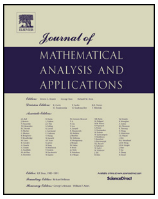
J. Math. Anal. Appl. 538 (2024) 128349

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa



Regular Articles

# In search of Newton-type inequalities

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ARTICLE INFO

*Article history:*
Received 30 January 2024
Available online 25 March 2024
Submitted by S. Hencl

To our friend and coauthor Vlado Petrov Kostov

*Keywords:*
Number of real zeros of real polynomial
Newton inequalities
Hutchinson's theorem
Descartes rule of signs

ABSTRACT

In this paper, we prove a number of results providing either necessary or sufficient conditions guaranteeing that the number of real roots of real polynomials of a given degree is either less or greater than a given number. We also provide counterexamples to two earlier conjectures refining Descartes rule of signs.

## 1. Introduction

The main motivation for the present paper comes from the classical Newton inequalities and Hutchinson's theorem, see [2,6] together with three conjectures formulated in [9], see § 5 and [1].

Our present set-up is as follows.

**Notation 1.** *Denote by* $\mathbb{R}[x]$ *the linear space of all polynomials with real coefficients, and by* $\mathbb{R}^+[x]$ *the cone of all polynomials with positive coefficients. Denote by* $\mathbb{R}_n[x]$ *the (affine) space of monic degree* $n$ *polynomials with real coefficients and by* $\mathbb{R}_n^+[x] \subset \mathbb{R}_n[x]$ *the orthant of monic polynomials of degree* $n$ *with positive coefficients.*

*Denote by* $\widehat{\mathbb{R}}_n[x]$ *the (affine) space of monic degree* $n$ *polynomials with real coefficients and constant term* $1$ *and by* $\widehat{\mathbb{R}}_n^+[x] \subset \mathbb{R}_n[x]$ *the orthant of monic polynomials of degree* $n$ *with positive coefficients and constant term* $1$.

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https://doi.org/10.1016/j.jmaa.2024.128349

*Now let* $\Delta_n$ *denote the discriminant of the family of polynomials* $x^n + a_{n-1}x^{n-1} + \cdots + a_0$, *i.e.,* $\Delta_n \in \mathbb{Z}[a_{n-1}, \ldots, a_0]$ *is the unique (up to constant factor) irreducible polynomial such that* $x^n+a_{n-1}x^{n-1}+\cdots+a_0$ *has a root of multiplicity greater than* 1 *implies that* $\Delta_k(a_{n-1}, \ldots, a_0) = 0$. *Denoting by* $\mathfrak{D}_n \subset \mathbb{R}_n[x]$ *the zero locus of* $\Delta_n$, *we will be mainly interested in its subset* $\mathcal{D}_n \subset \mathfrak{D}_n$ *consisting of polynomials in* $\mathbb{R}_n[x]$ *having a multiple real zero. (It is well-known that* $\mathfrak{D}_n \setminus \mathcal{D}_n$ *has codimension* 1 *in* $\mathfrak{D}_n$. *Indeed the complement* $\mathfrak{D}_n \setminus \mathcal{D}_n$ *consists of real monic polynomials of degree* $n$ *having at least one pair of complex conjugated roots each having multiplicity at least* 2 *and therefore has real codimension* 2 *in* $\mathbb{R}_n[x]$ *and real codimension* 1 *in* $\mathfrak{D}_n$*).*

*Further, let* $\mathcal{D}_n^+ \subset \mathbb{R}_n^+[x]$ *denote the positive discriminantal locus which is the set of monic polynomials of degree* $n$ *with positive coefficients having at least one real multiple root. In other words,* $\mathcal{D}_n^+$ *is the restriction of* $\mathcal{D}_n$ *to* $\mathbb{R}_n^+[x]$. *Finally, let* $\widehat{\mathcal{D}}_n$ *be the restriction of* $\mathcal{D}_n$ *to* $\widehat{\mathbb{R}}_n[x]$ *and* $\widehat{\mathcal{D}}_n^+$ *be the restriction of* $\mathcal{D}_n$ *to* $\widehat{\mathbb{R}}_n^+[x]$.

*For each polynomial* $P(x) = \sum_{k=0}^{n} a_k x^k$ *with positive coefficients, we define the sequence of (positive) numbers*

$$q_k = q_k(P) = \frac{a_k^2}{a_{k-1}a_{k+1}}, \; k = 1, \ldots, n-1. \tag{1.1}$$

In what follows we will try to generalize and reinterpret the following classical results of J. I. Hutchinson [2] and I. Newton, see also Proposition 1 below.

**Theorem H.** *An entire function* $p(x) = a_0 + a_1x + \cdots + a_nx^n + \ldots$ *with strictly positive coefficients has the property that all of its finite segments* $a_ix^i + a_{i+1}x^{i+1} + \ldots + a_jx^j$ *have only real roots if and only if for* $k = 1, 2, \ldots,$

$$q_k \geq 4. \tag{1.2}$$

**Theorem N.** *Let* $P(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}_+[x]$ *be a polynomial with all real roots. Then for* $k = 1, 2, \ldots, n-1$,

$$q_k \geq \frac{k+1}{k} \cdot \frac{n-k+1}{n-k}. \tag{1.3}$$

*An easy to prove statement is that for any positive integer* $n$, *the set* $\mathbb{R}_n[x] \setminus \mathcal{D}_n$ *of all real monic degree* $n$ *polynomials with all simple real roots consists of* $\left[\frac{n}{2}\right] + 1$ *contractible connected components enumerated by the number of simple real zeros of polynomials belonging to the respective component.*

Similar results hold for $\mathbb{R}_k^+[x]$ and $\widehat{\mathbb{R}}_k^+[x]$, see Lemma 2 and Corollary 1 below.

**Notation 2.** *Fix a sequence of (strict) inequality signs* $\bar{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_{n-1})$ *of length* $n-1$, *where each entry is either* $<$ *or* $>$. *(One can interpret* $\bar{\sigma}$ *as a binary sequence). Further let* $\bar{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})$ *be a sequence of positive numbers. Given a pair* $(\bar{\sigma}, \bar{\epsilon})$, *define the subset* $\widehat{\mathbb{R}}_{\bar{\sigma},\bar{\epsilon}}^+ \subset \widehat{\mathbb{R}}_n^+[x]$ *as the set of all polynomials* $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + 1$ *satisfying the system of inequalities*

$$\{q_j \lessgtr_{\bar{\sigma}} \epsilon_j\}, \; j = 1, \ldots, n-1$$

*where the inequality sign in defined by* $\sigma_j$.

*Consider the map* $Log|\cdot| : \widehat{\mathbb{R}}_n^+[x] \to \mathbb{R}^{n-1}$ *sending the polynomial* $P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + 1$ *with positive coefficients to the* $(n-1)$*-tuple* $(\log a_1, \log a_2, \ldots, \log a_{k-1})$. *(The map Log is a diffeomorphism between the source and the target spaces). Denote the coordinates in the image space* $\mathbb{R}^{n-1}$ *by* $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$.

**Remark 1.** Observe that logarithmizing the inequality in the Hutchinson theorem we obtain the linear inequality

$$2\alpha_k - \alpha_{k-1} - \alpha_{k+1} \geq \ln 4$$

and logarithmizing Newton's inequality we get

$$2\alpha_k - \alpha_{k-1} - \alpha_{k+1} \geq \ln((k+1)(n-k+1)) - \ln(k(n-k)).$$

The prototype result which we want to generalize in this paper is as follows. Denote by $\Sigma_n^+ \subset \mathbb{R}_n^+[x]$ the set of all degree $n$ real-rooted polynomials with positive coefficients.

**Proposition 1** *(see Proposition 7 of [5]).* (i) *The polyhedral cone described by (the logarithm of) Hutchinson's inequalities* (1.2) *is the maximal polyhedral cone contained in* $Log(\Sigma_n^+)$.
(ii) *The minimal polyhedral cone containing* $Log(\Sigma_n^+)$ *is given by (the logarithm of) Newton's inequalities.*

**Notation 3.** *For $n$ even, consider two filtrations of* $\mathbb{R}_n[x]$,

$$\mathcal{O}_{<2} \subset \mathcal{O}_{<4} \subset \cdots \subset \mathcal{O}_{<n} \subset \mathbb{R}_n[x] \supset \mathcal{O}_{\geq 2} \supset \mathcal{O}_{\geq 4} \cdots \supset \mathcal{O}_{\geq n},$$

*where* $\mathcal{O}_{<2\ell}$ *(resp.* $\mathcal{O}_{\geq 2\ell}$*) is the set of all polynomials in* $\mathbb{R}_n[x]$ *with fewer than (resp. at least)* $2\ell$ *real roots (counting multiplicities).*

*Analogously,*

$$\widehat{\mathcal{O}}_{<2} \subset \widehat{\mathcal{O}}_{<4} \subset \cdots \subset \widehat{\mathcal{O}}_{<n} \subset \widehat{\mathbb{R}}_n[x] \supset \widehat{\mathcal{O}}_{\geq 2} \supset \widehat{\mathcal{O}}_{\geq 4} \cdots \supset \widehat{\mathcal{O}}_{\geq n}$$

*is the restriction of the above filtrations to* $\widehat{\mathbb{R}}_n[x]$;

$$\mathcal{O}^+_{<2} \subset \mathcal{O}^+_{<4} \subset \cdots \subset \mathcal{O}^+_{<n} \subset \mathbb{R}^+_n[x] \supset \mathcal{O}^+_{\geq 2} \supset \mathcal{O}^+_{\geq 4} \cdots \supset \mathcal{O}^+_{\geq n}$$

*is the restriction of the above filtrations to* $\mathbb{R}_n^+[x]$;

$$\widehat{\mathcal{O}}^+_{<2} \subset \widehat{\mathcal{O}}^+_{<4} \subset \cdots \subset \widehat{\mathcal{O}}^+_{<n} \subset \widehat{\mathbb{R}}^+_n[x] \supset \widehat{\mathcal{O}}^+_{\geq 2} \supset \widehat{\mathcal{O}}^+_{\geq 4} \cdots \supset \widehat{\mathcal{O}}^+_{\geq n}$$

*is the restriction of the above filtrations to* $\widehat{\mathbb{R}}_n^+[x]$.

*Similarly, for $n$ odd, we have the following two filtrations of* $\mathbb{R}_n[x]$,

$$\mathcal{O}_{<3} \subset \mathcal{O}_{<5} \subset \cdots \subset \mathcal{O}_{<n} \subset \mathbb{R}_n[x] \supset \mathcal{O}_{\geq 3} \supset \mathcal{O}_{\geq 5} \cdots \supset \mathcal{O}_{\geq n},$$

*where* $\mathcal{O}_{<2\ell+1}$ *(resp.* $\mathcal{O}_{\geq 2\ell+1}$*) is the set of all polynomials in* $\mathbb{R}_n[x]$ *with fewer than (resp. at least)* $2\ell+1$ *real roots (counting multiplicities) and their restrictions to* $\widehat{\mathbb{R}}_n[x]$, $\mathbb{R}_n^+[x]$, *and* $\widehat{\mathbb{R}}_n^+[x]$.

**Remark 2.** One can easily see that the terms of the left filtration are open while the terms of the right filtration are closed in $\mathbb{R}_n[x], \widehat{\mathbb{R}}_n[x], \mathbb{R}_n^+[x]$, and $\widehat{\mathbb{R}}_n^+[x]$ resp. Furthermore, the first and the second filtrations are complementary in the following sense. For any positive integer $n$ and any $2 \leq \ell \leq n$, $\ell \equiv n \mod 2$ one has,

$$\mathcal{O}_{<\ell} \cup \mathcal{O}_{\geq \ell} = \mathbb{R}_n[x] \quad \text{and} \quad \mathcal{O}_{<\ell} \cap \mathcal{O}_{\geq \ell} = \emptyset.$$

The intersection $H_\ell = \overline{\mathcal{O}}_{<\ell} \cap \mathcal{O}_{\geq \ell}$ is a semi-algebraic hypersurface in $\mathbb{R}_n[x]$ consisting of all polynomials having exactly $\ell$ real roots counting multiplicities with at least one non-simple real root. (Here $\overline{\mathcal{O}}_{<\ell}$ stands for the closure of the open domain $\mathcal{O}_{<\ell}$).

The hypersurface $H_\ell$ splits into $\ell - 1$ smooth parts depending on the location of the double real root among all the real roots.

**Definition 1.** By a closed non-convex polyhedral cone we mean the union of an arbitrary finite collection of closed polyhedral cones.

**Definition 2.** We say that a closed (probably non-convex) polyhedral cone $\mathcal{C} \subset \mathbb{R}_n[x]$ contained in $\mathcal{O}_{\geq \ell}$ (resp. $\mathcal{O}_{<\ell}$) is inscribed in $\mathcal{O}_{\geq \ell}$ (resp. $\mathcal{O}_{<\ell}$) if one cannot freely deform any of its vertices and still obtain a cone contained in the domain.

Analogously, we say that a closed (probably non-convex) polyhedral cone $\mathcal{C} \subset \mathbb{R}_n[x]$ containing $\mathcal{O}_{\geq \ell}$ (resp. $\mathcal{O}_{<\ell}$) is circumscribed if one can not freely deform any of its vertices and still obtain a cone containing the domain.

The main question we consider below is as follows.

**Problem 1.** *Find interesting examples of inscribed and circumscribed polyhedral cones for the domains* $\mathcal{O}_{<\ell}$ *and* $\mathcal{O}_{\geq \ell}$.

Observe that any such cone provides a generalization of either Hutchinson theorem or Newton inequalities respectively. Below we present some partial, but non-trivial results related to Problem 1. We are currently looking for a more conceptual approach to this question.

**Acknowledgments.** The research of the second named author was supported by the Swedish Research Council grant 2021-04900. He is grateful to Robin Stoll for topological consultations. All three authors want to acknowledge the importance of their long-term contacts with the Professor V. Kostov of Université Côte d'Azur to whom this note is dedicated. We highly appreciate the relevant comments of the anonymous referee which allowed us to improve the overall exposition and correct a number of small mistakes.

## 2. Preliminary results

**Lemma 2.** *The complement* $\Omega_n^+ := \mathbb{R}_n^+[x] \backslash \mathcal{D}_n^+$ *is the union of* $\left[\frac{n}{2}\right] + 1$ *open contractible components* $\Omega_{n,\ell}^+$, $0 \leq \ell \leq n$, $\ell \equiv n \mod 2$, *where* $\Omega_{n,\ell}^+$ *is the set of all monic degree* $n$ *polynomials with positive coefficients having* $\ell$ *simple real (negative) roots.*

**Proof.** Observe that $\Omega_n^+ := \mathbb{R}_n^+[x] \setminus \mathcal{D}_n^+$ is an open set and therefore its connected components are open as well. Obviously the number of real and simple roots is constant within each such connected component. Let us show that the set $\Omega_{n,\ell}^+$ consisting of all monic degree $n$ polynomials with positive coefficients and exactly $\ell$ distinct real roots where $\ell \leq n$ and $\ell \equiv n \mod 2$ is open, non-empty, and contractible. This fact will settle our lemma.

To prove non-emptiness observe that the polynomial

$$p_{n,\ell}(x) = (x+1)(x+2)\cdots(x+\ell)(x^2+1)^{\frac{n-\ell}{2}}$$

belongs to $\Omega_{n,\ell}^+$. To prove contractibility we will show that any compact subset of $\Omega_{n,\ell}^+$ is contractible. (The fact that contractibility of any compact subset in an open subset $S \subseteq \mathbb{R}^n$ implies contractibility of

$S$ follows from Whitehead's theorem together with Corollary IV.5.5 of [7]). To do this we first observe the contractibility of the subset $\widetilde{\Omega^+_{n,\ell}} \subset \Omega^+_{n,\ell}$ consisting of all polynomials of the form

$$p(x) = (x + x_1)(x + x_2) \dots (x + x_\ell)(x^2 + p_1 x + q_1) \dots (x^2 + p_\kappa x + q_\kappa)$$

where $\kappa = \frac{n-\ell}{2}$, $x_1 > 0, x_2 > 0, \dots, x_\ell > 0$, $p_1 > 0, p_2 > 0, \dots, p_\kappa > 0$, and $q_1 > \frac{p_1^2}{4}, \dots, q_\kappa > \frac{p_\kappa^2}{4}$.

Indeed, we can contract $\widetilde{\Omega^+_{n,\ell}}$ to the polynomial $p_{n,\ell}(x) \in \widetilde{\Omega^+_{n,\ell}}$ given by

$$p_{n,\ell}(x) = (x + 1)^\ell (x^2 + x + 1)^\kappa.$$

To do this we (linearly) deform each factor $u_i(x) = (x + x_i)$ into $(x + 1)$ by using the family $u_i(x, t) = t(x + 1) + (1 - t)u_i(x),\ t \in [0, 1]$. Analogously, we can linearly deform each factor $v_i(x) = x^2 + p_i x + q_i$ into $x^2 + x + 1$ by using the family $v_i(x, t) = t(x^2 + x + 1) + (1 - t)v_i(x),\ t \in [0, 1]$. One can easily check that the latter deformation preserves the condition $p > 0$ and $q > \frac{p^2}{4}$.

Now given any compact subset of $S \subset \Omega^+_{n,\ell}$, we can move it within $\Omega^+_{n,\ell}$ into $\widetilde{\Omega^+_{n,\ell}}$ where it can be contracted to $p_{n,\ell}(x)$. Indeed, for any polynomial $p(x)$ consider its deformation $p_t(x) = p(x + t)$ where $t \in [0, +\infty)$. One can check that for any $p(x) \in \Omega^+_{n,\ell}$, there exists $t_p$ such that for all $t > t_p$ the polynomial $p_t(x)$ belongs to $\widetilde{\Omega^+_{n,\ell}}$. One can easily see that $t_p$ depends continuously on $p$. Thus we can move any compact set $S \subset \Omega^+_{n,\ell}$ in $\widetilde{\Omega^+_{n,\ell}}$ and contract it. This fact along with a number of similar statements can be found in [4]. □

**Corollary 1.** *The complement* $\widehat{\Omega}^+_n := \widehat{\mathbb{R}}^+_n[x] \setminus \widehat{\mathcal{D}}^+_n$ *is the union of* $\left[\frac{n}{2}\right] + 1$ *open contractible components* $\widehat{\Omega}^+_{n,\ell},\ 0 \le \ell \le n,\ \ell \equiv n \mod 2$, *where* $\widehat{\Omega}^+_{n,\ell}$ *is the set of all monic degree $n$ polynomials with positive coefficients and constant term 1 having $\ell$ simple real (negative) roots.*

**Proof.** Use the quasihomogeneous action of $\mathbb{R}^+$ on $\mathbb{R}_n[x]$. □

**Remark 3.** Observe that any real polynomial with positive coefficients has only negative real roots.

**Lemma 3.** *(i) In the above notation, the map* $Log|\cdot|$ *sends* $\widehat{\mathbb{R}}^+_{\bar\sigma,\bar\epsilon}$ *to the affine cone given by affine inequalities*

$$2\alpha_j - \alpha_{j-1} - \alpha_{j+1} \lessgtr_{\bar\sigma} \log \epsilon_j,\ j = 1, \dots, n - 1,$$

*where the inequality sign of the $j$-th inequality is given by $\sigma_j$. (Here $\alpha_0 = \alpha_n = 0$).*
*(ii) the affine cone* $Log\left(\widehat{\mathbb{R}}^+_{\bar\sigma,\bar\epsilon}\right)$ *is an (affine) orthant in* $\mathbb{R}^{n-1}$ *in appropriate coordinates;*
*(iii) For two $(n-1)$-tuples $\bar\epsilon^{(1)}$ and $\bar\epsilon^{(2)}$ of positive numbers, the set* $\widehat{\mathbb{R}}^+_{\bar\sigma,\bar\epsilon^{(1)}}$ *is contained in the set* $\widehat{\mathbb{R}}^+_{\bar\sigma,\bar\epsilon^{(2)}}$ *if and only if* $\bar\epsilon^{(1)} \lessgtr_{\bar\sigma} \bar\epsilon^{(2)}$, *which means that* $\epsilon_j^{(1)} \lessgtr_{\bar\sigma} \epsilon_j^{(2)},\ j = 1, \dots, n - 1$ *and the sign of the inequality is determined by $\sigma_j$.*

**Proof.** Item (i) follows immediately by taking the logarithm of the inequalities defining $\widehat{\mathbb{R}}^+_{\bar\sigma,\bar\epsilon}$.

To settle (ii), introduce $\kappa_j := \log q_j = 2\alpha_j - \alpha_{j-1} - \alpha_{j+1},\ j = 1, \dots, n - 1$. We show that $(\kappa_1, \dots, \kappa_{n-1})$ is a coordinate system in $\mathbb{R}^{n-1}$. Indeed, one has the following relation

$$\begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \vdots \\ \kappa_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}.$$

The determinant of the $(n-1)\times(n-1)$-matrix in the right-hand side of the latter equation equals $n$ which can be easily proved by induction which implies item (ii) since $\operatorname{Log}\left(\widehat{\mathbb{R}}^{+}_{\bar\sigma,\bar\epsilon}\right)$ is given by the system of inequalities $\kappa_j \lessgtr_{\bar\sigma} \log\epsilon_j,\ j=1,\dots,n-1$.

Item (iii) follows immediately from comparison of shifted coordinate orthants. □

The following lemma is straightforward.

**Lemma 4.** *Let* $P(x)=\sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$.

*(1) If* $\mathfrak{P}(x)=aP(bx)$ *for some non-zero real values of* $a$ *and* $b$, *then*

$$q_j(\mathfrak{P})=q_j(P),\quad j=1,2,\dots,n-1 \tag{2.1}$$

*(2) If* $\widetilde{P}(x)=x^n P\left(\frac{1}{x}\right)$, *then*

$$q_j(\widetilde{P})=q_{n-j}(P),\quad j=1,2,\dots,n-1. \tag{2.2}$$

## 3. Main results

### 3.1. Previously known facts

In [3] the first and the third authors proved the following statements.

**Theorem A.** *For* $P(x)=\sum_{k=0}^{2m} a_k x^k \in \mathbb{R}_+[x]$, *if the inequalities*

$$q_{2k+1} < \frac{1}{\cos^2\left(\frac{\pi}{m+2}\right)} \tag{3.1}$$

*hold for all* $k=0,1,\dots,m$, *then* $P(x)>0$ *for each real value of* $x$, *i.e.* $P(x)\in\mathcal{O}^{+}_{<2}$.

**Theorem B.** *For* $P(x)=\sum_{k=0}^{2m+1} a_k x^k \in \mathbb{R}_+[x]$, *if the inequalities*

$$q_{2k} < \frac{4k^2-1}{4k^2}\frac{1}{\cos^2\left(\frac{\pi}{m+2}\right)} \tag{3.2}$$

*hold for all* $k=1,2,\dots,m$, *then* $P(x)$ *has exactly one real (negative) root (counting its multiplicity), i.e.* $P(x)\in\mathcal{O}^{+}_{<3}$.

**Theorem C.** *The constants* $\frac{1}{\cos^2\left(\frac{\pi}{m+2}\right)}$ *in Theorem A and* $\frac{4k^2-1}{4k^2}\frac{1}{\cos^2\left(\frac{\pi}{m+2}\right)}$ *in Theorem B are sharp for every* $m\in\mathbb{N}$.

**Corollary 2.** *The assumptions of Theorems A and B can be interpreted as the fact that the logarithmic image of the polynomials under consideration belongs to the polyhedral cone given as follows:*
*In case of Theorem A:* $2\alpha_{2k+1}-\alpha_{2k}-\alpha_{2k+2} < -2\ln(\cos\left(\frac{\pi}{m+2}\right)),\ k=0,1,\dots,m.$
*In case of Theorem B:* $2\alpha_{2k}-\alpha_{2k-1}-\alpha_{2k+1} < -2\ln(\cos\left(\frac{\pi}{m+2}\right))+\ln(\frac{4k^2-1}{4k^2}),\ k=1,2,\dots,m.$

*Theorem C guarantees that there exist small deformations of the above polyhedral cones which are not contained in the logarithmic image of* $\mathcal{O}^{+}_{<2}$ *and* $\mathcal{O}^{+}_{<3}$ *respectively.*

Theorems A and B can be generalized to the case when some coefficients are allowed to be negative. Namely, the following modifications of these results hold.

**Theorem D.** *For* $P(x)=\sum_{k=0}^{2m} a_k x^k \in \mathbb{R}[x]$, *assume that* $a_{2k}>0, \quad k=0,1,\ldots,m$. *If the inequalities*

$$q_{2k+1} < \frac{1}{\cos^2\left(\frac{\pi}{m+2}\right)} \tag{3.3}$$

*hold for all* $k=0,1,\ldots,m$, *then* $P(x)>0$ *for each real value of* $x$, *i.e.* $P(x)\in\mathcal{O}_{<2}$.

**Theorem E.** *Given* $P(x)=\sum_{k=0}^{2m+1} a_k x^k \in \mathbb{R}[x]$, *assume that* $a_{2k+1}>0, \quad k=0,1,\ldots,m$. *If the inequalities*

$$q_{2k} < \frac{4k^2-1}{4k^2}\frac{1}{\cos^2\left(\frac{\pi}{m+2}\right)} \tag{3.4}$$

*hold for all* $k=1,2,\ldots,m$, *then* $P(x)$ *has exactly one real root (counting its multiplicity), i.e.* $P(x)\in\mathcal{O}_{<3}$.

The next result follows from Theorem E and the second statement of Lemma 4.

**Theorem F.** *Given* $P(x)=\sum_{k=0}^{2m+1} a_k x^k \in \mathbb{R}[x]$, *assume that* $a_{2k+1}>0, \quad k=0,1,\ldots,m$. *If the inequalities*

$$q_{2m+1-2k} < \frac{4k^2-1}{4k^2}\frac{1}{\cos^2\left(\frac{\pi}{m+2}\right)} \tag{3.5}$$

*hold for all* $k=1,2,\ldots,m$, *then* $P(x)$ *has exactly one real root (counting its multiplicity), i.e.* $P(x)\in\mathcal{O}_{<3}$.

### *3.2. New results*

We will use the following notation.

For each real polynomial $P(x)$, we will denote by $\sharp_r(P)$ the number of its real roots (counting their multiplicities).

**Theorem 5.** *Assume that* $P(x)=\sum_{k=0}^{n} a_k x^k \in \mathbb{R}_n^+[x]$, *where* $n\geq 4$. *If* $\sharp_r(P)\geq n-2$, *then*

$$q_1+q_{n-1}\geq 4\frac{n^2-3n}{(n-2)^2}. \tag{3.6}$$

*In particular,*

$$\textit{either} \quad q_1\geq\frac{2n^2-6n}{(n-2)^2} \quad \textit{or} \quad q_{n-1}\geq\frac{2n^2-6n}{(n-2)^2}. \tag{3.7}$$

The polyhedral domain given by inequalities (3.7) contains $\mathcal{O}^+_{\geq n-2}$. Below we will split Theorem 5 into Theorems 8 and 9 covering the cases of even and odd degrees respectively.

As an immediate consequence of Theorem 5 we get the following corollary.

**Corollary 3.** *If*

$$\max(q_1,q_{n-1})<\frac{2n^2-6n}{(n-2)^2}, \tag{3.8}$$

*then*

$$\sharp_r(P) \leq n-4.$$

The polyhedral domain given by these inequalities is contained in $\mathcal{O}^+_{<n-2}$.

**Corollary 4.** *Assume that* $P_n(x) = \sum_{k=0}^n a_k x^k \in \mathbb{R}_+[x]$, *where* $n \geq 4$. *If for some* $m = 2, 3, \dots, n-2$ *and for some* $j = 1, 2, \dots, n-m-1$, *the following two estimations are valid*

$$q_j < \frac{(m-1)(m+1)}{m^2} \cdot \frac{n-j+1}{n-j} \cdot \frac{j+1}{j}, \tag{3.9}$$

*and*

$$q_{m+j} < \frac{(m-1)(m+1)}{m^2} \cdot \frac{n-m-j+1}{n-m-j} \cdot \frac{m+j+1}{m+j}, \tag{3.10}$$

*then*

$$\sharp_r(P_n) \leq n-4.$$

For the proof of Corollary 4 see § 4.

**Theorem 6.** *The estimate in Theorem 5 is sharp.*

**Proof.** The following example

$$P_n(x) = (x+1)^{n-2}\left(x^2 + \frac{n-4}{n-2}x + 1\right) \tag{3.11}$$

with $\sharp_r(P_n) = n-2$ and $q_1 = q_{n-1} = \frac{2n(n-3)}{(n-2)^2}$ settles Theorem 6. □

**Proposition 7.** *The constants in Corollary 4 are sharp.*

For the proof of Proposition 7 see § 4.

Theorem 5 is equivalent to the following 2 statements whose rather lengthy proofs can be found in § 4.

**Theorem 8.** *If*

$$P_{2n} = \left(x^2 + 2tx + \frac{1}{b_1 b_2 \cdots b_{n-1}}\right) \prod_{j=1}^{n-1} (x^2 + 2a_j x + b_j), \tag{3.12}$$

*where* $a_j > 0, \quad b_j > 0, \quad a_j^2 \geq b_j, \quad j = 1, 2, \dots, n-1,$ *then*

$$q_1 + q_{2n-1} \geq 2 \cdot \frac{2n^2 - 3n}{(n-1)^2}. \tag{3.13}$$

*In particular,*

$$\text{either} \quad q_1 \geq \frac{2n^2-3n}{(n-1)^2} \quad \text{or} \quad q_{2n-1} \geq \frac{2n^2-3n}{(n-1)^2}. \tag{3.14}$$

**Theorem 9.** *If*

$$P_{2n+1} = (x+c)\left(x^2 + 2tx + \frac{1}{b_1 b_2 \cdots b_{n-1} c}\right) \prod_{j=1}^{n-1} (x^2 + 2a_j x + b_j), \tag{3.15}$$

*where* $c > 0, \quad a_j > 0, \quad b_j > 0, \quad a_j^2 \geq b_j, \quad j = 1, 2, \ldots, n-1,$ *then*

$$q_1 + q_{2n} \geq 8 \cdot \frac{2n^2 - n - 1}{(2n-1)^2}. \tag{3.16}$$

*In particular,*

$$\textit{either} \;\; q_1 \geq 4 \cdot \frac{2n^2 - n - 1}{(2n-1)^2} \;\; \textit{or} \;\; q_{2n} \geq 4 \cdot \frac{2n^2 - n - 1}{(2n-1)^2}. \tag{3.17}$$

**Notation 4.** *We will use the following standard notation for the symmetric functions. For* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$*, set*

$$\sigma_1(\alpha) = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}; \tag{3.18}$$

$$\sigma_2(\alpha) = \prod_{1 \leq i < j \leq n-1} \alpha_i \alpha_j; \tag{3.19}$$

$$S_2(\alpha) = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_{n-1}^2. \tag{3.20}$$

The next claim is standard.

**Lemma 10.** *In the above notation,* $\sigma_1^2(\alpha) = S_2(\alpha) + 2\sigma_2(\alpha)$ *and* $2\sigma_2(\alpha) \leq (n-2) S_2(\alpha)$*.*

To formulate our next result, notice that for $P(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}_+[x]$, one obtains

$$a_k = a_1 \left(\frac{a_1}{a_0}\right)^{k-1} \frac{1}{q_1^{k-1} q_2^{k-2} \cdot \ldots \cdot q_{k-2}^2 q_{k-1}}, \quad \forall k, 2 \leq k \leq n-1. \tag{3.21}$$

The following proposition is an analog of the Hutchinson theorem.

**Proposition 11.** *Let* $P(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}_+[x]$ *and suppose that* $q_k(P) \geq 1$ *for all* $k, 1 \leq k \leq n-1$*. If for some* $j, 1 \leq j \leq n-1$*, we have* $q_j(P) \geq 4$*, then there exists a point* $x_j \in \left(-\frac{a_j}{a_{j+1}}, -\frac{a_{j-1}}{a_j}\right)$ *such that* $(-1)^j P(x_j) \geq 0$*.*

For the proof of Proposition 11 see § 4.

## 4. Proofs

**Proof of Theorem 8.** We will deal with the sequences:

$$\begin{gathered} a = (a_1, a_2, \ldots, a_{n-1}), \quad b = (b_1, b_2, \ldots, b_{n-1}), \\ \frac{1}{b} = \left(\frac{1}{b_1}, \frac{1}{b_2}, \ldots, \frac{1}{b_{n-1}}\right), \quad \frac{a}{b} = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_{n-1}}{b_{n-1}}\right). \\ ab^{n-2} = (a_1 b_2 b_3 \cdots b_{n-1}, \; a_2 b_1 b_3 \cdots b_{n-1}, \ldots, \; a_{n-1} b_1 b_2 \cdots b_{n-2}). \end{gathered} \tag{4.1}$$

Hence $\sigma_1(\frac{a}{b})b_1b_2\cdots b_{n-1} = \sigma_1(ab^{n-2})$. We have

$$\begin{aligned} P_{2n} = \left(x^2 + 2tx + \frac{1}{b_1b_2\cdots b_{n-1}}\right)\prod_{j=1}^{n-1}(x^2+2a_jx+b_j) \\ = x^{2n} + 2x^{2n-1}(\sigma_1(a)+t) \\ +x^{2n-2}\left(4t\sigma_1(a) + 4\sigma_2(a) + \sigma_1(b) + \frac{1}{b_1b_2\cdots b_{n-1}}\right) \\ +\cdots + x^2\left(4t\sigma_1(ab^{n-2}) + 4\sigma_2\left(\frac{a}{b}\right) + \sigma_1\left(\frac{1}{b}\right) + b_1b_2\cdots b_{n-1}\right) \\ +2x\left(\sigma_1\left(\frac{a}{b}\right) + b_1b_2\cdots b_{n-1}t\right) + 1. \end{aligned}$$

Therefore,

$$q_{2n-1} = \frac{(\sigma_1(a)+t)^2}{t\sigma_1(a) + \sigma_2(a) + \frac{1}{4}\cdot\sigma_1(b) + \frac{1}{4}\frac{1}{b_1b_2\cdots b_{n-1}}} \tag{4.2}$$

and

$$q_1 = \frac{\left(\sigma_1\left(\frac{a}{b}\right) + b_1b_2\cdots b_{n-1}t\right)^2}{t\sigma_1(ab^{n-2}) + \sigma_2\left(\frac{a}{b}\right) + \frac{1}{4}\cdot\sigma_1\left(\frac{1}{b}\right) + \frac{1}{4}b_1b_2\cdots b_{n-1}}. \tag{4.3}$$

Identities (4.2) and (4.3) can be rewritten in the form:

$$\begin{aligned} t^2 + (2-q_{2n-1})\sigma_1(a)t + \sigma_1^2(a) - q_{2n-1}\sigma_2(a) \\ -\frac{1}{4}q_{2n-1}\cdot\sigma_1(b) - \frac{q_{2n-1}}{4b_1b_2\cdots b_{n-1}} = 0 \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} (b_1b_2\cdots b_{n-1})^2t^2 + (2-q_1)\sigma_1(ab^{n-2})t + \sigma_1^2\left(\frac{a}{b}\right) \\ -q_1\sigma_2\left(\frac{a}{b}\right) - \frac{1}{4}q_1\cdot\sigma_1\left(\frac{1}{b}\right) - \frac{q_1}{4}\cdot b_1b_2\cdots b_{n-1} = 0. \end{aligned} \tag{4.5}$$

We have two quadratic equations with respect to the variable $t$. Given that these equations have real solutions, i.e. that their discriminants are non-negative, we obtain the following two inequalities:

$$(2-q_{2n-1})^2\sigma_1^2(a) - 4\sigma_1^2(a) + 4q_{2n-1}\sigma_2(a) + q_{2n-1}\cdot\sigma_1(b) + \frac{q_{2n-1}}{b_1b_2\cdots b_{n-1}} \geq 0 \tag{4.6}$$

and

$$\begin{aligned} (2-q_1)^2\sigma_1^2(ab^{n-2}) - 4(b_1b_2\cdots b_{n-1})^2\sigma_1^2\left(\frac{a}{b}\right) \\ +q_1\cdot(b_1b_2\cdots b_{n-1})^2\left(4\sigma_2\left(\frac{a}{b}\right) + \sigma_1\left(\frac{1}{b}\right) + b_1b_2\cdots b_{n-1}\right) \geq 0. \end{aligned} \tag{4.7}$$

The inequality (4.6) is equivalent to the estimate

$$q_{2n-1} \geq 4 - \frac{4\sigma_2(a) + \sigma_1(b) + \frac{1}{b_1b_2\cdots b_{n-1}}}{\sigma_1^2(a)}. \tag{4.8}$$

Note that

$$(b_1 b_2 \cdots b_{n-1})^2 \sigma_1^2\left(\frac{a}{b}\right) = \sigma_1^2(ab^{n-2}), \tag{4.9}$$

and

$$(b_1 b_2 \cdots b_{n-1})^2 \sigma_2\left(\frac{a}{b}\right) = \sigma_2(ab^{n-2}). \tag{4.10}$$

So, we can rewrite the inequality (4.7) in the form

$$q_1 \geq 4 - \frac{4\sigma_2(ab^{n-2}) + (b_1 b_2 \cdots b_{n-1})^2 \sigma_1\left(\frac{1}{b}\right) + (b_1 b_2 \cdots b_{n-1})^3}{\sigma_1^2(ab^{n-2})}. \tag{4.11}$$

Given that $a_j^2 \geq b_j, \quad j = 1, 2, \ldots, n-1$, the following inequalities hold:

$$\sigma_1(b) \leq S_2(a) \tag{4.12}$$

and

$$(b_1 b_2 \cdots b_{n-1})^2 \sigma_1\left(\frac{1}{b}\right) \leq S_2(ab^{n-2}). \tag{4.13}$$

Thus, it follows from (4.8) and the first statement of Lemma 10 that

$$q_{2n-1} \geq 4 - \frac{4\sigma_2(a) + S_2(a) + \frac{1}{b_1 b_2 \cdots b_{n-1}}}{S_2(a) + 2\sigma_2(a)} \geq 3 - \frac{2\sigma_2(a) + \frac{1}{b_1 b_2 \cdots b_{n-1}}}{S_2(a) + 2\sigma_2(a)} \tag{4.14}$$

and from (4.11) that

$$\begin{aligned} q_1 &\geq 4 - \frac{4\sigma_2(ab^{n-2}) + S_2(ab^{n-2}) + (b_1 b_2 \cdots b_{n-1})^3}{S_2(ab^{n-2}) + 2\sigma_2(ab^{n-2})} \\ &\geq 3 - \frac{2\sigma_2(ab^{n-2}) + (b_1 b_2 \cdots b_{n-1})^3}{S_2(ab^{n-2}) + 2\sigma_2(ab^{n-2})}. \end{aligned} \tag{4.15}$$

Inequalities (4.14) and (4.15) imply that the statement of Theorem 8 would follow from the fact that one of the two inequalities below is valid: either

$$\frac{2\sigma_2(a) + \frac{1}{b_1 b_2 \cdots b_{n-1}}}{S_2(a) + 2\sigma_2(a)} \leq \frac{n^2 - 3n + 3}{(n-1)^2} \tag{4.16}$$

or

$$\frac{2\sigma_2(ab^{n-2}) + (b_1 b_2 \cdots b_{n-1})^3}{S_2(ab^{n-2}) + 2\sigma_2(ab^{n-2})} \leq \frac{n^2 - 3n + 3}{(n-1)^2}. \tag{4.17}$$

The inequality (4.16) is equivalent to

$$1 \leq \frac{b_1 b_2 \cdots b_{n-1}}{(n-1)^2}\left((n^2 - 3n + 3) S_2(a) - (n-2) \cdot 2\sigma_2(a)\right), \tag{4.18}$$

while (4.17) is equivalent to the inequality

$$1 \le \frac{1}{(n-1)^2(b_1b_2\cdots b_{n-1})^3}\left((n^2-3n+3)S_2(ab^{n-2}) - (n-2)\cdot 2\sigma_2(ab^{n-2})\right). \tag{4.19}$$

Note that applying the second statement of Lemma 10 and after that inequalities $a_j^2 \ge b_j, \quad j = 1, 2, \ldots, n-1$, we get

$$\begin{aligned}
&\frac{b_1b_2\cdots b_{n-1}}{(n-1)^2}\left((n^2-3n+3)S_2(a) - (n-2)\cdot 2\sigma_2(a)\right)\\
&\ge \frac{b_1b_2\cdots b_{n-1}}{(n-1)^2}\left((n^2-3n+3)S_2(a) - (n-2)^2S_2(a)\right)\\
&= \frac{b_1b_2\cdots b_{n-1}}{(n-1)}S_2(a) \ge \frac{b_1b_2\cdots b_{n-1}\sigma_1(b)}{(n-1)}\\
&= \frac{1}{n-1}\left(b_1^2b_2\cdots b_{n-1} + b_1b_2^2\cdots b_{n-1} + b_1b_2\cdots b_{n-1}^2\right),
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&\frac{1}{(n-1)^2(b_1b_2\cdots b_{n-1})^3}\left((n^2-3n+3)S_2(ab^{n-2}) - (n-2)\cdot 2\sigma_2(ab^{n-2})\right)\\
&\ge \frac{1}{(n-1)^2(b_1b_2\cdots b_{n-1})^3}\left((n^2-3n+3)S_2(ab^{n-2}) - (n-2)^2S_2(ab^{n-2})\right)\\
&= \frac{1}{(n-1)(b_1b_2\cdots b_{n-1})^3}S_2(ab^{n-2})\\
&\ge \frac{\left(b_1b_2^2\cdots b_{n-1}^2 + b_1^2b_2\cdots b_{n-1}^2 + b_1^2b_2^2\cdots b_{n-1}^2\right)}{(n-1)(b_1b_2\cdots b_{n-1})^3}\\
&= \frac{1}{n-1}\left(b_1^{-2}b_2^{-1}\cdots b_{n-1}^{-1} + b_1^{-1}b_2^{-2}\cdots b_{n-1}^{-1} + b_1^{-1}b_2^{-1}\cdots b_{n-1}^{-2}\right).
\end{aligned} \tag{4.21}$$

Denote by

$$B_1 =: \frac{1}{n-1}\left(b_1^2b_2\cdots b_{n-1} + b_1b_2^2\cdots b_{n-1} + b_1b_2\cdots b_{n-1}^2\right) \tag{4.22}$$

and by

$$B_2 =: \frac{1}{n-1}\left((b_1^2b_2\cdots b_{n-1})^{-1} + (b_1b_2^2\cdots b_{n-1})^{-1} + (b_1b_2\cdots b_{n-1}^2)^{-1}\right). \tag{4.23}$$

Theorem 8 will be proved if we show that $B_1 + B_2 \ge 2$. Indeed, since for each $x > 0$, the inequality $x + \frac{1}{x} \ge 2$ is valid, we get

$$B_1 + B_2 \ge \frac{1}{n-1}\cdot 2(n-1) = 2.$$

In particular, at least one of the numbers $B_1$ and $B_2$ must be bigger than 1.

Theorem 8 follows. □

**Proof of Theorem 9.** (The logic of this proof is very similar to that of Theorem 8).

We have

$$\begin{aligned}
P_{2n+1} &= (x+c)\left(x^2 + 2tx + \frac{1}{b_1b_2\cdots b_{n-1}c}\right)\prod_{j=1}^{n-1}(x^2 + 2a_jx + b_j)\\
&= x^{2n+1} + 2x^{2n}(\sigma_1(a) + \frac{c}{2} + t) +
\end{aligned}$$

$$x^{2n-1}\left(4t\left(\sigma_1(a)+\frac{c}{2}\right)+4\sigma_2(a)+2c\sigma_1(a)+\sigma_1(b)+\frac{1}{b_1b_2\cdots b_{n-1}c}\right)+\cdots+$$
$$x^2\left(4t\left(c\sigma_1(ab^{n-2})+\frac{b_1b_2\dots b_{n-1}}{2}\right)+4\sigma_2\left(\frac{a}{b}\right)+\frac{2}{c}\sigma_1\left(\frac{a}{b}\right)+\sigma_1\left(\frac{1}{b}\right)+cb_1b_2\cdots b_{n-1}\right)$$
$$+2x\left(\sigma_1\left(\frac{a}{b}\right)+\frac{1}{2c}+cb_1b_2\cdots b_{n-1}t\right)+1.$$

Therefore,

$$q_{2n}=\frac{(\sigma_1(a)+\frac{c}{2}+t)^2}{t(\sigma_1(a)+\frac{c}{2})+\sigma_2(a)+\frac{c}{2}\sigma_1(a)+\frac{1}{4}\cdot\sigma_1(b)+\frac{1}{4}\frac{1}{cb_1b_2\cdots b_{n-1}}} \tag{4.24}$$

and

$$q_1=\frac{\left(\sigma_1\left(\frac{a}{b}\right)+\frac{1}{2c}+cb_1b_2\cdots b_{n-1}t\right)^2}{t\left(c\sigma_1(ab^{n-2})+\frac{b_1b_2\dots b_{n-1}}{2}\right)+\sigma_2\left(\frac{a}{b}\right)+\frac{\sigma_1\left(\frac{a}{b}\right)}{2c}+\frac{1}{4}\cdot\sigma_1\left(\frac{1}{b}\right)+\frac{c}{4}b_1b_2\cdots b_{n-1}}. \tag{4.25}$$

The identities (4.24) and (4.25) can be rewritten in the form:

$$\begin{gathered}t^2+(2-q_{2n})\left(\sigma_1(a)+\frac{c}{2}\right)t+\left(\sigma_1(a)+\frac{c}{2}\right)^2-q_{2n}\sigma_2(a)\\ -\frac{c}{2}q_{2n}\sigma_1(a)-\frac{1}{4}q_{2n}\cdot\sigma_1(b)-\frac{q_{2n}}{4cb_1b_2\cdots b_{n-1}}=0\end{gathered} \tag{4.26}$$

and

$$\begin{gathered}(cb_1b_2\cdots b_{n-1})^2t^2+(2-q_1)\left(c\sigma_1(ab^{n-2})+\frac{b_1b_2\cdots b_{n-1}}{2}\right)t\\ +\left(\sigma_1\left(\frac{a}{b}\right)+\frac{1}{2c}\right)^2-q_1\sigma_2\left(\frac{a}{b}\right)-\frac{q_1}{2c}\sigma_1\left(\frac{a}{b}\right)\\ -\frac{1}{4}q_1\cdot\sigma_1\left(\frac{1}{b}\right)-\frac{cq_1}{4}\cdot b_1b_2\cdots b_{n-1}=0.\end{gathered} \tag{4.27}$$

We have two quadratic equations with respect to the variable $t$. Given that these equations have real solutions, i.e. that their discriminants are non-negative, we obtain the inequalities:

$$\begin{gathered}(2-q_{2n})^2\left(\sigma_1(a)+\frac{c}{2}\right)^2-4\left(\sigma_1(a)+\frac{c}{2}\right)^2\\ +4q_{2n}\sigma_2(a)+2cq_{2n}\sigma_1(a)+q_{2n}\cdot\sigma_1(b)+\frac{q_{2n}}{cb_1b_2\cdots b_{n-1}}\geq 0\end{gathered} \tag{4.28}$$

and

$$\begin{gathered}(2-q_1)^2\left(c\sigma_1(ab^{n-2})+\frac{b_1b_2\dots b_{n-1}}{2}\right)^2-4(cb_1b_2\cdots b_{n-1})^2\left(\sigma_1\left(\frac{a}{b}\right)+\frac{1}{2c}\right)^2\\ +q_1\cdot(b_1b_2\cdots b_{n-1})^2\left(4c^2\sigma_2\left(\frac{a}{b}\right)+2c\sigma_1\left(\frac{a}{b}\right)+c^2\sigma_1\left(\frac{1}{b}\right)+c^3b_1b_2\cdots b_{n-1}\right)\geq 0.\end{gathered} \tag{4.29}$$

Inequality (4.28) is equivalent to the estimate

$$q_{2n}\geq 4-\frac{4\sigma_2(a)+2c\sigma_1(a)+\sigma_1(b)+\frac{1}{cb_1b_2\cdots b_{n-1}}}{\left(\sigma_1(a)+\frac{c}{2}\right)^2}. \tag{4.30}$$

Using (4.9) and (4.10) we can rewrite (4.29) in the form

$$\begin{aligned} q_1 \geq 4 &- \frac{4c^2\sigma_2(ab^{n-2}) + 2(cb_1b_2\cdots b_{n-1})\sigma_1(ab^{n-2})}{\left(c\sigma_1(ab^{n-2}) + \frac{b_1b_2\ldots b_{n-1}}{2}\right)^2} \\ &- \frac{(cb_1b_2\cdots b_{n-1})^2\sigma_1\left(\frac{1}{b}\right) + (cb_1b_2\cdots b_{n-1})^3}{\left(c\sigma_1(ab^{n-2}) + \frac{b_1b_2\ldots b_{n-1}}{2}\right)^2}. \end{aligned} \tag{4.31}$$

Applying (4.12), (4.13) and Lemma 10 to (4.30) we obtain

$$\begin{aligned} q_{2n} &\geq 4 - \frac{4\sigma_2(a) + 2c\sigma_1(a) + S_2(a) + \frac{1}{cb_1b_2\cdots b_{n-1}}}{S_2(a) + 2\sigma_2(a) + c\sigma_1(a) + \frac{c^2}{4}} \\ &\geq 3 - \frac{2\sigma_2(a) + c\sigma_1(a) + \frac{1}{cb_1b_2\cdots b_{n-1}} - \frac{c^2}{4}}{S_2(a) + 2\sigma_2(a) + c\sigma_1(a) + \frac{c^2}{4}}. \end{aligned} \tag{4.32}$$

Using the same idea we can derive the inequality below from (4.31)

$$\begin{aligned} q_1 \geq 4 &- \frac{4c^2\sigma_2(ab^{n-2}) + 2(cb_1b_2\cdots b_{n-1})\sigma_1(ab^{n-2})}{c^2S_2(ab^{n-2}) + 2c^2\sigma_2(ab^{n-2}) + (cb_1b_2\cdots b_{n-1})\sigma_1(ab^{n-2}) + \frac{1}{4}(b_1b_2\cdots b_{n-1})^2} \\ &- \frac{c^2S_2(ab^{n-2}) + (cb_1b_2\cdots b_{n-1})^3}{c^2S_2(ab^{n-2}) + 2c^2\sigma_2(ab^{n-2}) + (cb_1b_2\cdots b_{n-1})\sigma_1(ab^{n-2}) + \frac{1}{4}(b_1b_2\cdots b_{n-1})^2} \\ \geq 3 &- \frac{2\sigma_2(ab^{n-2}) + \frac{b_1b_2\cdots b_{n-1}}{c}\sigma_1(ab^{n-2}) + c(b_1b_2\cdots b_{n-1})^3 - \frac{(b_1b_2\cdots b_{n-1})^2}{4c^2}}{S_2(ab^{n-2}) + 2\sigma_2(ab^{n-2}) + \frac{b_1b_2\cdots b_{n-1}}{c}\sigma_1(ab^{n-2}) + \frac{(b_1b_2\cdots b_{n-1})^2}{4c^2}}. \end{aligned} \tag{4.33}$$

It follows from (4.32) and (4.33) that the statement of Theorem 9 would follow from the fact that one of the two inequalities below is valid: either

$$\frac{2\sigma_2(a) + c\sigma_1(a) + \frac{1}{cb_1b_2\cdots b_{n-1}} - \frac{c^2}{4}}{S_2(a) + 2\sigma_2(a) + c\sigma_1(a) + \frac{c^2}{4}} \leq \frac{4n^2 - 8n + 7}{(2n-1)^2} \tag{4.34}$$

or

$$\begin{aligned} &\frac{2\sigma_2(ab^{n-2}) + \frac{b_1b_2\cdots b_{n-1}}{c}\sigma_1(ab^{n-2}) + c(b_1b_2\cdots b_{n-1})^3 - \frac{(b_1b_2\cdots b_{n-1})^2}{4c^2}}{S_2(ab^{n-2}) + 2\sigma_2(ab^{n-2}) + \frac{b_1b_2\cdots b_{n-1}}{c}\sigma_1(ab^{n-2}) + \frac{(b_1b_2\cdots b_{n-1})^2}{4c^2}} \\ &\qquad\leq \frac{4n^2 - 8n + 7}{(2n-1)^2}. \end{aligned} \tag{4.35}$$

The inequality (4.34) is equivalent to the following one

$$\begin{aligned} 1 \leq \frac{cb_1b_2\cdots b_{n-1}}{(2n-1)^2}&((4n^2 - 8n + 7)S_2(a) \\ -(4n-6)\cdot 2\sigma_2(a) - (4n-6)c\sigma_1(a) &+ (2n^2 - 3n + 2)c^2) \end{aligned} \tag{4.36}$$

while (4.35) is equivalent to the inequality

$$1 \leq \frac{1}{(2n-1)^2c(b_1b_2\cdots b_{n-1})^3}((4n^2 - 8n + 7)S_2(ab^{n-2}) - (4n-6)\cdot 2\sigma_2(ab^{n-2})$$

$$-(4n-6)\frac{b_1b_2\cdots b_{n-1}}{c}\sigma_1(ab^{n-2})+(2n^2-3n+2)\frac{(b_1b_2\cdots b_{n-1})^2}{c^2}). \tag{4.37}$$

Applying the second statement of Lemma 10 and $2c\sigma_1(a)\le S_2(a)+(n-1)c^2$, and after that inequalities $a_j^2\ge b_j,\quad j=1,2,\dots,n-1$, to the right-hand part of the above inequalities we will have

$$\begin{aligned}
&\frac{cb_1b_2\cdots b_{n-1}}{(2n-1)^2}((4n^2-8n+7)S_2(a)-(4n-6)\cdot 2\sigma_2(a)\\
&-(4n-6)c\sigma_1(a)+(2n^2-3n+2)c^2))\ge\frac{cb_1b_2\cdots b_{n-1}}{(2n-1)^2}((4n^2-8n+7)S_2(a)\\
&-(4n-6)(n-2)S_2(a)-(2n-3)(S_2(a)+(n-1)c^2)+(2n^2-3n+2)c^2)\\
&=\frac{cb_1b_2\cdots b_{n-1}}{(2n-1)^2}((4n-2)S_2(a)+(2n-1)c^2)\\
&\ge\frac{cb_1b_2\cdots b_{n-1}}{(2n-1)^2}((4n-2)\sigma_1(b)+(2n-1)c^2)\\
&\ge\frac{2}{2n-1}\left(cb_1^2b_2\cdots b_{n-1}+cb_1b_2^2\cdots b_{n-1}+cb_1b_2\cdots b_{n-1}^2+\frac{c^3b_1b_2\cdots b_{n-1}}{2}\right)
\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}
&\frac{1}{(2n-1)^2c(b_1b_2\cdots b_{n-1})^3}((4n^2-8n+7)S_2(ab^{n-2})-(4n-6)\cdot 2\sigma_2(ab^{n-2})\\
&-(4n-6)\frac{b_1b_2\cdots b_{n-1}}{c}\sigma_1(ab^{n-2})+(2n^2-3n+2)\frac{(b_1b_2\cdots b_{n-1})^2}{c^2})\\
&\ge\frac{1}{(2n-1)^2c(b_1b_2\cdots b_{n-1})^3}((4n^2-8n+7)S_2(ab^{n-2})-(4n-6)(n-2)S_2(ab^{n-2})\\
&-(2n-3)\left((n-1)\frac{(b_1b_2\cdots b_{n-1})^2}{c^2}+S_2(ab^{n-2})\right)+(2n^2-3n+2)\frac{(b_1b_2\cdots b_{n-1})^2}{c^2})\\
&=\frac{1}{(2n-1)^2c(b_1b_2\cdots b_{n-1})^3}\left((4n-2)S_2(ab^{n-2})+(2n-1)\frac{(b_1b_2\cdots b_{n-1})^2}{c^2}\right)\\
&\ge\frac{2}{2n-1}\left(\frac{(b_1b_2^2\cdots b_{n-1}^2+b_1^2b_2b_3^2\cdots b_{n-1}^2+b_1^2b_2^2\cdots b_{n-2}^2b_{n-1})}{c(b_1b_2\cdots b_{n-1})^3}+\frac{1}{2c^3b_1b_2\cdots b_{n-1}}\right)\\
&=\tfrac{2}{2n-1}\left((cb_1^2b_2\cdots b_{n-1})^{-1}+(cb_1b_2^2b_3\cdots b_{n-1})^{-1}+(cb_1b_2\cdots b_{n-2}b_{n-1}^2)^{-1}\right.\\
&\qquad\left.+\tfrac{1}{2c^3b_1b_2\cdots b_{n-1}}\right).
\end{aligned} \tag{4.39}$$

Denote by

$$B_1=\frac{2}{2n-1}\left(cb_1^2b_2\cdots b_{n-1}+cb_1b_2^2\cdots b_{n-1}+cb_1b_2\cdots b_{n-1}^2+\frac{c^3b_1b_2\cdots b_{n-1}}{2}\right)$$

and

$$B_2=\frac{2}{2n-1}\left((cb_1^2b_2\cdots b_{n-1})^{-1}+(cb_1b_2^2\cdots b_{n-1})^{-1}+(cb_1b_2\cdots b_{n-1}^2)^{-1}+\frac{1}{2c^3b_1b_2\cdots b_{n-1}}\right).$$

Theorem 9 will be proved if we show that $B_1+B_2\ge 1$. Indeed, since for each $x>0$ the inequality $x+\frac{1}{x}\ge 2$ is valid, we have

$$B_1 + B_2 \geq \frac{2}{2n-1} \cdot (2(n-1)+1) = 2.$$

In particular, at least one of the numbers $B_1$ and $B_2$ must be bigger than 1.

Theorem 9 is proved. □

Next let us show how to derive Corollary 4 from Corollary 3.

**Proof of Corollary 4.** Consider the polynomial

$$\begin{aligned} P(x) = a_n x^n + \cdots + a_{m+j+1}x^{m+j+1} + a_{m+j}x^{m+j} + a_{m+j-1}x^{m+j-1} \\ + \cdots + a_{j+1}x^{j+1} + a_j x^j + a_{j-1}x^{j-1} + \cdots + a_0. \end{aligned} \tag{4.40}$$

Differentiating the polynomial $P(x)$ $(j-1)$ times we get:

$$\begin{aligned} P^{(j-1)}(x) = \frac{n!}{(n-j+1)!}a_n x^{n-j+1} + \cdots + \frac{(m+j+1)!}{(m+2)!}a_{m+j+1}x^{m+2} \\ + \frac{(m+j)!}{(m+1)!}a_{m+j}x^{m+1} + \frac{(m+j-1)!}{m!}a_{m+j-1}x^m \\ + \cdots + \frac{(j+1)!}{2!}a_{j+1}x^2 + \frac{j!}{1!}a_j x + (j-1)!a_{j-1}. \end{aligned} \tag{4.41}$$

Consider the polynomial

$$\begin{aligned} Q_{n-j+1}(x) = x^{n-j+1}P\left(\frac{1}{x}\right) \\ = \frac{n!}{(n-j+1)!}a_n + \cdots + \frac{(m+j+1)!}{(m+2)!}a_{m+j+1}x^{n-m-j-1} \\ + \frac{(m+j)!}{(m+1)!}a_{m+j}x^{n-m-j} + \frac{(m+j-1)!}{m!}a_{m+j-1}x^{n-m-j+1} \\ + \cdots + \frac{(j+1)!}{2!}a_{j+1}x^{n-j-1} + \frac{j!}{1!}a_j x^{n-j} + (j-1)!a_{j-1}x^{n-j+1}. \end{aligned} \tag{4.42}$$

Differentiating the above polynomial $(n-m-j-1)$ times we get:

$$\begin{aligned} Q_{n-j+1}^{(n-m-j-1)}(x) = (n-m-j-1)!\frac{(m+j+1)!}{(m+2)!}a_{m+j+1} \\ + \frac{(n-m-j)!}{1!} \cdot \frac{(m+j)!}{(m+1)!}a_{m+j}x + \frac{(n-m-j+1)!}{2!} \cdot \frac{(m+j-1)!}{m!}a_{m+j-1}x^2 \\ + \cdots + \frac{(n-j-1)!}{m!} \cdot \frac{(j+1)!}{2!}a_{j+1}x^m + \frac{(n-j)!}{(m+1)!} \cdot \frac{j!}{1!}a_j x^{m+1} \\ + \frac{(n-j+1)!}{(m+2)!} \cdot (j-1)!a_{j-1}x^{m+2}. \end{aligned} \tag{4.43}$$

Let's evaluate $q_{m+1}$ and $q_1$ for the last polynomial. We have

$$q_{m+1}\left(Q_{n-j+1}^{(n-m-j-1)}\right) = \frac{2(m+2)}{m+1} \cdot \frac{j}{j+1} \cdot \frac{n-j}{n-j+1} \cdot q_j \tag{4.44}$$

$$q_1\left(Q_{n-j+1}^{(n-m-j-1)}\right) = \frac{2(m+2)}{m+1} \cdot \frac{m+j}{m+j+1} \cdot \frac{n-m-j}{n-m-j+1} \cdot q_{m+j}. \tag{4.45}$$

Applying the given estimations (3.9) and (3.10), we obtain

$$\max\left(q_1\left(Q_{n-j+1}^{(n-m-j-1)}\right), q_{m+1}\left(Q_{n-j+1}^{(n-m-j-1)}\right)\right) < \frac{2(m+2)(m-1)}{m^2}. \tag{4.46}$$

Note that $\deg Q_{n-j+1}^{(n-m-j-1)} = m+2$. If we denote by $n = m+2$, we will obtain (3.8). Therefore, by Corollary 3

$$\sharp_r\left(Q_{n-j+1}^{(n-m-j-1)}\right) \le m-2. \tag{4.47}$$

By virtue of Rolle's theorem

$$\sharp_r(Q_{n-j+1}) \le n-j-3. \tag{4.48}$$

It is clear that for any polynomial $P(x) = \sum_{k=0}^n a_k x^k$, the following identity holds true

$$\sharp_r(P(x)) = \sharp_r\left(x^n P\left(\frac{1}{x}\right)\right). \tag{4.49}$$

So, by (4.42) we have

$$\sharp_r(P^{(j-1)}) \le n-j-3. \tag{4.50}$$

Applying Rolle's theorem we can conclude that

$$\sharp_r(P) \le n-4. \tag{4.51}$$

Corollary 4 is proved. □

**Proof of Proposition 7.** Let

$$f(x) = (x+1)^m(ax^2+bx+c)$$

be a polynomial with positive coefficients, such that the quadratic factor $ax^2+bx+c$ is irreducible. Firstly we will prove that there exists an anti-derivative $P$ of $f$ with positive coefficients such that $P(x) = (x+1)^{m+1}(Ax^2+Bx+C)$, where the quadratic factor $Ax^2+Bx+C$ is irreducible.

Let $F$ be an anti-derivative of $f$ such that $F(0) = 0$. Obviously, all coefficients of $F$ are positive. Define the polynomial $P$ as follows

$$P(x) = F(x) + C, \quad \text{where} \quad P(-1) = 0.$$

So,

$$C = -F(-1) = \int_{-1}^{0} (t+1)^m(at^2+bt+c)dt > 0.$$

Therefore,

$$P(x) = (x+1)^{m+1}(Ax^2+Bx+C) \in \mathbb{R}_+[x],$$

where by virtue of Rolle's theorem the polynomial $Ax^2+Bx+C$ is irreducible, and $P'(x)=f(x)$.

Now we can easily prove Proposition 7. Consider the polynomial

$$\begin{aligned} P_{m+2}(x) = (x+1)^m\left(x^2+\tfrac{m-2}{m}x+1\right) = x^{m+2} + \tfrac{(m+2)(m-1)}{m}x^{m+1} \\ + \tfrac{(m+2)(m-1)}{2}x^m + \ldots + \tfrac{(m+2)(m-1)}{2}x^2 + \tfrac{(m+2)(m-1)}{m}x + 1. \end{aligned} \tag{4.52}$$

Obviously, for the polynomial $P_{m+2}$ we have

$$q_1 = q_{m+1} = \frac{2(m+2)(m-1)}{m^2}.$$

Let us integrate the polynomial $P_{m+2}$ from (4.52) $(n-m-j-1)$ times. We will use the statement in the beginning of the proof to obtain the resulting polynomial with positive coefficients in the form

$$\begin{aligned} P_{n-j+1} = (x+1)^{n-j-1}(Ax^2+Bx+C) = \frac{(m+2)!}{(n-j+1)!}x^{n-j+1} \\ + \frac{(m+2)(m-1)}{m}\cdot\frac{(m+1)!}{(n-j)!}x^{n-j} + \frac{(m+2)(m-1)}{2}\cdot\frac{m!}{(n-j-1)!}x^{n-j-1} \\ + \ldots + \frac{(m+2)(m-1)}{2}\cdot\frac{2!}{(n-m-j+1)!}x^{n-m-j+1} \\ + \frac{(m+2)(m-1)}{m}\cdot\frac{1!}{(n-m-j)!}x^{n-m-j} + \frac{0!}{(n-m-j-1)!}x^{n-m-j-1} + \ldots \end{aligned}$$

Let us consider the polynomial $\tilde{P}_{n-j+1}(x) = x^{n-j+1}P_{n-j+1}(\frac{1}{x}) = (x+1)^{n-j-1}(Cx^2+Bx+A)$. We will integrate the polynomial $\tilde{P}_{n-j+1}$ $(j-1)$ times. We will use the statement in the beginning of the proof to obtain the resulting polynomial with positive coefficients in the form

$$\begin{aligned} \tilde{P}_n(x) = (x+1)^{n-2}(\tilde{A}x^2+\tilde{B}x+\tilde{C}) = \ldots + \frac{(m+2)!}{(n-j+1)!}\cdot\frac{0!}{(j-1)!}x^{j-1} \\ + \frac{(m+2)(m-1)}{m}\cdot\frac{(m+1)!}{(n-j)!}\cdot\frac{1!}{j!}x^j + \frac{(m+2)(m-1)}{2}\cdot\frac{m!}{(n-j-1)!}\cdot\frac{2!}{(j+1)!}x^{j+1} \\ + \ldots + \frac{(m+2)(m-1)}{2}\cdot\frac{2!}{(n-m-j+1)!}\cdot\frac{m!}{(m+j-1)!}x^{m+j-1} \\ + \frac{(m+2)(m-1)}{m}\cdot\frac{1!}{(n-m-j)!}\cdot\frac{(m+1)!}{(m+j)!}x^{m+j} \\ + \frac{0!}{(n-m-j-1)!}\cdot\frac{(m+2)!}{(m+j+1)!}x^{m+j+1} + \ldots \end{aligned}$$

If we evaluate $q_j$ and $q_{m+j}$ for the polynomial $\tilde{P}_n$ we obtain

$$q_j = \frac{(m-1)(m+1)}{m^2}\cdot\frac{(n-j+1)}{(n-j)}\cdot\frac{(j+1)}{j}$$

and

$$q_{m+j} = \frac{(m-1)(m+1)}{m^2}\cdot\frac{(n-m-j+1)}{(n-m-j)}\cdot\frac{(m+j+1)}{(m+j)}.$$

Proposition 7 is proved. □

**Proof of Proposition 11.** Following Hutchinson's idea, for a polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ with positive coefficients, we can assume without loss of generality, that $a_0 = a_1 = 1$, since we can consider a polynomial $T(x) = a_0^{-1}P(a_0 a_1^{-1}x)$ instead of $P$. (We use the fact that such rescaling of $P$ preserves the second quotients $q_k(T) = q_k(P)$ for all $k$.) For the sake of brevity, we further use notation $q_k$ instead of $q_k(P)$. Thereafter, we consider a polynomial

$$Q(x) = T(-x) = 1 - x + \sum_{k=2}^{n} \frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\cdot\ldots\cdot q_{k-1}^2 q_k} \tag{4.53}$$

instead of $P$ (see (3.21) for the formulas for coefficients).

Suppose that $x \in \left(\frac{a_{j-1}}{a_j}, \frac{a_j}{a_{j+1}}\right) = (q_1 q_2\cdot\ldots\cdot q_{j-1}, q_1 q_2\cdot\ldots\cdot q_{j-1}q_j)$. Since $q_k \geq 1$ for all $k, 1 \leq k \leq n-1$, it is easy to check that

$$1 < x < \frac{x^2}{q_1} < \frac{x^3}{q_1^2 q_2} < \cdots < \frac{x^j}{q_1^{j-1}q_2^{j-2}\cdot\ldots\cdot q_{j-1}}$$

and

$$\frac{x^j}{q_1^{j-1}q_2^{j-2}\cdot\ldots\cdot q_{j-1}} > \frac{x^{j+1}}{q_1^{j}q_2^{j-1}\cdot\ldots\cdot q_{j-1}^2 q_j} > \cdots > \frac{x^n}{q_1^{n-1}q_2^{n-2}\cdot\ldots\cdot q_{n-2}^2 q_{n-1}}.$$

We have

$$\begin{aligned}(-1)^j Q(x) = {} & \sum_{k=0}^{j-2} \frac{(-1)^{j+k}x^k}{q_1^{k-1}q_2^{k-2}\cdot\ldots\cdot q_{k-1}} + \left(-\frac{x^{j-1}}{q_1^{j-2}q_2^{j-3}\cdot\ldots\cdot q_{j-2}}\right. \\ & \left. + \frac{x^j}{q_1^{j-1}q_2^{j-2}\cdot\ldots\cdot q_{j-1}} - \frac{x^{j+1}}{q_1^{j}q_2^{j-1}\cdot\ldots\cdot q_j}\right) + \sum_{k=j+2}^{n} \frac{(-1)^{j+k}x^k}{q_1^{k-1}q_2^{k-2}\cdot\ldots\cdot q_{k-1}} \\ & =: \Sigma_1(x) + g(x) + \Sigma_2(x).\end{aligned}$$

We note that the terms in $\Sigma_1(x)$ are alternating in sign and increasing in moduli, wherein the last summand (for $k = j-2$) is positive, whence $\Sigma_1(x) \geq 0$. Analogously, the summands in $\Sigma_2(x)$ are alternating in sign and their moduli are decreasing, wherein the first summand (for $k = j+2$) is positive, whence $\Sigma_2(x) \geq 0$. Thus,

$$(-1)^j Q(x) \geq g(x) \text{ for all } x \in \left(\frac{a_{j-1}}{a_j}, \frac{a_j}{a_{j+1}}\right). \tag{4.54}$$

So we obtain

$$g(x) = \frac{x^{j-1}}{q_1^{j-2}q_2^{j-3}\cdot\ldots\cdot q_{j-2}} \cdot \left(-1 + \frac{x}{q_1 q_2\cdot\ldots\cdot q_{j-1}} - \frac{x^2}{q_1^2 q_2^2\cdot\ldots\cdot q_{j-2}^2 q_{j-1}^2 q_j}\right).$$

Set $x_j = q_1 q_2\cdot\ldots\cdot q_{j-1}\sqrt{q_j} \in (q_1 q_2\cdot\ldots\cdot q_{j-1}, q_1 q_2\cdot\ldots\cdot q_{j-1}q_j) = \left(\frac{a_{j-1}}{a_j}, \frac{a_j}{a_{j+1}}\right)$ and obtain

$$g(x) = \frac{x_j^{j-1}}{q_1^{j-2}q_2^{j-3}\cdot\ldots\cdot q_{j-2}} \cdot (-1 + \sqrt{q_j} - 1) \geq 0,$$

since, by our assumptions, $q_j \geq 4$. Proposition 11 is proved. □

For the sequence of real numbers $(b_0, b_1, \dots, b_m)$, let us denote by $\nu(b_0, b_1, \dots, b_m)$ the number of sign changes in this sequence (when counting the sign changes we omit zero terms).

An immediate consequence of Proposition 11 is the following statement.

**Corollary 5.** *Let $P(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}_+[x]$ and suppose that $q_k(P) \ge 1$ for all $k, 1 \le k \le n-1$. Suppose that there exists a sequence of indices $1 \le j_1 < j_2 < j_k \le n-1$ such that $q_{j_s}(P) \ge 4$ for $1 \le s \le k$. Then the number of real roots of $P$ counting multiplicities is not less than*

$$\nu\left((-1)^0, (-1)^{j_1}, (-1)^{j_2}, \dots, (-1)^{j_k}, (-1)^n\right).$$

**Proof.** To prove the statement, we apply Proposition 11, and for all $s, 1 \le s \le k$, we find the sequence of points $x_{j_s} \in \left(-\frac{a_{j_s}}{a_{j_s+1}}, -\frac{a_{j_s-1}}{a_{j_s}}\right)$ such that $(-1)^{j_s} P(x_{j_s}) \ge 0$. It remains to mention that, since $q_k(P) \ge 1$ for all $k, 1 \le k \le n-1$, we have

$$\frac{a_0}{a_1} \le \frac{a_1}{a_2} \le \frac{a_2}{a_3} \le \dots \le \frac{a_{n-1}}{a_n},$$

whence we obtain

$$-\infty < x_{j_k} < x_{j_{k-1}} < \dots < x_{j_2} < x_{j_1} < 0,$$

and

$$P(0) > 0, (-1)^{j_1} P(x_{j_1}) \ge 0, (-1)^{j_2} P(x_{j_2}) \ge 0, \dots, (-1)^{j_{k-1}} P(x_{j_{k-1}}) \ge 0,$$
$$(-1)^{j_k} P(x_{j_k}) \ge 0, (-1)^n P(-\infty) \ge 0.$$

Corollary 5 is proved. □

## 5. Related conjectures and some counterexamples

The following conjectures closely related to the topic under consideration were the original motivation for our study. However in the process of working on this paper we were able to disprove two of them. For the sake of completeness let us present them as well as our counterexamples.

**Conjecture 1** *(See Conjecture 8 of [9]). Let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial with positive coefficients, and consider the related (weighted) tropical polynomial*

$$ftrop(P) = \max_k \left(\log(a_k) + kt + \log\binom{n}{k}\right).$$

*The number of real zeros of $P(x)$ does not exceed the number of points in the tropical variety defined by $ftrop(t)$, i.e., the number of corners of the piecewise-linear continuous function $ftrop(t), t \in \mathbb{R}$.*

**Conjecture 2** *(See Conjecture 9 of [9]). Let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial with positive coefficients. Consider the differences*

$$\tilde{c}_k = (k+1)a_k^2 - k a_{k-1} a_{k+1},$$

*where $a_{-1} = a_{n+1} = 0$. Let $0 = k_1 < k_2 < \dots < k_m = n$ be the sequence of all indices $k_i$ such that $\tilde{c}_{k_i}$ is positive and let $v(P)$ be the number of changes in the (binary) sequence $\{k_i \mod 2\}_{i=0}^{m}$.*

*The number of real zeros of $P(x)$ does not exceed $v(P)$.*

**Conjecture 3** *(See Conjecture 10 of [9]). Let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial with positive coefficients. Consider the differences*

$$c_k = a_k^2 - a_{k-1}a_{k+1}, \quad k = 0, 1, \dots, n$$

*where $a_{-1} = a_{n+1} = 0$. Let $0 = k_1 < k_2 < \cdots < k_m = n$ be the sequence of all indices such that $c_{k_i}$ is non-negative, and let $v(P)$ be the number of changes in the sequence $\{k_i \mod 2\}_{i=0}^{m}$.*

*The number of real zeros of $P(x)$ does not exceed $v(P)$.*

Next we provide counterexamples to Conjectures 2 and 3. Consider the polynomial

$$Q_{15}(x) = (x+1)^{13}\left(\frac{19}{360360}x^2 - \frac{89}{720720}x + \frac{83}{720720}\right) = \frac{19}{360360}x^{15} + \frac{9}{16016}x^{14} + \frac{3}{1144}x^{13}$$
$$+\frac{1}{144}x^{12} + \frac{1}{88}x^{11} + \frac{1}{80}x^{10} + \frac{1}{72}x^{9} + \frac{3}{112}x^{8} + \frac{3}{56}x^{7} + \frac{11}{144}x^{6} + \frac{3}{40}x^{5} + \frac{9}{176}x^{4} + \frac{19}{792}x^{3} + \frac{17}{2288}x^{2}$$
$$+\frac{x}{728} + \frac{83}{720720}.$$

Obviously, $\deg Q_{15}(x) = 15$ and its number of real roots in 13. One has

$$\begin{cases} q_{10} = \frac{88\cdot 72}{80^2} = \frac{99}{100} < 1 \\ q_9 = \frac{88\cdot 112}{3\cdot 72^2} = \frac{140}{243} < 1 \\ q_8 = \frac{72\cdot 3^2\cdot 56}{3\cdot 112^2} = \frac{27}{28} < 1. \end{cases}$$

Observe that in the sequence of indices $0 = k_1 < k_2 < \cdots < k_m = n$ appearing in Conjecture 3 each $k_j$ corresponds exactly to the case $q_{k_j} \geq 1$. Therefore among $(k_1, k_2, \dots, k_m)$ the values $8, 9, 10$ are missing. Thus the number of changes in the sequence $\{k_i \mod 2\}_{i=0}^{m}$ is less than or equal to $15 - 4 = 11$. But $Q_{15}(x)$ has 13 real zeros which is a contradiction with Conjecture 3 in degree 15.

To obtain counterexamples in higher degrees let us introduce the sequence of primitives $\{Q_n(x)\}_{n=15}^{\infty}$ such that $Q_{15}(x)$ is given above and for $n > 15$ we set

$$Q_n(x) = \int_{-1}^{x} Q_{n-1}(t)dt.$$

One easily checks that all coefficients of $Q_n(x)$ are positive and the multiplicity of the real root at $-1$ increases provided that the number of real roots of $Q_n(x)$ equals $(n-2)$.

For the coefficients of the polynomial $Q_n(x)$, we obtain

$$q_{n-5} = \frac{99}{100}\cdot\frac{10\cdot 12}{11^2}\cdot\frac{11\cdot 13}{12^2}\cdots\frac{(n-1)\cdot(n+1)}{n^2} = \frac{99}{100}\cdot\frac{10}{11}\cdot\frac{n+1}{n} \to \frac{9}{10}, \quad n\to\infty$$
$$q_{n-6} = \frac{140}{243}\cdot\frac{9\cdot 11}{10^2}\cdot\frac{10\cdot 12}{11^2}\cdots\frac{(n-1)\cdot(n+1)}{n^2} = \frac{140}{243}\cdot\frac{9}{10}\cdot\frac{n+1}{n} \to \frac{14}{27}, \quad n\to\infty$$
$$q_{n-7} = \frac{27}{28}\cdot\frac{8\cdot 10}{9^2}\cdot\frac{9\cdot 11}{10^2}\cdots\frac{(n-1)\cdot(n+1)}{n^2} = \frac{27}{28}\cdot\frac{8}{9}\cdot\frac{n+1}{n} \to \frac{6}{7}, \quad n\to\infty.$$

Moreover, every $q_{n-5}, q_{n-6}, q_{n-7}$ given above is strictly less than 1. By the same reason as above the number of sign changes in $\{k_i \mod 2\}_{i=0}^{m}$ is less than or equal to $n-4$ while the number of real roots of $Q_n(x)$ is $n-2$. Contradiction.

The same sequence $\{Q_n(x)\}$ provides counterexamples to Conjecture 2 for all sufficiently large $n$. Indeed, we have that the additional factor $\frac{k}{k+1}$ tends to 1 for $k\to\infty$. In particular, we get $q_{n-5} < \frac{n-5}{n-4}$, $q_{n-6} < \frac{n-6}{n-5}$, $q_{n-7} < \frac{n-7}{n-6}$ for all large $n$. The same argument as above provides the required counterexamples.

## 6. Final remark and outlook

**1.** Our proofs of the main results are obtained by “brute force”. We are currently looking for their more conceptual proofs and generalizations of the Newton inequalitites and the Hutchinson theorem.
**2.** Theorems 5 and 8 provide relevant inequalities on the sum of two $q_i$'s. Such a domain will not transform into a polytope under the logarithmic map. Maybe to obtain sharper results more sophisticate domains should be used in the main Problem 1.
**3.** For the set of polynomials with positive coefficients and all real (negative) roots one can show that there exist and unique minimal with respect to inclusion inscribed and circumscribed polytopes, see [5,8]. In the paper [8] one can find connection of this topic with amoebas, resultants and multiplier sequences. It seems that for the majority of domains formed by polynomials with the number of real roots exceeding or below some given number of roots there is no uniqueness of inscribed and/or circumscribed polygons. This question seems to be very important in this area of research and should be answered.

## References

[1] J. Forsgård, D. Novikov, B. Shapiro, A Tropical Analog of Descartes’ Rule of Signs, Int. Math. Res. Not. 12 (2017) 3726–3750.
[2] J.I. Hutchinson, On a remarkable class of entire functions, Trans. Am. Math. Soc. 25 (1923) 325–332.
[3] O.M. Katkova, A.M. Vishnyakova, A remark about positive polynomials, Math. Inequal. Appl. 13 (4) (2010) 753–759, https://doi.org/10.7153/mia-13-54.
[4] V. Kostov, Univariate polynomials and the contractibility of certain sets, Annuaire Univ. Sofia Fac. Math. Inform. 107 (2020).
[5] V. Kostov, B. Shapiro, Hardy-Petrovich-Hutchinson’s problem and partial theta function, Duke Math. J. 162 (5) (2013) 889–924.
[6] D.C. Kurtz, A sufficient condition for all roots of a polynomial to be real, Am. Math. Mon. 99 (3) (1992) 259–263.
[7] A.T. Lundell, S. Weingram, The Topology of CW complexes, The University Series in Higher Mathematics, Van Nostrand Reinhold Co., New York, 1969, viii+216 pp.
[8] M. Passare, J.M. Rojas, B. Shapiro, New multiplier sequences via discriminantal amobae, Mosc. Math. J. 11 (3) (July–September 2011) 1–14.
[9] B. Shapiro, Problems with polynomials - the good, the bad, and the ugly, Arnold Math. J. 1 (1) (2015) 91–99.