1}$ consists of a single critical cell $\omega = (d)$, unless $d = 1$ and the stratum $\bar{\mathcal{P}}_d^{\mathbf{\Omega}_{|\sim|' \geq d-1}}$ is a circle S^1. So in this case, the map p reduces to a homeomorphism of two circles.

For $k \leq d - 2$, we know that $\bar{\mathcal{P}}_d^{\mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k}}$ is simply connected by [KSW1, Theorem 2.2]. By the Hurewicz theorem, p is a homotopy equivalence, which concludes the proof. □

The following corollary is an immediate consequence of Proposition 4.3.

Corollary 4.4. *Fix $1 \leq k \leq d$. Then the one-point compactification $\bar{H}_i(\bar{\mathcal{P}}_d^{\mathbf{\Omega}^{\neq 1}_{|\sim|' \geq k}}, \mathbb{Z}) = 0$ for $i < k$.*

5. Homology and the homotopy type of $\bar{\mathcal{P}}_d^{\langle \omega \rangle}$ for a single composition ω

The following detailed guess is based on our extensive computer experiments for polynomials of degrees up to 14.

Conjecture 5.1. (i) *For any single pattern ω with an entry exceeding 2 and any $d \geq |\omega|$ of the same parity as $|\omega|$, the one-point compactification $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is contractible or homotopy equivalent to a sphere of a certain dimension.*

(ii) *for a given d, all the strata which are non-contractible correspond to compositions ω whose entries can only be 1, 2 or 3 with the only exception. Namely, for $d = 8$, the stratum $\bar{\mathcal{P}}_d^{\langle(4)\rangle}$ is homotopy equivalent to S^3;*

(iii) *for a given d, all the strata of codimension at least 1 which are non-contractible satisfy the condition $|w| = d - 4$ with one exception. Namely, if d is even then the stratum $\bar{\mathcal{P}}_d^{\langle(2)\rangle}$ is homotopy equivalent (and in fact homemorphic) to S^{d-1};*

(iv) *for a given d, among the top-dimensional strata, i.e., $\dim \bar{\mathcal{P}}_d^{\langle\omega\rangle} = d$ in which case ω contains only 1's, the strata with the minimal and maximal possible number of real roots are contractible while the remaning $\left[\frac{d-2}{2}\right]$ top-dimensional strata are homotopy equivalent to S^{d-1}.*

(v) *for a given d, if a stratum $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is non-contractible then neither the first nor the last entry of ω can be a 2 with the same exception as above. Namely, if d is even then the stratum $\bar{\mathcal{P}}_d^{\langle(2)\rangle}$ is homotopy equivalent to S^{d-1}. In particular, for a given d, among the strata of codimension 1, i.e. ω contains a single 2 and some number of 1's which also satisfy the necessary condition $|w| = d - 4$, see* (iii) *then for any ω of the latter form in which the only included 2 is not in the first nor in the last positions the stratum $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is homeomorphic to S^{d-5}. In the remaining two cases, i.e. when ω either starts or ends with 2, $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is contractible. (Notice the special case $\bar{\mathcal{P}}_d^{\langle(2)\rangle}$ which appears for $d = 6$.)*

(vi) *for a given d, among the strata of codimension 2 satisfying the necessary condition $|w| = d - 4$, i.e. ω either contains a single 3 and $(d-7)$ 1's or two 2's and $(d-8)$ 1's , then* a) *every stratum with a single 3 is homotopy equivalent to S^{d-4}.*

(vii) *More to come...*

Remark 5.2. Recall that by Theorem 2.2 of [KSW1] for any closed poset $\Theta \subseteq \Omega_{\langle d]}$, the fundamental group $\pi_1(\bar{\mathcal{P}}_d^{\Theta})$ is trivial unless $\Theta = \{(d)\}$ in which case $\bar{\mathcal{P}}_d^{\langle(d)\rangle}) \simeq S^1$. Therefore in all cases in which we can prove the triviality of (the reduced) homology $\bar{H}_*(\bar{\mathcal{P}}_d^{\Theta}, \mathbb{Z})$ we also obtain that the 1-point compactification $\bar{\mathcal{P}}_d^{\Theta}$ itself is contractible. Additionally, in all cases when we can show that homology with integer coefficients live in single dimension bigger than one and the fundamental group is trivial we get the following. This is probably folklore, and a direct proof could be given by using the classical theorems of Hurewicz and JHC Whitehead.

Instead of trying to do that, let us mention a much more general result, due to David Anick (1984). The second author heard of this result from Matthias Kreck in Nov 1999 while on a RiP at MFO with Stefan Papadima (Kreck was director of MFO at the time), and it appears in the paper we wrote then (https://doi.org/10.1006/aima.2001.2023), see [PaSu] in Remark 2.14. Namely:

Every simply connected, finite-type CW-complex with no torsion in integral homology admits a minimal cell structure (that is, a cell structure with the exact number of cells predicted by the Betti numbers).

In the situation from your question, assuming K is of finite type (ie, has finitely many cells in each dimension), and $H_*(K,\mathbb{Z}) \cong H_*(S^k,\mathbb{Z})$, then Anick's result implies that K is homotopy equivalent to a cell complex with a single 0-cell and a single k-cell, and that is S^k.

5.1. **Spectral sequence of the double complex $(\mathbb{Z}[\Theta], \partial_{\mathsf{I}}, \partial_{\mathsf{M}})$ and its application to calculation of $H_*(\bar{\mathcal{P}}_d^{\langle\omega\rangle},\mathbb{Z})$.** By Theorem 2.2 and Proposition 2.5 for any closed poset $\Theta \subset \Omega_{\langle d]}$, the combinatorial differential complex $(\mathbb{Z}[\Theta], \partial)$ where $\partial = \partial_{\mathsf{M}} + \partial_{\mathsf{I}}$ calculates $\bar{H}_*(\bar{\mathcal{P}}_d^{\Theta},\mathbb{Z})$. Since by Lemma 2.3, ∂_{M} and ∂_{I} are anti-commuting differentials, one gets a double complex $(\mathbb{Z}[\Theta], \partial_{\mathsf{I}}, \partial_{\mathsf{M}})$ and one can apply one or both of the spectral sequences of this double complex to calculate the homology of $(\mathbb{Z}[\Theta], \partial)$ and therefore $\bar{H}_*(\bar{\mathcal{P}}_d^{\Theta},\mathbb{Z})$.

As we mentioned above, for any composition $\omega \in \Omega_{\langle d]}$, the differential ∂_{M} preserves $\kappa(\omega)$ and decreases $\nu(\omega)$ by 1 while ∂_{I} preserves $\nu(\omega)$ and decreases $\kappa(\omega)$ by 1 where $\kappa(\omega) := \frac{d-|\omega|}{2}$, $\nu(\omega) := \frac{d-|\omega|}{2} + s_\omega$ and s_ω stands for the cardinality of the support of a composition ω. The sum $\kappa(\omega)+\nu(\omega)$ equals the dimension of the stratum $\mathcal{P}_d^{\omega} \subset \mathcal{P}_d$.

For any composition $\omega \in \Omega_{\langle d]}$, denote by $\langle\omega\rangle_{\mathsf{M}} \subset \Omega_{\langle d]}$ the set of compositions obtained by applying to ω the merge operation one or several times. We include ω in $\langle\omega\rangle_{\mathsf{M}}$ as well. For example, for $\omega = (1,3,2)$ and $d = 8$, $\langle\omega\rangle_{\mathsf{M}}$ consists of 4 compositions $\langle\omega\rangle_{\mathsf{M}} = ((1,3,2);(4,2);((1,5);(6))$. Obviously, for any $d \geq |\omega|$ and $d \equiv |\omega| \mod 2$, $\langle\omega\rangle_{\mathsf{M}}$ is independent of d. Alternatively, one can describe $\langle\omega\rangle_{\mathsf{M}}$ as the lower interval of ω in $\Omega_{\langle d]}$ in the partial order induced by merging. The next statement is trivial.

Lemma 5.3. *In the above notation, ∂_{M} acts on $\langle\omega\rangle_{\mathsf{M}}$ and the complex $(\langle\omega\rangle_{\mathsf{M}}, \partial_{\mathsf{M}})$ is acyclic. In fact, this complex is isomorphic to the standard boundary complex calculating the reduced homology of a simplex of dimension $s_\omega - 2$.*

Similarly, for any composition $\omega \in \Omega_{\langle d]}$, denote by $\langle\omega\rangle_{\mathsf{I},d} \subset \Omega_{\langle d]}$ the set of compositions obtained by applying to ω the insertion operation one or several times. Observe that we only allow to insert the number of 2's such that the norm of the obtained composition does not exceed d. As above, we include ω in $\langle\omega\rangle_{\mathsf{I},d}$. For example, for $\omega = (1,3,2)$ and $d = 8$, then $\langle\omega\rangle_{\mathsf{I},8}$ consists of 4 compositions $\langle\omega\rangle_{\mathsf{M}} = ((1,3,2);(2,1,3,2);((1,2,3,2);(1,3,2,2))$. If $\omega = (1,3,2)$ and $d = 10$, then $\langle\omega\rangle_{\mathsf{I},10}$ consists of 10 compositions $\langle\omega\rangle_{\mathsf{I},10} = \langle\omega\rangle_{\mathsf{I},8} \cup ((2,2,1,3,2);(2,1,2,3,2);(2,1,3,2,2);(1,2,2,3,2);(1,2,3,2,2),(1,3,2,2,2))$. If $\omega = (1,3,2)$ and $d = 12$, then $\langle\omega\rangle_{\mathsf{I},12}$ consists of 20 compositions $\langle\omega\rangle_{\mathsf{I},12} = \langle\omega\rangle_{\mathsf{I},10} \cup ((2,2,2,1,3,2);$ $(2,2,1,2,3,2)$; $(2,2,1,3,2,2)$; $(2,1,2,2,3,2)$; $(2,1,2,3,2,2)$; $(2,1,3,2,2,2)$; $(1,2,2,2,3,2)$; $(1,2,2,3,2,2)$; $(1,2,3,2,2,2)$; $(1,3,2,2,2,2))$.

Proposition 5.4. *In the above notation, ∂_{I} acts on $\langle\omega\rangle_{\mathsf{I},d}$ and the complex $(\langle\omega\rangle_{\mathsf{I}}, \partial_{\mathsf{I}})$ has the following homology. If $\partial(\omega) \neq 0$, then the complex is torsion-free and all homology is concentrated in the bottom dimension. If $\partial(\omega) = 0$ then homology is torsion-free and has a copy of $\mathbb{Z}$ in the top dimension and the rest in the bottom dimension. CHECK!!!*

Definition 5.5. Given a composition $\omega = (\omega_1, \omega_2, \dots, \omega_s)$ we associate to it its *s*ymbol $\mathfrak{S}(\omega) = (x_1, x_1, \dots, x_s)$ where each x_j attains two values 2 and " ". Example, $(2,3,1,2,2,5)$ corresponds $(2, \quad , \quad , 2, 2, \quad)$ or, equivalently, $(1, \quad , \quad , 2, \quad)$ where the numbers count the lengths of the sequences of consecutive 2's.

Remark 5.6. Given a composition ω, $\partial_{|}(\omega) = 0$ if and only if ω starts and ends with odd numbers of 2's and any two consecutive non-two entries are separated by an odd number of 2's. In other words, the second type of symbol starts and ends with an even number and each second entry of it is an even number. A composition ω' is not in the image of $\partial_{|}$ (CLARIFY) if and only if it contains non-increasable sequences of 2's of even length.

We start with the following claim.

Lemma 5.7. *In the above notation, if $\partial_{|}(\omega) \neq 0$ then $(\langle\omega\rangle_{|}, \partial_{|})$ is acyclic except for the homology in the bottom dimension which equals $\mathbb{Z}^{|\chi|}$ where χ is the Euler characteristic of $\langle\omega\rangle_{|}$, see below.*

Proof. We will use the technique of discrete Morse theory and will find a matching of all compositions of neighbouring levels except for some elements of the bottom level. We will construct our matching by using the simple-minded procedure starting from with the unique composition ω lying on the top level. Assume that we have constructed our matching for all compositions of level $i-1$. Now, for each composition ω' of the level i which were not previously matched with some composition from the level $i-1$ we choose the composition from the level $i+1$ which is obtained by insertion of 2 in the leftmost possible position of ω' with an even number of 2's, i.e. the corresponding arrow does not vanish. See example below. We have to show that this procedure creates a matching for all compositions of all levels from the top to $j-1$ and that the "inversion of the directions of the arrows does not create cycles".

Base of induction. Indeed, assume that $\partial_{|}(\omega) \neq 0$, i.e. ω contains at least one group of 2's of even length (including possibly an empty group). Then we match ω to the composition obtained by inserting 2 in the leftmost possible position. Let us first discuss the case when ω contains no 2's. Then in the above procedure each composition which is not on the bottom level and which starts with an even group of 2's (possibly empty) will be the starting vertex of an edge of the above matching and each composition which starts with an odd group of 2's will be the ending vertex of an edge of the above matching.

Next consider the case when ω contains some even group of 2's (possibly empty). If ω starts with an even group of 2's (possibly empty) then exactly the same argument as above applies. Analogously, we can inserts 2's in this group and everything works out. Explain this better...

Finally, consider the case when ω contains only odd groups of 2's. BLA

□

Remark 5.8. If $s_\omega = s$ and $\frac{d-|\omega|}{2} = j$ then
$$|\chi| = \binom{s+j}{j} - \binom{s+j-1}{j-1} + \binom{s+j-2}{j-2} + \dots + (-1)^{j+1}.$$

5.2. **Proving Conjecture 5.1.**

Proposition 5.9. *The following statements are true.*
(1) *If d is even and $\omega = (\emptyset)$ then $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is contractible.*
(2) *If $d > 2$ is even and $\omega = (2)$ then $\bar{\mathcal{P}}_d^{\langle\omega\rangle} \simeq S^{d-1}$ (resp. S^{d-2} in the reduced setting).*
(3) *If d is even and $\omega = \underbrace{(1,1,\dots,1)}_{\ell \text{ ones}}$ where $1 < \ell < d$, $\ell \equiv d \bmod 2$ then $\bar{\mathcal{P}}_d^{\langle\omega\rangle} \simeq S^{d-1}$ (resp. S^{d-2} in the reduced setting).*
(4) *If d is odd and $\omega = \underbrace{(1,1,\dots,1)}_{\ell \text{ ones}}$ where $1 < \ell < d$, $\ell \equiv d \bmod 2$ then $\bar{\mathcal{P}}_d^{\langle\omega\rangle} \simeq S^{d-1}$ (resp. S^{d-2} in the reduced setting).*

Proof. To prove (1) notice that for d even, $\mathring{\mathsf{R}}_d^{\emptyset}$ coincides with the set of monic positive polynomials of even degree d. This set is open and convex in $\mathcal{P}_d$. Its closure $\mathcal{P}_d^{\langle\emptyset\rangle}$ coincides with the convex closed set of monic non-negative polynomials which is homeomorphic to a closed halfspace in $\mathcal{P}_d$. Thus its one-point compactification $\bar{\mathcal{P}}_d^{\langle\emptyset\rangle}$ is contractible.

To settle (2) notice that for d even, $\mathcal{P}_d^{\langle(2)\rangle}$ coincides with the set of non-negative polynomials with at least one real root, i.e. it is the boundary of $\mathcal{P}_d^{\langle\emptyset\rangle}$. Thus $\mathcal{P}_d^{\langle(2)\rangle}$ is homeomorphic to a hyperplane in $\mathcal{P}_d$ implying that $\bar{\mathcal{P}}_d^{\langle\emptyset\rangle}$ is homeomorphic to S^{d-1}.

To settle (3) notice that for d even, $\mathcal{P}_d^{\langle\omega\rangle}$ with $\omega = \underbrace{(1,1,\dots,1)}_{\ell \text{ ones}}$ coincides with the set of all monic polynomials of degree d with at least ℓ real zeros counting multiplicity of which at most ℓ are simple....

Analogously in case (4) for d odd, $\mathcal{P}_d^{\langle\omega\rangle}$ with $\omega = \underbrace{(1,1,\dots,1)}_{\ell \text{ ones}}$ coincides with the set of all monic polynomials of degree d with at least ℓ real zeros counting multiplicity of which at most ℓ are simple.... □

5.3. **Volkmar's results on $\bar{\mathcal{P}}_d^{\Theta}$.** The next result provides a substantial support for the validity of Conjecture 5.1.

Proposition 5.10. *Let ω be a composition and $d \geq |\omega|$ be a positive integer of the same parity as $|\omega|$. If ω contains at least two distinct entries and no entry ≤ 2, then the one-point compactification $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is contractible. In particular, $\bar{H}^*(\mathcal{P}_d^{\mathbf{c}\langle\omega\rangle};\mathbb{Z}) = 0$. If $\omega = (\ell)$ for some $2 < \ell \leq d$, then $\bar{\mathcal{P}}_d^{\langle\omega\rangle} \cong S^{d-\ell+1}$.*

Proof. If $\omega = (d)$ for some $d \geq 2$, then $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ consists of a single open 1-cell $\mathring{R}_d^{\omega}$ and a 0-cell at infinity. Obviously, $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is homeomorphic to S^1.

Now assume that ω contains at least two entries and no entry ≤ 2. We will show in these cases that the reduced homology of $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is trivial. By Theorem 2.2. of [KSW1] we know that $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is simply connected in this situation. If we can show that the reduced homology is trivial, then together with the latter fact this implies that $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ is contractible. The assertion about $\bar{H}^*(\mathcal{P}_d^{\mathbf{c}\langle\omega\rangle};\mathbb{Z})$ follows by the Alexander duality.

To verify homological triviality of $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ we need to consider several cases. We do so by exhibiting an acyclic matching on $\langle\omega\rangle_d$ which covers all its elements.

If $d = |\omega|$ then $\langle\omega\rangle_d$, ordered by $\preceq$, is isomorphic to a Boolean lattice of subsets of a finite set. Since ω contains at least two entries, the Boolean lattice is the lattice of subsets of a set of size at least one, or equivalently, the face lattice (including the empty face) of a full r-simplex for some $r \geq 0$, where $r = \sup(\omega) - 1 = |\omega| - |\omega|' - 1$. Since there is no insertion of a 2, it follows that the coefficients of the differential in (2.3), corresponding to the cover relations in the poset $\langle\omega\rangle_d$, are nonzero. Hence any cover relation can be part of a matching. It is well known that Boolean lattices have a perfect acyclic matching, i.e. an acyclic matching covering all elements of the Boolean lattice. In the lattice of subsets of $\{1,\dots,n\}$ take all edges $(A \to A\setminus\{n\})$ for subsets A containing n. It is easily checked that this is an acyclic matching. It follows that $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ has trivial homology.

Assume $d \geq |\omega| + 2$. Consider the following matching M_1 on the graph of $\langle\omega\rangle_d$. Let $\omega' = (\omega'_1,\dots,\omega'_r) \in \langle\omega\rangle_d$. Let $j(\omega)$ be the minimal index such that $\omega'_{j(\omega)} \neq 2$. By the assumptions on ω we have $1 \leq j(\omega) \leq r$.

For $\omega' \in \langle\omega\rangle$ such that $|\omega| < d$ and $j(\omega)$ is odd set

$$\omega'' := (2,\omega'_1,\dots,\omega'_r) \in \langle\omega\rangle_d.$$

By the choice of the parity of $j(\omega)$, we have that the edge $[\omega'' \to \omega']$ is in the graph of $\langle\omega\rangle_d$.

If $\omega' \in \langle\omega\rangle$ is not in one of the edges constructed before then either $|\omega'| = d$ and $j(\omega')$ is odd. If $I(\omega')$ is nonempty, we choose i to be minimal in $I(\omega')$ and add the edge $[\omega'' \to \omega']$, corresponding to that choice of i, to M_1. Note that, by this construction, we have $I(\omega'') = \emptyset$. Consider the case when $I(\omega')$ is empty and there is an $1 \leq i \leq r-1$ such that $\omega'_i = 2$. Let i be minimal with that property. Set $\omega'' = (\omega'_1,\dots,\omega_{i-1}, 2+\omega'_{i+1},\dots,\omega'_r)$. Then $[\omega' \to \omega'']$ is in M_1.

This shows that, so far, we have defined a matching M_1 on $\langle\omega\rangle_d$.

The edges from M_1 match all compositions, except for those compositions $\omega' = (\omega'_1,\dots,\omega'_r)$, for which the following holds

- if $\omega'_i = 2$ then $i = r$ and
- for every i such that $\omega_i > 2$, replacing ω'_i by by 2 and $\omega_i - 2$ yields a composition not contained in $\langle\omega\rangle_d$.

We claim that if $\omega' \in \langle\omega\rangle_d$ is not matched by M_1 then $|\omega'| = |\omega|$ or $|\omega'| = |\omega| + 2$.

Indeed if, $|\omega'| \geq |\omega| + 4$ then on the way from ω to ω' at least two 2s have been inserted. Thus even if $\omega'_r = 2$ then either one of $\omega'_1,\dots,\omega'_{r-1}$ is a 2 that can be removed while staying inside of $\langle\omega\rangle_d$ or for some i the block $\omega'_i > 2$ can be split into 2 and $\omega'_i - 2$.

Conversely, consider any composition $\omega' = (\omega'_1,\dots,\omega'_r)$ in $\langle\omega\rangle_d$ for which $|\omega'| = |\omega|$. Since ω contains no blocks of size ≤ 2, so does ω' and for the same reason no 2 can be split off either. It follows that ω' is unmatched by M_1.

Next consider a composition ω' in $\langle \omega \rangle_d$ with $|\omega'| = |\omega| + 2$. Then on the way from ω to ω' a single 2 must have been inserted. By our assumptions on ω this is the only possibility a 2 can arise in ω'. If this block 2 is still indeed still present in ω' then ω' is unmatched if and only if the 2 is the rightmost block. If the 2 has disappeared then it must have been merged and can be split off. In this case ω' is matched by M_1.

Now we define a matching M_2 covering the composition unmatched by M_1. In M_2 we collect the edges $[\omega' \to \omega'']$ where ω' is a composition with $|\omega| = |\omega'|$ and ω'' arise from ω' be adding a 2 as the rightmost block. Since ω does not contain blocks of size ≤ 2 it follows that ω' does not contain a 2. In particular, ω'' has nonzero coefficient in the differential of ω'.

By construction and the arguments above M_2 is a matching that does not touch any composition from an edge in M_1. It follows that $M = M_1 \cup M_2$ is a matching. Since M_2 matches all compositions not matched by M_1 it follows that all composition in $\langle \omega \rangle_d$ are matched by M.

It remains to be show that this matching is acyclic. Assume there is a directed cycle $\omega^1 \to \omega^2 \cdots \to \omega^{2\ell+1} = \omega^1$. We can assume that $[\omega^1 \to \omega^2]$ is an edge such that $[\omega^2 \to \omega^1] \in M$. Then, since directed cycles alternate between two homological degrees, it follows that in the edges $[\omega^{2j} \to \omega^{2j+1}]$, $j = 1, \ldots, \ell$, either two blocks are merged or a 2 is inserted. In the first case, $|\omega^{2j}| = |\omega^{2j+1}|$. In the second case, $|\omega^{2j+1}| = |\omega^{2j}| + 2$. It also follows that $[\omega^{2j} \to \omega^{2j-1}] \in M$ for $j = 1, \ldots, \ell$. If $[\omega^{2j} \to \omega^{2j-1}] \in M_1$, then $|\omega^{2j-1}| = |\omega^{2j}|$, and if $[\omega^{2j} \to \omega^{2j-1}] \in M_2$, then $|\omega^{2j}| = |\omega^{2j-1}| - 2$. Since edges in M_2 connect compositions of norm $|\omega|$ and norm $|\omega|+2$, it follows that in the directed cycle either only edges and inverted edges corresponding to mergings occur or that $|\omega^j| \in \{|\omega|, |\omega| + 2\}$ for $j = 1, \ldots, 2\ell + 1$.

First consider the case that only mergings occur. By construction if $\omega^2 = (\omega_1^2, \ldots, \omega_r^2)$ then

$$\omega^1 = (\omega_1^2, \ldots, \omega_{i-1}^2, 2, \omega_i^2 - 2, \omega_{i+1}^2, \ldots, \omega_r^2)$$

for the smallest index i from $I(\omega^2)$ and $I(\omega^1) = \emptyset$. Let $[\omega^{2\ell} \to \omega^1]$ be the preceding arrow in the directed cycle, where by our assumptions two blocks are merged. Then either $I(\omega^{2\ell}) = I(\omega^1) = \emptyset$ or $I(\omega^{2\ell}) \neq \emptyset$ and $\min I(\omega^{2\ell}) < \min I(\omega_2)$. If $I(\omega^{2\ell}) = \emptyset$ then $[\omega^{2\ell} \to \omega^{2\ell-1}]$ cannot be in M_1 and we arrive at a contradiction. Thus we must have $I(\omega^{2\ell}) \neq \emptyset$. But iterating the considerations in order to arrive back at ω^1 we obtain a decreasing sequence $\min I(\omega^2) > \min I(\omega^{2\ell}) > \cdots > \min I(\omega^2)$. Again we arrive at a contradiction.

It remains to consider the case when $|\omega^j| \in \{|\omega|, |\omega| + 2\}$, $j = 1, \ldots, 2\ell + 1$. Since we have also already covered the case when only mergins occurs along the directed cycle we may also assume that $[\omega^2 \to \omega^1] \in M_2$. Thus the last block of ω^1 is a 2 and since removing this block leads to a composition ω^2 of norm $|\omega^2| = |\omega|$ it follows by our assumptions on ω that this is the only 2 in ω^1. Hence ω^2 is a composition without any 2. Since neither edges in M_1 nor M_2 add blocks of size 2 to the right of the rightmost block of a comosition, it follows that there is a first edge $[\omega^{2j} \to \omega^{2j+1}]$ where the trailing 2 is added. Since $|\omega^{2j}|, |\omega^{2j+1}| \in \{|\omega|, |\omega| + 2\}$ it follows that $|\omega^{2j}| = |\omega|$ and $|\omega^{2j+1}| = |\omega| + 2$. But then this edge is part of the matching and an element of M_2. These edges can can only be traversed from $\omega^{2j} \to \omega^{2j+1}$. Again a contradiction.

Thus we have shown that the matching is acyclic. Since all element of $\langle \omega \rangle_d$ are covered the remaining assertion follows. □

6. (Co)homological stabilization

Theorem 6.1. *Let $\Delta = \{\omega^{(1)}, \ldots, \omega^{(m)}\}$ be a finite set of compositions such that $|\omega^{(1)}| \equiv \cdots \equiv |\omega^{(m)}| \mod 2$. Then the following claims hold:*

(i) *if no $\omega^{(i)}$, $i = 1, \ldots, m$, contains a 1, then for any non-negative integer i, the homology group $\bar{H}_i(\bar{\mathcal{P}}_d^{\langle \Delta \rangle}, \mathbb{Z})$ vanishes for any sufficiently large d,*

(ii) *if no $\omega^{(i)}$, $i = 1, \ldots, m$, contains a 1, then the space $\bar{\mathcal{P}}_\infty^{\langle \Delta \rangle} = \lim_{d \to \infty} \bar{\mathcal{P}}_d^{\langle \Delta \rangle}$, obtained as the direct limit (1.5), is weakly homotopically trivial,*

(iii) *for any non-negative integer j, we have $\bar{H}^j(\mathcal{P}_d^{\mathbf{c}\langle \Delta \rangle}, \mathbb{Z}) \cong \bar{H}^{j+2}(\mathcal{P}_{d+2}^{\mathbf{c}\langle \Delta \rangle}, \mathbb{Z})$ for d large enough.*

The following statement implies Theorem 6.1(i).

Proposition 6.2. *Let $\Delta = \{\omega^{(1)}, \ldots, \omega^{(m)}\} \subseteq \mathbf{\Omega}$ be a set of compositions such that $|\omega^{(1)}| \equiv \cdots \equiv |\omega^{(m)}| \mod 2$ and such that none of the compositions has an entry 1 and none of the composition equals (). We set*

$$r := \min\left\{|\omega^{(1)}|, \ldots, |\omega^{(m)}|\right\}.$$

Then $r \geq 2$ and $\bar{H}_i(\bar{\mathcal{P}}_d^{\langle \Delta \rangle}) = 0$ for all positive $i < \lfloor \frac{d}{r-1} \rfloor$.

In particular, $\lim_{d \to \infty} \bar{H}_i(\bar{\mathcal{P}}_d^{\langle \Delta \rangle}) = 0$ for all positive $i \geq 0$.

Proof. Since no $\omega^{(i)}$, $i = 1, \ldots, m$, equals () we have that $r > 0$. Furthermore if $r = 1$ then there is an i with $\omega^{(i)} = (1)$, but this contradicts our assumptions. Thus $r \geq 2$. Now take $\omega \in \mathbf{\Omega}_{\langle d]}$ with $|\omega| \equiv r \mod 2$, containing no 1 and such that its largest part is greater than or equal to r. Then $\omega \prec \omega^{(i)}$ for all i such that $|\omega^{(i)}| = r$. In particular, $\omega \in \langle \Delta \rangle_d$. Thus the shortest chain connecting the composition (d) to a composition all parts of which are smaller than r is the one where at each step one successively splits off parts of size $r-1$. For example, if all parts $r-1$ are split off to the left, one constructs the chain

$$\begin{aligned} (d) \prec (r-2, d-(r-1)) &\prec (r-1, r-1, d-2(r-1)) \prec \cdots \\ \cdots &\prec (\underbrace{r-1, \ldots, r-1}_{\lfloor \frac{d}{r-1} \rfloor}, d - \lfloor \frac{d}{r-1} \rfloor (r-1)). \end{aligned}$$

Note that the last element may not lie in $\langle \Delta \rangle$. When going up the chain, the reduced norm of the compositions decreases by 1 in each step. It follows that, for $k \leq d - \lfloor \frac{d}{r-1} \rfloor$, the poset $\mathbf{\Omega}_{|\sim|' \geq k}^{\neq 1}$ coincides with $\{\omega \in \langle \Delta \rangle_d \mid |\omega|' \geq k\}$. Thus $\bar{H}_i(\bar{\mathcal{P}}_d^{\langle \Delta \rangle}, \mathbb{Z}) \cong \bar{H}_i(\bar{\mathcal{P}}_d^{\mathbf{\Omega}_{|\sim|' \geq k}^{\neq 1}}, \mathbb{Z}) = 0$ for all $i < k$. Now Proposition 4.3 implies that $\bar{H}_i(\bar{\mathcal{P}}_d^{\langle \Delta \rangle}, \mathbb{Z}) = 0$ for all $i < \lfloor \frac{d}{r-1} \rfloor$. □

Remark 6.3. Note that if $() \in \Delta$ in Proposition 6.2 then $r = 0$, $\mathbf{\Omega}^{\neq 1}_{|\tilde{\ }|' \geq 0} = \Delta$ and $\bar{H}_i(\bar{\mathcal{P}}_d^{\langle\Delta\rangle}, \mathbb{Z}) = 0$ for $i < d$. The case $r = 1$ can occur only if we allow 1's as parts in the compositions. Even though experiments suggest that a similar vanishing occurs in these cases as well, we do not know how to handle this situation at present.

Proof of Theorem 6.1(ii). By [KSW1, Theorem 2.2] we know that $\bar{\mathcal{P}}_d^{\Theta}$ is simply connected if $\Theta \neq \{(d)\}$. Clearly for any $d > \max\{|\omega^{(1)}, \dots, |\omega^{(m)}|\}$, we have $\langle\Delta\rangle_d \neq \{(d)\}$. Thus $\bar{\mathcal{P}}_d^{\langle\Delta\rangle}$ is simply connected. Now by (i) and the Hurewicz theorem we have that $\bar{\mathcal{P}}_d^{\langle\Delta\rangle}$ is $(\lfloor \frac{d}{r-1} \rfloor - 1)$-connected. The result follows. $\square$

To settle Theorem 6.1(iii) we need the following statement.

Proposition 6.4. *Let $\Delta = \{\omega^{(1)}, \dots, \omega^{(m)}\} \subseteq \mathbf{\Omega}$ be a set of compositions such that $|\omega^{(1)}| \equiv \cdots \equiv |\omega^{(m)}| \mod 2$. Define*

$$t := \max\left\{|\omega^{(1)}|, \dots, |\omega^{(m)}|\right\}, \text{ and } s := \min\left\{|\omega^{(1)}|', \dots, |\omega^{(m)}|'\right\}.$$

Then for $d \geq t$ with $d \equiv t \mod 2$, we have

$$\langle\Delta\rangle_d \cap \mathbf{\Omega}_{|\sim|' \leq k} \quad = \quad \langle\Delta\rangle_{d+2} \cap \mathbf{\Omega}_{|\sim|' \leq k}, \tag{6.1}$$

where $k := s + \frac{1}{2}(d - t)$.

In particular, $\bar{H}_i(\bar{\mathcal{P}}_d^{\langle\Delta\rangle}) \cong \bar{H}_i(\bar{\mathcal{P}}_{d+2}^{\langle\Delta\rangle})$ for $i > k - 1$.

Proof. Assume that ω is in $\langle\Delta\rangle_{d+2} \setminus \langle\Delta\rangle_d$. It follows that $|\omega| = d + 2$ and there exists $\omega^{(i)} \in \Delta$ such that $\omega \prec \omega^{(i)}$. Hence $|\omega|' \geq |\omega^{(i)}|' \geq s$ and $d + 2 = |\omega| \geq t \geq |\omega^{(i)}|$. Thus in order to get from $\omega^{(i)}$ to ω at least $\frac{d+2-t}{2}$ insertions of 2 must be performed. Each insertion increases $|\tilde{\ }|'$ by one. This implies that

$$|\omega|' \geq \frac{d + 2 - t}{2} + s > \frac{d - t}{2} + s.$$

Now (6.1) follows. From this the homological consequence is immediate by Corollary 2.6. $\square$

Now we can finish the proof of Theorem 6.1(iii).

Proof of Theorem 6.1(iii). By Proposition 6.4 $\bar{H}_i(\bar{\mathcal{P}}_d^{\langle\Delta\rangle}) \cong \bar{H}_{i+2}(\bar{\mathcal{P}}_{d+2}^{\langle\Delta\rangle})$ of $i \geq s + \frac{1}{2}(d - t)$ for some numbers s and t depending only on Δ. By the Alexander duality we deduce $\bar{H}^{d-i-1}(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle}, \mathbb{Z}) \cong \bar{H}_{d+2-i-1}(\mathcal{P}_{d+2}^{\langle\Delta\rangle}, \mathbb{Z})$ if $i \geq s + \frac{1}{2}(d - t)$. Hence $\bar{H}^j(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle}, \mathbb{Z}) \cong \bar{H}_{j+2}(\mathcal{P}_{d+2}^{\langle\Delta\rangle}, \mathbb{Z})$ if $j \leq \frac{d}{2} - s + \frac{t}{2}$. $\square$

The next result claims that, for a large assortment of ideals $\langle\Delta\rangle$, generated by finite collections Δ, the limit $\mathcal{P}_\infty^{\mathbf{c}\langle\Delta\rangle}$ is also homologically trivial. Combining this fact with the results about $\pi_1(\mathcal{P}_\infty^{\mathbf{c}\langle\Delta\rangle})$ from [KSW1], we get the homotopy triviality of $\mathcal{P}_\infty^{\mathbf{c}\langle\Delta\rangle}$ as well.

Proposition 6.5. *Let $\Delta = \{\omega^{(1)}, \ldots, \omega^{(m)}\} \subseteq \mathbf{\Omega}$ be a finite set of compositions such that $|\omega^{(1)}| \equiv \cdots \equiv |\omega^{(m)}| \mod 2$. If none of the $\omega^{(i)}$ is of the form*

$$(\underbrace{2,\ldots,2,}_{odd}\ \underbrace{\omega',\cdots\cdots}_{possibly\ empty}\ )$$

for $\omega' \neq 2$, and none of the $\omega^{(i)}$ contains two consecutive 1's, then $H_i(\bar{\mathcal{P}}_d^{\langle\Delta\rangle}) = 0$ for all

$$i\ >\ \frac{1}{2}d + \max_{1\leq i\leq m}\Big\{\frac{1}{2}|\omega^{(i)}| - |\omega^{(i)}|'\Big\}.$$

By the Alexander duality, this implies $\bar{H}^j(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle}, \mathbb{Z}) = 0$ in the range

$$j\ <\ \frac{1}{2}d - 1 - \max_{1\leq i\leq m}\Big\{\frac{1}{2}|\omega^{(i)}| - |\omega^{(i)}|'\Big\}.$$

Proof. Let $\omega \in \langle\Delta\rangle_d$. Assume that $\omega = (\underbrace{2,\ldots,2}_{\ell},\omega',\ldots,)$ for $\omega' \neq 2$. If ℓ is even and $d - |\omega| \geq 2$, then $\gamma = (\underbrace{2,\ldots,2}_{\ell+1},\omega',\ldots,)$ lies in $\langle\Delta\rangle_d$. By assumption breaking up one of the leading $2's$ from γ into two $1's$ does not result in a a composition from $\langle\Delta\rangle_d$. Thus the $(\ell+1)$st 2 is inserted in one of the $\ell+1$ positions before, between or after a 2 from the leading block of 2's in ω. The coefficients of these insertions in the differential alternate from left to right and sum up to 1 since ℓ is even. Thus γ has a nonzero coefficient in $\partial\omega$. Hence the graph associated to $\langle\Delta\rangle$ contains the edge $[\omega \to \gamma]$. Let M be the collection of all those edges.

Assume now that ℓ is odd. Since by our assumption any $\omega^{(i)} \succ \omega$ contains no two consecutive 1s and starts with an even number of 2's it follows that $\gamma = (\underbrace{2,\ldots,2}_{\ell-1},\omega',\ldots,)$ lies in $\langle\Delta\rangle_d$. Moreover, the reasoning above shows that $[\gamma \to \omega]$ is in the graph associated to $\langle D\rangle$. In particular, $[\gamma \to \omega] \in M$.

We want to show that indeed M defines an acyclic matching (see § 3).

- (matching) This is obvious from the definition.
- (acyclic) If there is a directed cycle then it must contain an edge $[\gamma \to \omega]$ for some $[\omega \to \gamma] \in M$. Say ω has ℓ leading 2's. Then by definition of M we have that ℓ is even and γ has $\ell+1$ leading 2's. Let $\gamma' \in \langle D\rangle$ such that $[\omega \to \gamma']$ follows $[\gamma \to \omega]$ in a directed cycle. Then we have $|\gamma'|' = |\gamma|'$ and $\gamma' \neq \gamma$. If follows that either the number of leading 2's is ℓ and hence even or at least one of the leading 2s is merged with a neighboring block. If γ' has an even number leading 2's then there is no edge $[\omega' \to \gamma'] \in M$. In this case the directed cycle cannot be continued by an edge $[g' \to \omega']$. Note that directed cycles alternate between two homological degrees. Thus γ' must have an odd number of 2's and by $\gamma' \neq \gamma$ arises from ω by merging the rightmost of the leading 2's with its block to the right. In particular, there are $\ell - 1$ leading 2's in γ'. It follows that on a directed path the number of leading 2s is strictly decreasing. Thus there cannot be a directed cycle.

There is no critical cell in this matching of dimension greater than or equal to

$$\max_{1\le i\le m}\left\{d-|\omega^{(i)}|'-\frac{d-|\omega^{(i)}|}{2}\right\}=\frac{1}{2}d+\max_{1\le i\le m}\left\{\frac{1}{2}|\omega^{(i)}|-|\omega^{(i)}|'\right\}.$$

From this the claim follows. □

Proof. BLA □

7. Computer experiments and open problems

7.1. **Computer experiments.** Here we present a number of homological results obtained on computer. Our main goal is to illustrate the results and motivate the conjectures from the previous sections and to present some intriguing information about nontrivial non-stable homology, i.e. homology which disappears when the degree grows and whose existence does not follow from our previous considerations. These examples might serve as an inspiration for a future research in the area.

In order to calculate $\bar{H}_*(\bar{\mathcal{P}}_d^{\langle\Delta\rangle};\mathbb{Z})$, for a given finite set Δ of compositions having the same parity of their norms, the third author wrote a program in GAP language. Namely, given an arbitrary finite set Δ of compositions and a number j, the program creates the corresponding closed poset $\langle\Delta\rangle_d$. The poset comes with the corresponding height function, given by the reduced norm $|\sim|'$. Then the program calculates the homology of the differential complex $(\mathbb{Z}[\langle\Delta\rangle],\partial)$ which coincides with the reduced homology $\bar{H}_*(\bar{\mathcal{P}}_d^{\langle\Delta\rangle};\mathbb{Z})$ (resp. $\bar{H}_*(\bar{\mathcal{P}}_d^{\langle\langle\Delta\rangle\rangle};\mathbb{Z})$). The source can be found at **https://www.mathematik.uni-marburg.de/welker/**.

In the tables below, the sequence standing after the symbol i consists of the homology $H_i(\bar{\mathcal{P}}_d^{\langle\Delta\rangle};\mathbb{Z})$ *read backwards*, i.e., starting from the dimension of the complex and ending with dimension 0 (in which case, we always have the vanishing Betti number).

Notice that in our computer examples shown below we only present Δ with *at least two different patterns*. However, we performed many experiments with single patterns not shown below; these results were instrumental in the formulations of the above Conjecture 5.1.

In the following tables we will mark the entries in stabilization range of Proposition 6.2 with red and the entries in stabilization range of Proposition 6.4 with blue. Non-vanishing entries, shown in the usual black color, are non-stable. In all examples below we were not able to detect any torsion although it undoubtedly exists in more sophisticated examples, see discussions in the Introduction.

Example 1. $\Delta=\{[2,3],[3,2]\}$, $\bar{H}_*(\bar{\mathcal{P}}_{2k+5}^{\langle\Delta\rangle};\mathbb{Z})$ coincides with the homology of S^{2k+2}. That is, $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})=(\mathbb{Z},0,\mathbb{Z},\dots)$, where "…" stand for zeros. Checked for $k=0,\dots,7$.

Example 2. $\Delta=\{[2,4],[4,2]\}$.

$d\backslash i$	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
6															$\mathbb{Z}$	0	0
8													$\mathbb{Z}$	$\mathbb{Z}$	0	0	0
10											$\mathbb{Z}$	0	0	0	0	0	0
12									$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	0
14							$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0
16					$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0
18			$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	$\mathbb{Z}$	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z}) = (\mathbb{Z},0,0,\mathbb{Z},\dots)$, where "$\dots$" stands for the unstable cohomology, which appears and disappears sporadically in higher and higher dimensions.

Example 3. $\Delta = \{[3,4],[4,3]\}$.

$d\backslash i$	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
7																	$\mathbb{Z}$	0	0
9															0	0	0	0	0
11													0	0	$\mathbb{Z}$	0	0	0	0
13											0	0	0	0	0	0	0	0	0
15									0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0
17							0	0	0	0	0	0	0	0	0	0	0	0	0
19					0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0
21			0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ only has some unstable cohomology.

Example 4. $\Delta = \{[1,5],[5,1]\}$.

$d\backslash i$	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
6															$\mathbb{Z}$	0	0
8													0	0	0	0	0
10											0	0	0	$\mathbb{Z}$	0	0	0
12									0	0	0	0	0	$\mathbb{Z}$	0	0	0
14							0	0	0	0	0	0	0	0	0	0	0
16					0	0	0	0	0	0	0	0	0	0	0	0	0
18			0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The table shows that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ only has some unstable cohomology.

Example 5. $\Delta = \{[3,5],[5,3]\}$.

$d\backslash i$	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
8															$\mathbb{Z}$	0	0
10													0	$\mathbb{Z}$	0	0	0
12											0	0	0	0	0	0	0
14									0	0	0	0	0	0	0	0	0
16							0	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0
18					0	0	0	0	0	0	0	0	0	0	0	0	0
20			0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ only has some unstable cohomology.

Example 6. $\Delta = \{[1,2,3] + \text{all permutations}\}$.

$d\backslash i$	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
6													$\mathbb{Z}^2$	0	0	0
8											$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0
10									$\mathbb{Z}$	0	0	0	0	0	0	0
12							$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0	0	0	0
14					$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
16			$\mathbb{Z}$	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0
18	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The table suggests that starting with $d = 8$, $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z}) = (\mathbb{Z}, 0, \mathbb{Z}, \dots)$, where "$\dots$" stands for the unstable cohomology.

Example 7. $\Delta = \{[2,3,4] + \text{all permutations}\}$.

$d\backslash i$	13	12	11	10	9	8	7	6	5	4	3	2	1	0
9											$\mathbb{Z}$	0	0	0
11									$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
13							$\mathbb{Z}$	0	0	0	0	0	0	0
15					$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	0	0
17			$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
19	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z}) = (\mathbb{Z}, 0, 0, 0, 0, \mathbb{Z}, \dots)$ with some unstable cohomology.

Example 8. $\Delta = \{[1,3,5] + \text{all permutations}\}$.

$d\backslash i$	13	12	11	10	9	8	7	6	5	4	3	2	1	0
9											$\mathbb{Z}$	0	0	0
11									0	$\mathbb{Z}^2$	0	0	0	0
13							0	0	0	$\mathbb{Z}$	0	0	0	0
15					0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^3$	0	0	0	0
17			0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0	0	0	0
19	0	0	0	0	0	0	0	0	$\mathbb{Z}^3$	0	0	0	0	0

The above suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ has no stable, but a lot of unstable cohomology.

Example 9. $\Delta = \{[2,4,6] + \text{all permutations}\}$.

$d\backslash i$	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
12																	$\mathbb{Z}^2$	0	0	0
14															$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
16													$\mathbb{Z}$	0	0	0	0	0	0	0
18											$\mathbb{Z}$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
20									$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
22							$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0
24					$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0
26			$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0
28	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z}) = (\mathbb{Z},0,0,0,0,0,0,0,\mathbb{Z},\dots)$ with some unstable cohomology.

Example 10. $\Delta = \{[3,5,7,9] + \text{all permutations}\}$.

$d\backslash i$	10	9	8	7	6	5	4	3	2	1	0
24							$\mathbb{Z}$	$\mathbb{Z}$	0	0	0
26					0	$\mathbb{Z}^3$	$\mathbb{Z}^2$	0	0	0	0
28			0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
30	0	0	0	0	$\mathbb{Z}^2$	0	0	$\mathbb{Z}$	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ has no stable, but a lot of unstable cohomology.

Example 11. $\Delta = \{[3,4,5,6] + \text{all permutations}\}$.

$d\backslash i$	10	9	8	7	6	5	4	3	2	1	0
18							$\mathbb{Z}$	$\mathbb{Z}$	0	0	0
20					0	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}$	0	0	0
22			0	0	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	0	0	0	0
24	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^8$	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ has no stable, but a lot of unstable cohomology.

Example 12. $\Delta = \{[2,2,3,3,4,4] + \text{all permutations}\}$.

$d\backslash i$	10	9	8	7	6	5	4	3	2	1	0
18					0	$\mathbb{Z}^2$	$\mathbb{Z}^3$	0	0	0	0
20			0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	0	0	0	0	0
22	0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}^6$	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ has no stable, but a lot of unstable cohomology.

Example 13. $\Delta = \{[2,2,4] + \text{all permutations}\}$.

$d\backslash i$	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
8													$\mathbb{Z}$	0	0	0
10											0	0	0	0	0	0
12									0	0	$\mathbb{Z}$	0	0	0	0	0
14							0	0	0	0	0	0	0	0	0	0
16					0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0
18			0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ has no stable, but some unstable cohomology.

Example 14. $\Delta = \{[2,3,4] + \text{all permutations} + [2,5] + [5,2]\}$.

$d\backslash i$	12	11	10	9	8	7	6	5	4	3	2	1	0
9									$\mathbb{Z}$	0	0	0	0
11							$\mathbb{Z}$	0	0	0	0	0	0
13					$\mathbb{Z}$	0	0	0	0	0	0	0	0
15			$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
17	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z}) = (\mathbb{Z},0,0,0,\mathbb{Z})$.

Example 15. $\Delta = \{[2,3,4] + \text{all permutations} + [1,6] + [6,1]\}$.

$d\backslash i$	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
9											0	0	0	0	0
11									0	0	0	0	0	0	0
13							0	0	0	0	$\mathbb{Z}$	0	0	0	0
15					0	0	0	0	0	0	0	0	0	0	0
17			0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ only has some unstable cohomology.

Example 16. $\Delta = \{[1,2,3] + \text{all permutations except for } [3,1,2]\}$.

$d\backslash i$	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
6													$\mathbb{Z}$	0	0	0
8											0	0	$\mathbb{Z}$	0	0	0
10									0	0	0	0	0	0	0	0
12							0	0	0	0	$\mathbb{Z}$	0	0	0	0	0
14					0	0	0	0	0	0	0	0	0	0	0	0
16			0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z})$ only has some unstable cohomology.

Example 17. $\Delta = \{[2,3] + [3,2] + [4,1] + [1,4]\}$.

$d\backslash i$	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
5																	$\mathbb{Z}^3$	0	0
7															$\mathbb{Z}$	$\mathbb{Z}^2$	0	0	0
9													$\mathbb{Z}$	0	$\mathbb{Z}^3$	0	0	0	0
11											$\mathbb{Z}$	0	0	$\mathbb{Z}^3$	0	0	0	0	0
13									$\mathbb{Z}$	0	0	0	$\mathbb{Z}^4$	0	0	0	0	0	0
15							$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}^4$	0	0	0	0	0	0	0
17					$\mathbb{Z}$	0	0	0	0	0	$\mathbb{Z}^5$	0	0	0	0	0	0	0	0
19			$\mathbb{Z}$	0	0	0	0	0	0	$\mathbb{Z}^6$	0	0	$\mathbb{Z}$	0	0	0	0	0	0
21	$\mathbb{Z}$	0	0	0	0	0	0	0	$\mathbb{Z}^6$	0	0	0	0	0	0	0	0	0	0

The table suggests that $H^*(\mathcal{P}_d^{\mathbf{c}\langle\Delta\rangle};\mathbb{Z}) = (\mathbb{Z}, 0, \mathbb{Z}, \dots)$ and the unstable cohomology behaves regularly.

7.2. **Open problems.** Here we collect some natural questions about which we do not have any relevant information.

Question 1. What about the unstable part? When can the sum of Betti numbers grow to infinity under the direct stabilization?

Question 2. What about the torsion? Does it always disappear under the stabilization?

It is well-known that any partially ordered set $\mathcal{S}$ canonically generates a simplicial complex $\boldsymbol{\Delta}(\mathcal{S})$. In particular, for any closed $\Theta \subset \boldsymbol{\Omega}_{\langle d]}$, we may consider the simplicial complex $\boldsymbol{\Delta}(\Theta)$ and its homology $H_*(\boldsymbol{\Delta}(\Theta);\mathbb{Z})$.

Question 3. What is the relation between $H_*(\bar{\mathcal{P}}_d^\Theta;\mathbb{Z})$ and $H_*(\boldsymbol{\Delta}(\Theta);\mathbb{Z})$? Recall that a standard construction delivers the canonical homomorphism $\pi_* : H_*(\bar{\mathcal{P}}_d^\Theta;\mathbb{Z}) \to H_*(\boldsymbol{\Delta}(\Theta);\mathbb{Z})$. Is π_* an epimorphism? an isomorphism?

8. Appendix. List of compositions ω with non-contractable $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ for $d \leq 13$.

Below we list all nontrivial compositions obtained using the same program indicating the topology of $\bar{\mathcal{P}}_d^{\langle\omega\rangle}$ in the space of reduced polynomials, i.e. with the sum of all roots vanishing.

$d=4$. codim=0: 1) $\omega=(1,1)$, S^2;
codim=1: 2) $\omega=(2)$, S^2;
codim=max: 3) $\omega=(4)$, S^0.

$d=5$. codim=0: 1) $\omega=(1,1,1)$, S^3;
codim=max: 2) $\omega=(5)$, S^0.

$d=6$. codim=0: 1) $\omega=(1,1,1,1)$, S^4; 2) $\omega=(1,1)$, S^4;
codim=1: 3) $\omega=(2)$, S^4;
codim=max: 4) $\omega=(6)$, S^0;.

$d=7$. codim=0: 1) $\omega=(1,1,1,1,1)$, S^5; 2) $\omega=(1,1,1)$, S^5;
codim=max: 3) $\omega=(7)$, S^0.

$d = 8$. codim=0: 1) $\omega = (1,1,1,1,1,1)$, S^6; 2) $\omega = (1,1,1,1)$, S^6; 3) $\omega = (1,1)$, S^6;
codim=1: 4) $\omega = (1,2,1)$, S^2; 5) $\omega = (2)$, S^6;
codim=2: 6) $\omega = (1,3)$, S^2; 7) $\omega = (3,1)$, S^2;
codim=3: 8) $\omega = (4)$, S^2;
codim=max: 9) $\omega = (8)$, S^0.

$d = 9$. codim=0: 1) $\omega = (1,1,1,1,1,1,1)$, S^7; 2) $\omega = (1,1,1,1,1)$, S^7; 3) $\omega = (1,1,1)$, S^7;
codim=1: 4) $\omega = (1,1,2,1)$, S^3; 5) $\omega = (1,2,1,1)$, S^3;
codim=2: 6) $\omega = (1,1,3)$, S^3; 7) $\omega = (1,3,1)$, S^3; 8) $\omega = (3,1,1)$, S^3;
codim=max: 9) $\omega = (9)$, S^0.

$d = 10$. codim=0: 1) $\omega = (1,1,1,1,1,1,1,1)$, S^8; 2) $\omega = (1,1,1,1,1,1)$, S^8; 3) $\omega = (1,1,1,1)$, S^8; 4) $\omega = (1,1)$, S^8;
codim=1: 5) $\omega = (1,1,1,2,1)$, S^4; 6) $\omega = (1,1,2,1,1)$, S^4; 7) $\omega = (1,2,1,1,1)$, S^4; 8) $\omega = (2)$, S^8;
codim=2: 9) $\omega = (1,1,1,3)$, S^4; 10) $\omega = (1,1,3,1)$, S^4; 11) $\omega = (1,3,1,1)$, S^4; 12) $\omega = (3,1,1,1)$, S^4;
codim=max: 13) $\omega = (10)$, S^0.

$d = 11$. codim=0: 1) $\omega = (1,1,1,1,1,1,1,1,1)$, S^9; 2) $\omega = (1,1,1,1,1,1,1)$, S^9; 3) $\omega = (1,1,1,1,1)$, S^9; 4) $\omega = (1,1,1)$, S^9;
codim=1: 5) $\omega = (1,1,1,1,2,1)$, S^5; 6) $\omega = (1,1,1,2,1,1)$, S^5; 7) $\omega = (1,1,2,1,1,1)$, S^5; 8) $\omega = (1,2,1,1,1,1)$, S^5;
codim=2: 9) $\omega = (1,1,1,1,3)$, S^5; 10) $\omega = (1,1,1,3,1)$, S^5; 11) $\omega = (1,1,3,1,1)$, S^5; 12) $\omega = (1,3,1,1,1)$, S^5; 13) $\omega = (3,1,1,1,1)$, S^5; 14) $\omega = (1,2,1,2,1)$, S^3;
codim=3: 15) $\omega = (1,2,1,3)$, S^3; 16) $\omega = (3,1,2,1)$, S^3;
codim=4: 17) $\omega = (3,1,3)$, S^3;
codim=max: 18) $\omega = (11)$, S^0.

$d = 12$. codim=0: 1) $\omega = (1,1,1,1,1,1,1,1,1,1)$, S^{10}; 2) $\omega = (1,1,1,1,1,1,1,1)$, S^{10}; 3) $\omega = (1,1,1,1,1,1)$, S^{10}; 4) $\omega = (1,1,1,1)$, S^{10}; 5) $\omega = (1,1)$, S^{10};
codim=1: 6) $\omega = (1,1,1,1,1,2,1)$, S^6; 7) $\omega = (1,1,1,1,2,1,1)$, S^6; 8) $\omega = (1,1,1,2,1,1,1)$, S^6; 9) $\omega = (1,1,2,1,1,1,1)$, S^6; 10) $\omega = (1,2,1,1,1,1,1)$, S^6; 11) $\omega = (1,2,1)$, S^4; 12) $\omega = (2)$, S^{10};
codim=2: 13) $\omega = (1,1,1,1,1,3)$, S^6; 14) $\omega = (1,1,1,1,3,1)$, S^6; 15) $\omega = (1,1,1,3,1,1)$, S^6; 16) $\omega = (1,1,3,1,1,1)$, S^6; 17) $\omega = (1,3,1,1,1,1)$, S^6; 18) $\omega = (3,1,1,1,1,1)$, S^6; 19) $\omega = (1,1,2,1,2,1)$, S^4; 20) $\omega = (1,2,1,1,2,1)$, S^4; 21) $\omega = (1,2,1,2,1,1)$, S^4; 22) $\omega = (1,2,2,1)$, S^2; 23) $\omega = (1,3)$, S^4; 24) $\omega = (3,1)$, S^4;
codim=3: 25) $\omega = (1,1,2,1,3)$, S^4; 26) $\omega = (1,2,1,1,3)$, S^4; 27) $\omega = (1,2,1,3,1)$, S^4; 28) $\omega = (1,3,1,2,1)$, S^4; 29) $\omega = (3,1,1,2,1)$, S^4; 30) $\omega = (3,1,2,1,1)$, S^4; 31) $\omega = (1,2,3)$, S^2; 32) $\omega = (3,2,1)$, S^2; 33) $\omega = (1,4,1)$, S^2; 34) $\omega = (4)$, S^4;
codim=4: 35) $\omega = (1,3,1,3)$, S^4; 36) $\omega = (3,1,3,1)$, S^4 37) $\omega = (1,5)$, S^2; 38) $\omega = (5,1)$, S^2; 39) $\omega = (3,3)$, S^2;
codim=5: 40) $\omega = (6)$, S^2;

codim=max: 41) $\omega = (12)$, S^0.

$d = 13$. codim=0: 1) $\omega = (1,1,1,1,1,1,1,1,1,1,1)$, S^{11}; 2) $\omega = (1,1,1,1,1,1,1,1,1)$, S^{11}; 3) $\omega = (1,1,1,1,1,1,1)$, S^{11}; 4) $\omega = (1,1,1,1,1)$, S^{11}; 5) $\omega = (1,1,1)$, S^{11};
codim=1: 6) $\omega = (1,1,1,1,1,1,2,1)$, S^7; 7) $\omega = (1,1,1,1,1,2,1,1)$, S^7; 8) $\omega = (1,1,1,1,2,1,1,1)$, S^7; 9) $\omega = (1,1,1,2,1,1,1,1)$, S^7; 10) $\omega = (1,1,2,1,1,1,1,1)$, S^7; 11) $\omega = (1,2,1,1,1,1,1,1)$, S^7; 12) $\omega = (1,1,2,1)$, S^5; 13) $\omega = (1,2,1,1)$, S^5;
codim=2: 14) $\omega = (1,1,1,1,1,1,3)$, S^7; 15) $\omega = (1,1,1,1,1,3,1)$, S^7; 16) $\omega = (1,1,1,1,3,1,1)$, S^7; 17) $\omega = (1,1,1,3,1,1,1)$, S^7; 18) $\omega = (1,1,3,1,1,1,1)$, S^7; 19) $\omega = (1,3,1,1,1,1,1)$, S^7; 20) $\omega = (3,1,1,1,1,1,1)$, S^7; 21) $\omega = (1,1,3)$, S^5; 22) $\omega = (1,3,1)$, S^5; 23) $\omega = (3,1,1)$, S^5; 24) $\omega = (1,1,1,2,1,2,1)$, S^5; 25) $\omega = (1,1,2,1,1,2,1)$, S^5; 26) $\omega = (1,1,2,1,2,1,1)$, S^5; 27) $\omega = (1,2,1,1,1,2,1)$, S^5; 28) $\omega = (1,2,1,1,2,1,1)$, S^5; 29) $\omega = (1,2,1,2,1,1,1)$, S^5; 30) $\omega = (1,1,2,2,1)$, S^3; 31) $\omega = (1,2,2,1,1)$, S^3;
codim=3: 32) $\omega = (1,1,1,2,1,3)$, S^5; 33) $\omega = (1,1,2,1,1,3)$, S^5; 34) $\omega = (1,1,2,1,3,1)$, S^5; 35) $\omega = (1,1,3,1,2,1)$, S^5; 36) $\omega = (1,2,1,1,3,1)$, S^5; 37) $\omega = (1,2,1,3,1,1)$, S^5; 38) $\omega = (1,3,1,1,2,1)$, S^5; 39) $\omega = (1,3,1,2,1,1)$, S^5; 40) $\omega = (3,1,1,2,1,1)$, S^5; 41) $\omega = (3,1,2,1,1,1)$, S^5; 42) $\omega = (1,1,2,3)$, S^3; 43) $\omega = (1,2,3,1)$, S^3; 44) $\omega = (1,3,2,1)$, S^3; 45) $\omega = (3,2,1,1)$, S^3; 46) $\omega = (1,1,4,1)$, S^3; 47) $\omega = (1,4,1,1)$, S^3;
codim=4: 48) $\omega = (1,1,3,1,3)$, S^5; 49) $\omega = (1,3,1,3,1)$, S^5 50) $\omega = (3,1,3,1,1)$, S^5 51) $\omega = (1,3,3)$, S^3; 52) $\omega = (3,3,1)$, S^3; 53) $\omega = (1,1,5)$, S^3; 54) $\omega = (1,5,1)$, S^3; 55) $\omega = (5,1,1)$, S^3;
codim=max: 56) $\omega = (13)$, S^0.

9. Appendix. Spaces $\mathcal{P}_d^{\mathbf{c}\Theta}$ as Grassmanians for vector flows with the Θ-constrained tangency patterns

The next Proposition 9.1 is instrumental in producing interesting examples of traversing vector fields on compact manifolds X with boundary, vector fields with a priori prescribed combinatorial tangency patterns of their trajectories to ∂X. These applications belongs to a different paper; the proposition below gives just a hint of these future investigations, which employ our computations of the cohomology $H^*(\mathcal{P}_d^{\mathbf{c}\Theta};\mathbb{Z})$.

For each element $\omega \in \mathbf{\Omega}_{\langle d]}$, let $\omega_{\succ}$ denote the set of elements in $\mathbf{\Omega}_{\langle d]}$ that are smaller than ω. The complementary set $\mathbf{c}(\omega_{\succ}) = \omega_{\preceq}$ consists of elements that are bigger than or equal to ω.

Proposition 9.1. *For each element $\omega \in \mathbf{\Omega}_{\langle d]}$ the boundary $\partial(\omega) \in \mathbb{Z}[\mathbf{\Omega}_{\langle d]}]$, given by formula* (2.3)*, produces, with the help of the Alexander duality $A_{\mathcal{P}}$ in the ambient space $\bar{\mathcal{P}}_d$, a cohomology class*

$$\theta_\omega := A_{\mathcal{P}}([\partial \mathsf{R}_d^\omega]) \in H^{|\omega|'}(\mathcal{P}_d^{\mathbf{c}\{\omega_\succ\}};\mathbb{Z}).$$

The value of θ_ω *on each homology class* $[h] \in H_{|\omega|'}(\mathcal{P}_d^{\mathbf{c}\{\omega_\succ\}}; \mathbb{Z})$*, represented by a singular cycle* $h : \Sigma \hookrightarrow \mathcal{P}_d^{\mathbf{c}\{\omega_\succ\}}$*, is the linking number* $\mathsf{lk}(h(\Sigma),\ \partial\bar{\mathsf{R}}_d^\omega)$*. The later number may be interpreted as the algebraic intersection between the cycle* $h(\Sigma)$ *and the cell* $\mathring{\mathsf{R}}_d^\omega$*.*

Proof. For any $\omega \in \mathbf{\Omega}_{\langle d]}$, take the closed poset $\omega_\succ = \langle\omega\rangle \setminus \omega \subset \mathbf{\Omega}_{\langle d]}$ for the role of Θ in Corollary 2.6. Then the boundary $\partial(\omega)$, given by the formula (2.3), represents the cycle $\partial\bar{\mathsf{R}}_d^\omega$ in $\bar{C}_{d-|\omega|'-1}(\bar{\mathcal{P}}_d^{\omega_\succ}; \mathbb{Z})$, and thus defines an element $[\partial\bar{\mathsf{R}}_d^\omega] \in \bar{H}_{d-|\omega|'-1}(\bar{\mathcal{P}}_d^{\omega_\succ}; \mathbb{Z})$. By the Alexander duality $A_\mathcal{P}$, this element produces a cohomology class $\theta_\omega = A_\mathcal{P}([\partial\bar{\mathsf{R}}_d^\omega]) \in H^{|\omega|'}(\mathcal{P}_d^{\mathbf{c}\{\omega_\succ\}}; \mathbb{Z})$.

The last claim of the proposition spells out the nature of the Alexander duality. □

In general, evaluating the pull-back $h^*(\theta_\omega)$ of the characteristic class θ_ω on the $|\omega|'$-dimensional cycle Σ, gives an oriented count of the trajectories of the combinatorial type ω on an appropriate $(|\omega|' + 1)$-dimensional manifold $X \subset \Sigma \times \mathbb{R}$.

Example 9.2. For $d = 6$, we get the following cohomology classes:

- $\theta_{121} = A_\mathcal{P}(\bar{\mathsf{R}}_6^{31} - \bar{\mathsf{R}}_6^{13} - \bar{\mathsf{R}}_6^{2121} + \bar{\mathsf{R}}_6^{1212}) \in H^1(\mathcal{P}_6^{\mathbf{c}\{121_\succ\}}; \mathbb{Z})$,
- $\theta_{3111} = A_\mathcal{P}(\bar{\mathsf{R}}_6^{411} - \bar{\mathsf{R}}_6^{321} + \bar{\mathsf{R}}_6^{312}) \in H^2(\mathcal{P}_6^{\mathbf{c}\{3111_\succ\}}; \mathbb{Z})$,
- $\theta_{31} = A_\mathcal{P}(\bar{\mathsf{R}}_6^{4} - \bar{\mathsf{R}}_6^{231} + \bar{\mathsf{R}}_6^{321} - \bar{\mathsf{R}}_6^{312}) \in H^2(\mathcal{P}_6^{\mathbf{c}\{31_\succ\}}; \mathbb{Z})$,
- $\theta_{1221} = A_\mathcal{P}(\bar{\mathsf{R}}_6^{321} - \bar{\mathsf{R}}_6^{141} + \bar{\mathsf{R}}_6^{123}) \in H^2(\mathcal{P}_6^{\mathbf{c}\{1221_\succ\}}; \mathbb{Z})$.

For example, take $\omega = (3111)$. By Proposition 9.1, the value of θ_{3111} on any singular surface $h : \Sigma^2 \to \mathcal{P}_6^{\mathbf{c}\{3111_\succ\}}$ is its linking number $\mathsf{lk}\big(h(\Sigma^2),\ \partial\bar{\mathsf{R}}_6^{3111}\big)$ with the 3-cycle $\partial\bar{\mathsf{R}}_6^{3111} = \bar{\mathsf{R}}_6^{411} - \bar{\mathsf{R}}_6^{321} + \bar{\mathsf{R}}_6^{312}$. In particular, consider a small ball $D^3_{411} \subset \mathcal{P}_6$, normal to the stratum $\mathring{\mathsf{R}}_6^{411}$, and take its boundary $S^2_{411} \subset \mathcal{P}_6^{\mathbf{c}\{411_\succeq\}}$ for the role of Σ^2. By analyzing the poset $(411)_\prec$, one can check that

$$\langle\theta_{3111}, S^2_{411}\rangle = \mathsf{lk}(S^2_{411},\ \partial\bar{\mathsf{R}}_6^{3111}) = S^2_{411} \circ \mathsf{R}_6^{3111} = 1.$$

As a result, along the lines of the proof of **??**, we get an example of a flow on a 3-manifold $X \subset S^2 \times \mathbb{R}$ with a *single* trajectory of the combinatorial type (3111). Similar examples of flows are provided by the normal spheres $S^2_{321} \subset \mathcal{P}_6^{\mathbf{c}\{321_\succeq\}}$ and $S^2_{312} \subset \mathcal{P}_6^{\mathbf{c}\{312_\succeq\}}$.

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