HYPERGEOMETRIC-TYPE INTEGRALS AND FUCHSIAN DIFFERENTIAL OPERATORS

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Abstract. In this paper we consider integrals of the form

$$\mathcal{J}(w)=\int_{a_1}^{a_k}\frac{dz}{(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_k)^{\alpha_k}(w-z)^{\lambda}}.$$

One can show that if $\Re(\alpha_i)<1$, $1\le i\le k$, and $\lambda\notin\mathbb{Z}$ then the multi-valued function $\mathcal{J}(w)$ is well-defined in $\mathbb{C}\setminus\{a_1,\dots,a_k\}$ and belongs to the Nilsson class, cf. [De] and [Ni2]. In this paper we establish an explicit formula for the minimal differential operator with polynomial coefficients annihilating $\mathcal{J}(w)$ and we prove that Nilsson class functions with prescribed local monodromy are uniquely determined up to conjugation by an appropriate Fuchsian differential operator.

Introduction

Definition 1. *Let S be a finite subset of $\mathbb{P}^1=\mathbb{C}\cup\infty$. A multi-valued holomorphic function F defined in $\mathbb{P}^1\setminus S$ is called a Nilsson class function if (i) it is finitely determined, i.e., for any $x\in\mathbb{P}^1\setminus S$ the germs of all branches of F near x span a finite-dimensional linear space, and (ii) each branch of F has temperate growth near every singular point $s\in S$.*

A detailed account of Nilsson class functions can be found in Nilsson's original papers [Ni1,Ni2] as well as §4 in Chap. 6 of [Bj2] and Chap. 3 of Deligne's book [De]. The importance of this class stems from its close connection with the class of global Fuchsian univariate differential operators on $\mathbb{P}^1$ (a standard reference for background material on the latter class of operators is [Po]). Indeed, the following fundamental result provides precise information that considerably strengthens the classical Riemann-Hilbert correspondence:

Proposition 1. *Let S be a finite subset of $\mathbb{P}^1$ and $\mathcal{H}$ be a Nilsson class function in $\mathbb{P}^1\setminus S$ of some rank k. Then there exists a unique Fuchsian differential operator*

$$Q=q_k(w)\cdot\partial^k+\ldots+q_0(w)$$

annihilating $\mathcal{H}$, i.e., $Q(\mathcal{H})=0$, and such that $q_k(w)$ is monic and the $k+1$ polynomials $q_i(w)$, $0\le i\le k$, have no common factor.

Here $Q(\mathcal{H})=0$ means that whenever $w\in\mathbb{P}^1\setminus\mathbb{C}$ and $h_1,\dots,h_k$ is some $\mathbb{C}$-basis of the stalk $\mathcal{H}(w)$ one has $Q(h_1)=\ldots=Q(h_k)=0$ in $\mathcal{O}(w)$. One refers to the Fuchsian differential operator Q in Proposition 1 as the *minimal differential operator* annihilating $\mathcal{H}$.

Remark 1. Although Proposition 1 is certainly well known to specialists in D-module theory, to our surprise its proof does not seem to appear anywhere in the existing literature. For this reason we give a proof of Proposition 1 in the Appendix, where we also briefly recall some results on Fuchsian differential operators needed in the process.

Let us now introduce the main object of our study. Fix a k-tuple of points $a_1, \dots, a_k$ in $\mathbb{C}$, where $k \geq 2$, and take a k-tuple of complex numbers $\alpha_1, \dots, \alpha_k$ such that $\mathfrak{Re}(\alpha_\nu) < 1$ and $\alpha_\nu \notin \mathbb{Z}$ for $1 \leq \nu \leq k$. Let further $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ and w be a new complex variable. Using this data we construct a *Nilsson class function* defined in $X = \mathbb{P}^1 \setminus (a_1, \dots, a_k, \infty)$ by the following procedure:

Take a rectifiable Jordan arc Γ_0 with the end-points at a_1 and a_k and choose some single-valued branches of the functions $(z - a_1)^{\alpha_1}, \dots, (z - a_k)^{\alpha_k}$ along Γ_0. (Notice that we allow some of the points $a_\nu \in \Gamma$ with $2 \leq \nu \leq k-1$ to belong to Γ_0 since this does not prevent us from choosing our single-valued branches.) Since $\mathfrak{Re}(\alpha_\nu) > 1$ one has that the absolute value of the function

$$\rho(z) = \frac{1}{(z - a_1)^{\alpha_1} \cdots (z - a_k)^{\alpha_k}} \tag{1}$$

is integrable with respect to the arc-length measure along Γ_0. Assuming that our independent variable w stays in a small disc D that does not intersect Γ_0 we can additionally choose a single-valued branch of $z \mapsto (w - z)^\lambda$ along Γ_0 and consider the holomorphic function $J_0(w)$ in D defined by:

$$J_0(w) = \int_{\Gamma_0} \frac{\rho(z) \cdot dz}{(w - z)^\lambda}. \tag{2}$$

Considering all possible Jordan arcs in $X = \mathbb{P}^1 \setminus (a_1, \dots, a_k, \infty)$ with the same two end-points a_1, a_k, we obtain the complete analytic continuation of the given germ $J_0(w)$ which is in general a multi-valued analytic function $J(w)$ defined in the connected open set X. (See [Ni1], where both the existence and various properties of analytic continuations are proved in a more general setting in dimension ≥ 2.)

By the main result of [Ni2] the local branches of $J(w)$ generate in X a *Nilsson class function* denoted by $\mathcal{J}$. More precisely, for each $w \in X$ denote by $\mathcal{J}(w)$ the complex vector space generated by the local branches of J in the ring $\mathcal{O}(w)$ of germs of holomorphic functions at w. Then $\mathcal{J}(w)$ is a family of finite-dimensional complex vector spaces as w varies in X. Moreover, $\mathcal{J}$ is a locally constant sheaf of $\mathbb{C}_X$-modules. In fact, by the classical monodromy theorem $\mathcal{J}$ is a trivial local system over simply connected open subsets of X. The *rank* of $\mathcal{J}$ is the common dimension of the vector spaces $\mathcal{J}(w)$ as w varies in X. Finally, in small punctured discs centered at one of the ramification points $a_1, \dots, a_k$ or ∞, the local branches of $\mathcal{J}$ have *temperate growth.*

The first main goal of this paper is to find a distinguished differential operator Q with polynomial coefficients – that is, an element of the Weyl algebra $\mathcal{A}_1 = \mathbb{C}\langle w, \partial \rangle$, where $\partial = \partial/\partial w$ – annihilating $\mathcal{J}$, i.e., such that

$$Q(\mathcal{J}) = 0. \tag{3}$$

This means that for any $w \in X$ one has $Q(g) = 0$ in $\mathcal{O}(w)$ for every germ g in the stalk $\mathcal{J}(w)$.

In order to formulate our first main result we need the following construction.

Definition 2. *To each polynomial*

$$S(w) = s_m \cdot w^m + \ldots + s_0$$

of some degree $m \geq 1$ we associate a linear differential operator with polynomial coefficients $\mathbf{S}$ *depending on an additional parameter λ and given by*

$$\mathbf{S} = \sum_{\nu=1}^{m} s_\nu \cdot \partial^{m-\nu} \cdot (\nabla + \lambda + \nu - 1) \cdots (\nabla + \lambda) + s_0 \cdot \partial^m,$$

where $\nabla = w \cdot \partial$.

For example, if $m = 2$ we get $\mathbf{S} = s_2(\nabla + \lambda + 1)(\nabla + \lambda) + s_1 \partial \cdot (\nabla + \lambda) + s_0 \cdot \partial^2$. Finally, given two k-tuples $a_1, \ldots, a_k$ and $\alpha_1, \ldots, \alpha_k$ of complex numbers we define the polynomials

$$A(w) = \prod_{\nu=1}^{k} (w - a_\nu) \text{ and } B(w) = \sum_{\nu=1}^{k} \frac{1 - \alpha_\nu}{w - a_\nu} \cdot A(w).$$

We can now formulate our first main result.

Theorem 1. *Set $\rho = \lambda + \alpha_1 + \ldots + \alpha_k - 2$. Then*
(a) if ρ is not a non-negative integer then rank$(\mathcal{J}) = k$ and $\mathbf{A} - \mathbf{B}$ is the minimal annihilating operator of $\mathcal{J}$;
(b) if ρ is a non-negative integer then there exist a unique monic polynimal $\phi(w)$ and a differential operator R of order $k - 1$ such that

$$\phi(w) \cdot (\mathbf{A} - \mathbf{B}) = \partial \cdot R$$

and R is the minimal annihilating operator of $\mathcal{J}$.

Remark 2. Above we treat $\phi(w)$ as a differential operator of order zero and therefore the equation $\phi \cdot (\mathbf{A} - \mathbf{B}) = \partial \cdot R$ should be understood as an equality in the Weyl algebra $\mathcal{A}_1$.

Our second main result show that Nilsson class functions with prescribed local monodromy are in a sense uniquely determined up to conjugation by some Fuchsian differential operator.

Theorem 2. *Formulate here the general theorem on "uniqueness up to conjugation by appropriate Fuchsian operators", i.e., if $\mathcal{P}$ and $\mathcal{Q}$ are Nilsson class functions with the same local monodromy then there exists an operator S such that $S \cdot \mathcal{P} = \mathcal{Q}$, etc.*

Corollary 1. *Formulate here the result about the uniqueness of the integral representation in the case of second order diff op $q_2 \partial^2 + q_1 \partial + (az + b)$ (so that we get a complete picture in this case).*

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1. Proofs

Since the proof of Theorem 1 requires several steps let us first sketch it. As a first step we prove the inequality

$$\text{rank}(\mathcal{J}) \geq k-1. \tag{4}$$

(This fact does not require any use of differential operators.) The second step is to show that $\mathbf{A}-\mathbf{B}$ annihilates $\mathcal{J}$, which follows from integration by parts. Thus, since $\mathbf{A}-\mathbf{B}$ has order k, we get $\text{rank}(\mathcal{J}) \leq k$ and it remains to decide when this rank equals k and when it equals $k-1$. To do this we first settle Case (b) of the Theorem 1. Using some basic facts about polynomial solutions of finite order differential operators with polynomial coefficients we factorize the operator $\mathbf{A}-\mathbf{B}$ and show that R annihilates $\mathcal{J}$. Since R has order $k-1$ by (4) it must be the minimal annihilating operator of $\mathcal{J}$. To settle Case (a) we use factorization properties of Fuchsian operators in $\mathcal{A}_1$ combined with some algebraic calculations.

Let us carry out the above mentioned steps.

Lemma 1.1. *rank*$(\mathcal{J}) \geq k-1$.

Proof. It suffices to show that there exist at least $k-1$ distinct $\mathbb{C}$-linearly independent local branches of $\mathcal{J}$ near ∞. With $\zeta = 1/w$ as a local coordinate at ∞ one has the holomorphic function in the domain $|\zeta| < 1$ defined by:

$$(1-\zeta)^{\lambda} = e^{\lambda\cdot\text{Log}(1-\zeta)} = 1 + \sum_{\nu\geq 1} c_{\nu}(\lambda)\cdot\zeta^{\nu}.$$

Fix a Jordan arc Γ_0 which passes through all the points $a_1, \dots, a_k$. Now Γ_0 is the union of $k-1$ subarcs $\gamma_1, \dots, \gamma_{k-1}$, where γ_ν has a_ν and $a_{\nu+1}$ as its end-points. Next, let $\dot{D}$ be a small punctered disc centered at $\zeta = 0$ that does not intersect Γ_0. By considering all analytic continuations of $\mathcal{J}$ it follows that the restriction of $\mathcal{J}$ to $\dot{D}$ contains at least $k-1$ branches of the form

$$\rho_\nu(\zeta) = \zeta^{\lambda}\cdot\phi_\nu(\zeta), \quad 1 \leq \nu \leq k-1,$$

where $\phi_\nu(\zeta)$ is holomorphic in D and given by

$$\phi_\nu(\zeta) = \sum_{\nu\geq 0}\int_{\gamma_\nu}\frac{dz}{(z-a_1)^{\alpha_1}\cdots(z-a_k)^{\alpha_k}}\cdot c_\nu(\lambda)\cdot\zeta^{\nu}.$$

(Here we assume that we have chosen certain single-valued branches of $(z-a_1)^{\alpha_1}, \dots, (z-a_k)^{\alpha_k}$ along γ_ν for each ν.) Since every continuous function on Γ_0 can be uniformly approximated by an appropriate polynomial in z, we see that the ϕ_ν's must be $\mathbb{C}$-linearly independent and the lemma follows. □

Lemma 1.2. *The operator* $\mathbf{A}-\mathbf{B}$ *annihilates* $\mathcal{J}$*, i.e.,* $(\mathbf{A}-\mathbf{B})(\mathcal{J}) = 0$.

The proof of Lemma 1.2 requires a number of additional statements. Let us define

$$J\cdot z^{\nu} = \int_{\Gamma}\frac{z^{\nu}\cdot dz}{(w-z)^{\lambda}\cdot(z-a_1)^{\alpha_1}\cdots(z-a_k)^{\alpha_k}}.$$

Notice that

$$\partial(w-z)^{-\lambda} = -\lambda\cdot(w-z)^{-\lambda+1} \implies$$

$$(w\partial+\lambda)((w-z)^{-\lambda}) = z\cdot\partial(w-z)^{-\lambda}. \tag{5}$$

(Here $\partial = \partial_w$, as above.) Thus, by using $\nabla = w\cdot\partial$ we obtain

$$(\nabla+\lambda)(J) = \partial\,(J\cdot z)\,.$$

As a corollary of (5) we get the following statement.

Lemma 1.3. *For each integer $\nu \geq 1$ one has*

$$(\nabla + \lambda + \nu) \cdot (J \cdot z^\nu) = \partial \left(J \cdot z^{\nu+1} \right).$$

Next, consider the polynomials $A(z) = (z - a_1) \cdots (z - a_k)$ and $B(z) = \sum_{\nu=1}^{k} \frac{\alpha_\nu}{w - a_\nu} \cdot A(z)$.

Lemma 1.4. *The following equality holds:*

$$\partial \left(J \cdot A(z) \right) = J \cdot B(z). \tag{6}$$

Proof. Set $\rho(z) = (z - a_1)^{\alpha_1} \cdots (z - a_k)^{\alpha_k}$. The obvious equality

$$\partial (w - z)^{-\lambda} = -\partial_z (w - z)^{-\lambda}$$

gives

$$\partial \left(J \cdot A(z) \right) = - \int_\Gamma \partial_z (w - z)^{-\lambda} \cdot \frac{A(z) \cdot dz}{\rho(z)}.$$

Since the terms $(z - a_1)^{1-\alpha_1}$ and $(z - a_k)^{1-\alpha_k}$ occur in $A(z)/\rho(z)$, an integration by parts gives

$$\partial \left(J \cdot A(z) \right) = \int_\Gamma (z - w)^{-\lambda} \cdot \partial_z (A(z)/\rho(z)) \cdot dz.$$

However, by construction one has $B(z) = \partial_z (A(z)/\rho(z))$ and the lemma follows. □

Finally, to settle Lemma 1.2 we apply ∂^{k-1} to both sides of (6) to get

$$\partial^k \left(J \cdot A(z) \right) = \partial^{k-1} \left(J \cdot B(z) \right).$$

Now repeated use of Lemma 1.3 combined with the properties satisfied by the differential operators $\mathbf{A}$ and $\mathbf{B}$ (by construction) implies that $\mathbf{A}(J) = \mathbf{B}(J)$. □

1.5. Definition. *The adjoint involution on the Weyl algebra $\mathcal{A}_1$ is defined by*

$$Q = \sum q_\nu(w) \cdot \partial^\nu \mapsto Q^* = \sum (-1)^\nu \cdot \partial^\nu \cdot q_\nu(w).$$

Notice that this map has the properties

$$(Q \cdot P)^* = P^* \cdot Q^* \text{ and } Q^{**} = Q$$

for each pair of operators Q, P in $\mathcal{A}_1$.

1.6. On polynomial solutions. Let $Q \in \mathcal{A}_1$ be a finite order differential operator and $\phi(w)$ be some polynomial. Applying the differential operator Q to ϕ we get the image polynomial $Q(\phi)$. At the same time we can regard ϕ as a differential operator of order zero in $\mathcal{A}_1$ and construct the product $Q \cdot \phi$. One can easily see that

$$Q \cdot \phi - Q(\phi) = R \cdot \partial \tag{7}$$

holds for some $R \in \mathcal{A}_1$, i.e., $Q \cdot \phi - Q(\phi)$ belongs to the left ideal generated by ∂. Applying (7) with Q replaced by Q^* and using the adjoint involution one gets

$$\phi \cdot Q - Q^*(\phi) = \partial \cdot R^*. \tag{8}$$

We conclude that the following statement holds.

1.7. Proposition. *The adjoint operator Q^* has a polynomial solution ϕ if and only if $\phi \cdot Q$ belongs to the right ideal in $\mathcal{A}_1$ generated by ∂.*

1.8. Important observation. Consider an operator $Q \in \mathcal{A}_1$ of the form

$$Q = (\nabla - \beta_1) \cdots (\nabla - \beta_k) + R, \tag{9}$$

where $R \in \mathcal{A}_1$ is a $\mathbb{C}$-linear sum of elements of the form

$$\partial^\nu \cdot \nabla^j, \quad \nu \geq 1 \text{ and } \nu + j \leq k. \tag{10}$$

Above we assume that $\beta_1, \ldots, \beta_k$ are different complex numbers. Notice that (9)-(10) imply that whenever $k \geq 0$ is an integer one has

$$Q(w^k) = (k - \beta_1) \cdots (k - \beta_k) \cdot w^k + \text{a polynomial of degree at most } k.$$

Indeed, this follows from the fact that ∂^ν lowers while ∇ preserves the degree of polynomials. So if Q is as in (9) then the equation $Q(\phi) = 0$ has a polynomial solution if and only if at least some β_ν is a non-negative integer. Moreover, if this occurs for a unique β, say $\beta_1 = m \geq 0$, then there exists a unique monic polynomial ϕ of degree m such that $Q(\phi) = 0$.

1.9. The case of $\mathbf{A} - \mathbf{B}$. By construction

$$\mathbf{A} - \mathbf{B} = (\nabla + \lambda + k - 2) \cdots (\nabla + \lambda) \cdot (\nabla + \lambda + \alpha_1 + \ldots + \alpha_k - 1) + R,$$

where R is a $\mathbb{C}$-linear sum of operators as in (10). In other words, $\mathbf{A} - \mathbf{B}$ is an operator of the special form considered in the observation made in §1.8 above. Passing to the adjoint operator and using the identity $\nabla^* = -\partial \cdot w = -\nabla - 1$ we obtain

$$(\mathbf{A} - \mathbf{B})^* = (-1)^k \cdot (\nabla - \lambda - k + 2) \cdots (\nabla - \lambda + 1) \cdot (\nabla - (\lambda + \alpha_1 + \ldots + \alpha_k) + 2) + R^*.$$

So when λ is not an integer we conclude that $(\mathbf{A} - \mathbf{B})^*$ has a unique monic polynomial solution $\phi(w)$ of degree $m \geq 0$ if and only if

$$\lambda + \alpha_1 + \ldots + \alpha_k = 2 + m. \tag{11}$$

By Proposition 1.7 this is equivalent to the existence of a factorisation

$$\phi(w) \cdot (\mathbf{A} - \mathbf{B}) = \partial \cdot R,$$

where R is a differential operator of order $k - 1$.

At this stage we can finish proving part (b) of Theorem 1. Namely, assume that (11) above holds for some $m \geq 0$. Then we get a factorisation

$$\phi(w) \cdot (\mathbf{A} - \mathbf{B}) = \partial \cdot R.$$

We have already proved that $(\mathbf{A} - \mathbf{B})(\mathcal{J}) = 0$. So if g is a local branch of $\mathcal{J}$ in a small punctured disc at infinity, then

$$\partial(R(g)) = 0 \implies R(g) \equiv \text{ constant.}$$

It remains to see why the relation $R(g) \equiv 0$ actually holds. To prove this we notice that each local branch of $\mathcal{J}$ near ∞ (where $\zeta = 1/w$ is a local coordinate) has the form

$$g(\zeta) = \zeta^\lambda \cdot G(\zeta),$$

where $G(\zeta)$ is a single-valued meromorphic function in a punctured ζ-disc. It follows that

$$R(g) = \zeta^\lambda \cdot S(\zeta)$$

for some meromorphic function S. Now $\partial = \partial_w = -\zeta^2 \cdot \partial_\zeta$ and hence

$$\partial_w(R(g)) = -\zeta^2 \cdot \partial_\zeta(\zeta^\lambda \cdot S) = -\zeta^{\lambda+1}(\lambda \cdot S - \zeta \cdot \partial_\zeta(S).$$

Since λ is not an integer there is no non-zero meromorphic function $S(\zeta)$ satisfying

$$\lambda \cdot S - \zeta \cdot \partial_\zeta(S) = 0.$$

Hence $R(g) \equiv 0$ and since this holds for every local branch of $\mathcal{J}$ we conclude that

$$R(\mathcal{J}) = 0.$$

Now R has order $k-1$ and since $\mathrm{rank}(\mathcal{J}) \geq k-1$ by Lemma 1.1, it follows that $\mathrm{rank}(\mathcal{J}) = k-1$ and that R is its minimal annihilating operator. $\square$

1.10. Proving part (a) of Theorem 1. As explained in the sketch it remains to show that if $\rho = \lambda + \alpha_1 + \ldots + \alpha_k - 2$ is not a non-negative integer, then $\mathrm{rank}(\mathcal{J}) = k$. To obtain this we need a certain result about factorisation in $\mathcal{A}_1$.

1.11. Factorisation of Fuchsian operators. Consider an element Q in $\mathcal{A}_1$ given by

$$Q = q_k(w) \cdot \partial^k + \ldots + q_0(w).$$

Introduce the field $\mathbb{C}(w)$ of rational functions and the $\mathbb{C}$-algebra

$$\mathbb{C}(w)\langle\partial\rangle = \mathbb{C}(w) \otimes_{\mathbb{C}[w]} \mathcal{A}_1$$

whose elements are differential operators with rational functions as coefficients. Suppose that there exists a factorisation

$$Q = (\rho_0 + \rho_1 \cdot \partial) \cdot R,$$

where ρ_0, ρ_1 are rational functions and $R \in \mathcal{A}_1$ is given by

$$R = r_{k-1}(w) \cdot \partial^{k-1} + \ldots + r_0(w).$$

Here $r_0, \ldots, r_{k-1}$ have no common zero and the leading polynomial equals $r_{k-1}(w) = \phi(w) \cdot q_k(w)$ for some monic polynomial $\phi(w)$. Since $\mathbb{C}[w]$ is a unique factorisation domain there exist polynomials γ, τ, a, b, where γ, τ have no common factor and similarly a, b have no common factor, such that

$$\gamma \cdot Q = \tau(a + b \cdot \partial) \cdot R.$$

Identifying the coefficients of ∂^k in the left- and right-hand sides we get

$$\gamma \cdot q_k = \tau \cdot b \cdot \phi \cdot q_k \implies \gamma = \tau \cdot b \cdot \phi.$$

Since γ, τ have no common factor it follows that $\tau = 1$. Thus

$$b \cdot \phi \cdot Q = (a + b \cdot \partial) \cdot R.$$

Next, since a, b have no common factor this gives

$$R = b \cdot S, \text{ where } S \in \mathcal{A}_1.$$

By assumption $r_0, \dots, r_{k-1}$ have no common factor and hence b is a constant. Dividing by b we obtain

$$\phi \cdot Q = \left(\partial + \frac{a(w)}{b}\right) \cdot R. \tag{12}$$

At this stage we shall impose a further condition on the pair of operators Q and R. Namely, *assume* that Q is the minimal operator of some Nilsson class function $\mathcal{G}$ of rank k. It is well-known that each operator in $\mathcal{A}_1$ sends a Nilsson class function to another Nilsson class function. In particular, $\mathcal{H} = R(\mathcal{G})$ is a Nilsson class function. Since $Q(\mathcal{G}) = 0$ it follows from (12) that the first order differential operator

$$P = \partial + \frac{a(w)}{b}$$

annihilates $\mathcal{H}$. Notice that this implies that $\mathcal{H}$ has rank k by Lemma 1.1. Now a local branch $h(\zeta)$ of $\mathcal{H}$ in a punctured disc $\dot{D}$ at ∞ is in the Nilsson class, i.e., it has temperate growth at $\zeta = 0$. At the same time we recall that $\partial_w = -\zeta^2 \partial_\zeta$ and hence

$$b \cdot \zeta^2 \cdot \partial_\zeta(h) = a(1/\zeta) \cdot h. \tag{13}$$

If the polynomial $a(w)$ is not identically zero and has degree $j \geq 0$ we notice that

$$a(1/\zeta) = \zeta^{-j} \cdot f(\zeta),$$

where $f(\zeta)$ is a holomorphic function in the small ζ-disc D and $f(0) \neq 0$. Then (13) gives

$$b \cdot \zeta^{2+j} \cdot \partial_\zeta(h) = f(\zeta) \cdot h.$$

But this is impossible since $2 + j \geq 2$. In fact, the solution h cannot have temperate growth by the classical *Fuchsian criterion*. Hence $a \equiv 0$ and we obtain the factorisation

$$\phi \cdot Q = \partial \cdot R.$$

When this is applied to the differential operator $Q = \mathbf{A} - \mathbf{B}$ we obtain the required contradiction. This completes the proof of Theorem 1. □

1.12. Proof of Theorem 2. ETT KORT BEVIS HÄR FÖR **Theorem 2**, D V S "the general theorem on uniqueness up to conjugation by appropriate Fuchsian operators"......

1.13. Proof of Corollary 1. HÄR SKULLE MAN HA ETT BEVIS FÖR **Corollary 1**, D V S "uniqueness of the integral representation in the case of $q_2\partial^2 + q_1\partial + (az + b)$".....

2. Examples

Example 1. Consider three points $(0, a, b)$ in $\mathbb{C}$. Let Γ_0 be a Jordan arc with end-points at 0 and b. Using deformations of Γ_0 as in the introduction we obtain a Nilsson class function $\mathcal{J}$ with local branches given by

$$J(w) = \int_\Gamma \frac{dz}{\sqrt{w-z}\cdot\sqrt{z\cdot(z-a)\cdot(z-b)}}.$$

Theorem 1 shows that the Nilsson class function $\mathcal{J}$ has rank 2. This function has special local germs in the punctured discs at the ramification points $(0, a, b)$ and ∞. Indeed, in a small punctured disc $\dot{D}$ centered at $z = 0$ one can see by direct inspection that $\mathcal{J}$ restricted to $\dot{D}$ is generated by a pair

$$g_0(z) \text{ and } g_0(z)\cdot \operatorname{Log}(z) + g_1(z),$$

where g_0, g_1 are holomorphic in D and $g_0(0) = 1$. A similar local description holds for $\mathcal{J}$ in small punctured discs centered at a and b. In other words, close to a the function $\mathcal{J}$ is generated by a pair

$$h_0(z) \text{ and } h_0(z)\cdot \operatorname{Log}(z-a) + h_1(z),$$

where $h_0(a) = 1$ and both h_0, h_1 are holomorphic in a disc centered at a. A similar fact holds with a replaced by b. Finally, at ∞ (where $\zeta = 1/w$ is a local coordinate) we notice that

$$J(\zeta) = \sqrt{\zeta}\cdot\int_\Gamma \frac{dz}{\sqrt{1-\zeta\cdot z}\cdot\sqrt{z\cdot(z-a)\cdot(z-b)}}.$$

Here $\frac{1}{\sqrt{1-\zeta\cdot z}}$ is expanded when $|\zeta|$ is small while z stays on a compact Jordan arc Γ. Using this one finds by inspection that the local branches of $\mathcal{J}$ close to ∞ are generated by a pair

$$f_0(\zeta)\cdot\sqrt{\zeta} \text{ and } f_1(\zeta)\cdot\sqrt{\zeta},$$

where f_0, f_1 are holomorphic in a disc $|\zeta| < \delta$, $f_0(0) = 1$, and $f_1(0\zeta) = \zeta+$ higher order terms.

Remark 3. Given the local descriptions of $\mathcal{J}$ at its ramification points one can show that if

$$P = w(w-a)(w-b)\cdot\partial^2 + p_1(w)\cdot\partial + p_0(w)$$

is a second order differential operator that annihilates $\mathcal{J}$, then P is equal to the minimal differential operator found in Theorem 1. Easy calculations show that $p_1(w) = Q'(w)/2$ and $p_0(w) = Q''(w)/8$, where $Q(w) = w(w-a)(w-b)$. Moreover, one can recover P by direct calculations using general $\mathcal{D}$-module theoretic results. Thus one has an interesting uniqueness result in this situation – DETTA HAR JU MED (DET NYA) **Corollary 1** ATT GÖRA!!

Example 2. Consider the function

$$J(z) = \int_0^1 \frac{dt}{\sqrt{t^4 - t^2 + z^2}}$$

After a variable substitution we find that

$$J(z) = \frac{1}{\sqrt{2}}\cdot\int_{-1}^1 \frac{du}{\sqrt{u+1}\cdot\sqrt{u^2+4z^2-1}}.$$

Let us put $w = 4z^2 - 1$ and study the w-function

$$I(w) = \frac{1}{\sqrt{2}} \cdot \int_{-1}^{1} \frac{du}{\sqrt{u+1} \cdot \sqrt{u^2 + w}}.$$

We obtain the following second order differential equation satisfied by $I(w)$:

$$(*) \qquad w(w+1)\partial^2(I) + (2w+1)\partial(I) + \frac{3I}{16} + \frac{1}{4 \cdot \sqrt{2} \cdot \sqrt{w+1}} = 0$$

Proof of (*). Set as before $\nabla = w \cdot \partial$ (cf §1). By partial integration we find the following:

$$(14) \qquad (\nabla + 1/2)(I \cdot s^{\nu}) = -\partial(I) \cdot s^{\nu+2} \text{ for each integer } \nu \geq 0,$$

$$(15) \qquad \partial(I) \cdot (s^3 - s) = -3/4 \cdot I \cdot s - I/4, \text{ and}$$

$$(16) \qquad \partial(I) \cdot (s^2 + s) = \frac{1}{\sqrt{2} \cdot \sqrt{w+1}} - I/4.$$

Now (16) and (14) with $\nu = 1$ give

$$(1 + w)\partial(I) \cdot s = \xi(w) + J/4 + \nabla(I).$$

Putting these relations together we obtain

$$\sqrt{1+w}/\sqrt{2} + wI/4 + (1+w)\nabla(I) = J \cdot s/4.$$

Then we apply ∂ on both sides and as a result we arrive at the second order equation (*). □

Next, we have

$$J(z) = I(4z^2 - 1) \implies$$

$$J'(z) = 8z \cdot \partial(I) \text{ and } J''(z) = 8 \cdot \partial(I) + 64z^2\partial^2(I).$$

Thus

$$\partial(I) = \frac{J'}{8z} \text{ and } \partial^2(I) = \frac{J''}{64z^2} - \frac{J'}{64z^3}.$$

Inserting this in (i) we get the following differential equation:

$$4z^2(4z^2 - 1) \cdot (\frac{J''}{64z^2} - \frac{J'}{64z^3}) + (8z^2 - 1) \cdot \frac{J'}{8z} + \frac{3J}{16} + \frac{1}{4\sqrt{2} \cdot z} = 0 \implies$$

$$\frac{1}{64}(4z^2 - 1) \cdot J'' + \frac{12z^2 - 1}{16z} \cdot J' + \frac{3J}{16} + \frac{1}{4\sqrt{2} \cdot z} = 0 \implies$$

$$\frac{z}{4}(4z^2 - 1) \cdot J'' + (12z^2 - 1) \cdot J' + 3z \cdot J + 2\sqrt{2} = 0.$$

3. Further remarks and open questions

Problem 1. An exciting question is to describe what properties/restrictions on the monodromy group of a Fuchsian differential operator must hold if such an operator annihilates a hypergeometric-type integral of the form (2) considered above. Similar questions were considered in [Var] where the appearing monodromy groups are related to the R-matrix of an appropriate quantum algebra. This problem is open already in the first nontrivial case of the Heun equation. In particular, one may ask the following

Question. Is it true that the monodromy group of a hypergeometric-type integral (2) is always irreducible?

Problem 2. What is the rank and minimal annihilating operator for a similar integral, namely

$$J(w)=\int_{p_1}^{p_2}\frac{dz}{(z-a_1)^{\alpha_1}\cdots(z-a_k)^{\alpha_k}(w-z)^{\lambda}},$$

where p_1 and p_2 are arbitrary points in $\mathbb{C}$ (not necessarily coinciding with some of a_j's)? Integrals of this form often appear in various applications.

Problem 3. Suppose that $\mathcal{G}$ is a Nilsson class function of rank 2 with ramification points at $0, a, b$ and ∞. Assume also that local branches around each of these four ramification points satisfy the same *monodromy conditions* as in Example 1 in §2. Thus, local branches at $z=0$ are generated by a pair

$$g_0 \text{ and } g_0\cdot\log(z)+g_1(z), \text{ where } g_0,g_1\in\mathcal{O}(D),$$

and similarly at a and b. However, here we do not necessarily require that $g_0(0)=1$. Finally, at ∞ local branches are generated by a pair

$$h_0\sqrt{\zeta} \text{ and } h_1\cdot\sqrt{\zeta},$$

where h_0, h_1 are $\mathbb{C}$-linearly independent holomorphic functions in a disc $|\zeta|<\delta$. But no special conditions on the vanishing of these h-functions at $\zeta=0$ are imposed. By Theorem 1 there exists a unique minimal differential operator

$$R=p_2(w)\partial^2+p_1(w)\cdot\partial+p_0(w)$$

such that $R(\mathcal{G})=0$. It would be nice to give a complete decription of all Nilsson class functions $\mathcal{G}$ as above with prescribed local monodromy at the four ramification points.

Appendix. Proof of Proposition 1

HÄR KAN DU KORT SAMMANFATTA DET DU SKREV TIDIGARE I INTRODUKTIONEN OM Fuchsian differential operators OCH GE SEN ETT KONCIST BEVIS FÖR Proposition 1...

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