

**COUNTING REAL RATIONAL FUNCTIONS
WITH ALL REAL CRITICAL VALUES**

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ABSTRACT. We study the number $\sharp_n^{\mathbb{R}}$ of real rational degree n functions (considered up to a linear fractional transformation of the independent variable) with a given set of $2n - 2$ distinct real critical values. We present a combinatorial interpretation of these numbers and pose a number of related questions.

§1. INTRODUCTION

This paper continues the series [A2–A6, S, SV] dealing with the topology of the sets of ordinary, trigonometric, and exponential polynomials with the maximal possible number of real critical values. Recall that a degree n rational function is any holomorphic degree n map $\mathbb{C}P^1 \rightarrow \mathbb{C}P^1$. A rational function f is called *real* if both the image and the preimage are equipped with an antiholomorphic involution, which is preserved by f . In analogy with [A2–A4], we call a real rational function with the maximal possible number of real critical values a *rational M-function*. We say that a rational M-function is *generic* if all its critical values are distinct. Thus, any degree n generic rational M-function has exactly $2n - 2$ real and distinct critical values. We call two rational functions f_1 and f_2 *equivalent* if $f_1 = f_2 \circ L$, where L is an arbitrary linear fractional transformation of the independent variable. In the present paper we study the number $\sharp_n^{\mathbb{R}}$ of nonequivalent generic degree n rational M-functions with a given set of $2n - 2$ real distinct critical values. The following result is a prototype for the main theorem of this note.

Theorem A (see [A3]). *The number \sharp_n^{pol} of nonequivalent real degree n polynomials with the same set of $n - 1$ real distinct critical values equals the number K_{n-1} of up-down permutations of length $n - 1$. These numbers are called Euler-Bernoulli numbers and their generating function is*

$$\sum_{n=0}^{\infty} K_n \frac{t^n}{n!} = \sec t + \tan t.$$

REMARK. Note that if we drop the condition that the polynomials are real, then the number of nonequivalent degree n polynomials with the same set of $n - 1$

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distinct critical values equals n^{n-3} and is one of the simplest Hurwitz numbers, see [L]. Thus, even if all the critical values are real, only an exponentially small fraction of all classes of complex polynomials with these critical values contains real representatives.

Denote the set of all generic rational M-functions of degree n by MRat_n . A *planar chord diagram* of order $2m$ is a circle \mathcal{C} embedded in \mathbb{R}^2 with fixed $2m$ points $\{v_i\}$ called *vertices* connected pairwise by m nonintersecting curve segments $\{\chi_j\}$ called *chords*. The domain \mathcal{D} bounded by \mathcal{C} is called *the basic disk*. The $2m$ arcs $\{b_k\}$ in $\mathcal{C} \setminus \bigcup_{j=1}^{2m} v_j$ are called *boundary arcs*. The $m+1$ connected components $\{f_i\}$ of $\mathcal{D} \setminus \bigcup_{i=1}^m \chi_i$ are called *faces*. An *automorphism* of a planar chord diagram is a homeomorphism of its basic disk sending vertices to vertices and chords to chords. An alternative way to represent planar chord diagrams is provided by planar trees. The *associated planar tree* of a given planar chord diagram is the tree whose vertices are in 1-1-correspondence with the faces of the diagram, and are connected if and only if the corresponding faces are adjacent.

A planar chord diagram is called *properly oriented*, or *directed*, if all its chords and boundary arcs are provided with an orientation in such a way that the boundary of each face becomes a directed cycle. Obviously, in order to direct a planar chord diagram it suffices to direct any one of its edges. Therefore, there exist exactly two possible ways of directing a diagram, which are opposite to each other, i.e. the second one is obtained from the first one by reversing the direction of every edge. Once and for all fixing the standard orientation of the plane, we call a face of a directed planar chord diagram *positive* if the face lies to the left when we traverse its boundary according to the chosen direction, and *negative* otherwise.

Consider a cyclically ordered labeling set $\mathcal{S} = \{1 \prec 2 \prec 3 \prec \dots \prec 2m \prec 1\}$. A properly oriented planar chord diagram is called *properly labeled* if its $2m$ vertices are labeled by pairwise distinct elements from \mathcal{S} so that the labels arising on the boundary of each face traversed according to its orientation form a cyclically ordered subset of \mathcal{S} , see Fig 1. Finally, two properly oriented and properly labeled planar chord diagrams are called *equivalent* if there exists a homeomorphism of the basic disks (in general, not preserving the orientation of the preimage disk) sending vertices to vertices, chords to chords, and preserving labels and orientations.

REMARK. The group C_{2m} of cyclic shifts of the labels in the set \mathcal{S} acts on the equivalence classes of properly oriented and labeled planar diagrams of order $2m$ (preserving the class of the underlying properly oriented diagram). Note that this action is free if one forbids automorphisms of the underlying diagram.

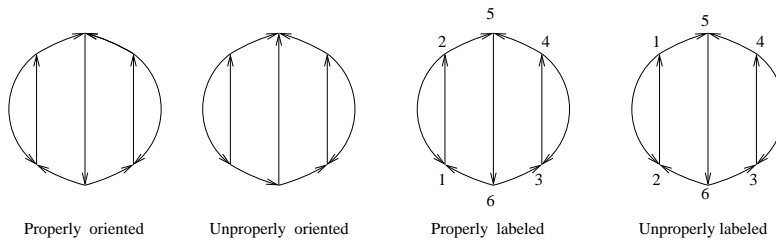


FIG. 1. PROPERLY AND UNPROPERLY ORIENTED, PROPERLY AND UNPROPERLY LABELED PLANAR DIAGRAMS

Now we can formulate the main results of the present note.

Theorem 1. *For $n \geq 3$ the number $\sharp_n^{\mathbb{R}}$ of nonequivalent generic degree n rational M -functions with a given set of $2n - 2$ real distinct critical values equals the number of nonequivalent properly oriented and labeled planar chord diagrams of order $2n - 2$. (The cases $n = 1, 2$ are exceptional and $\sharp_1^{\mathbb{R}} = \sharp_2^{\mathbb{R}} = 1$.)*

REMARK. Note that if we drop the condition that rational functions are real then the number of nonequivalent degree n rational functions with the same set of $2n - 2$ distinct critical values equals $n^{n-3}(2n - 2)!/n!$ and is one of the simplest Hurwitz numbers, see e.g. [CrT]. Thus even if all the critical values are real it seems plausible that only the exponentially small part of all classes of complex rational with these critical values contains real representatives.

The numbers $\sharp_n^{\mathbb{R}}$ can be considered as real analogs of the classical Hurwitz numbers, but they seem to be more difficult to calculate. The first interesting values of $\sharp_n^{\mathbb{R}}$, $n = 3, 4, 5$ are 2, 20, 406 resp. (see examples below).

Theorem 1 presents the number $\sharp_n^{\mathbb{R}}$ as the sum of the number of proper labelings over the set of all properly oriented planar chord diagrams of order $2n - 2$. The set of all properly oriented planar chord diagrams (considered up to plane homeomorphisms) is rather an inconvenient object. There is a more convenient base of summation for $\sharp_n^{\mathbb{R}}$ described below.

A planar chord diagram D is said to be *rooted* if one of its vertices is distinguished; the latter is then called the *root*. A chord of D is called a *diameter* if there exists a nontrivial automorphism of D that fixes both endpoints of this chord; evidently, any diagram has at most one diameter. Given a rooted planar chord diagram D , let $\sharp_r(D)$ denote the number of proper labelings of D with label 1 placed at the root and one of the two possible orientations D^+ or D^- fixed. The action of the cyclic group on the set of all labels shows that $\sharp_r(D)$ is independent of the choice of orientation and of the root, provided the root is not an endpoint of a diameter (in the latter case the number of proper labelings is twice smaller). In other words, $\sharp_r(D)$ depends only on the planar chord diagram, or equivalently, on the associated planar tree $T(D)$. For this reason instead of $\sharp_r(D)$ we can write $\sharp(T)$, where $T = T(D)$.

Theorem 1'. *For $n \geq 3$ the number $\sharp_n^{\mathbb{R}}$ of nonequivalent generic degree n rational M -functions with a given set of $2n - 2$ real distinct critical values equals*

$$\sharp_n^{\mathbb{R}} = \sum_{T \in \mathcal{P}_n} \sharp(T),$$

where \mathcal{P}_n is the set of all planted trees on n vertices.

The number of connected components of the set MRat_n is given by the following result.

Theorem 2. *For $n \geq 3$ the number $\widetilde{\sharp}_n^{\mathbb{R}}$ of connected components of the set MRat_n of all generic degree n rational M -functions equals the number of C_{2n-2} -orbits on the set of nonequivalent properly oriented and labeled planar chord diagrams of order $2n - 2$.*

The first interesting values of $\widetilde{\sharp}_n^{\mathbb{R}}$, $n = 3, 4, 5$ are 1, 4, 55 resp. (see examples below).

REMARK. The above Theorems 1 and 2 can be extended to a more general situation of generic rational functions whose critical values are not necessary real, see [NSV].

EXAMPLES. We start from the first nontrivial case $n = 3$. In this case there is only one planar chord diagram; its associated planar tree contains two edges. This diagram is *orientation-symmetric* (that is, it has an automorphism sending one proper orientation to the other). It has 2 distinct proper labelings that belong to the same unique C_4 -orbit. So, there are two different rational M-functions of degree 3, and the space MRat_3 is connected.

For $n = 4$ there exist two different planar chord diagrams whose trees are given in the right-hand side of Figure 2. Both diagrams are orientation-symmetric. The first of them has 18 proper labelings split into three full C_6 -orbits (i.e. of length 6) presented on the left upper part of Fig.2. The second one has 2 proper labelings forming a single C_6 -orbit. The total number of distinct real M-functions of degree 4 is thus 20, and the number of connected components in MRat_4 equals 4.

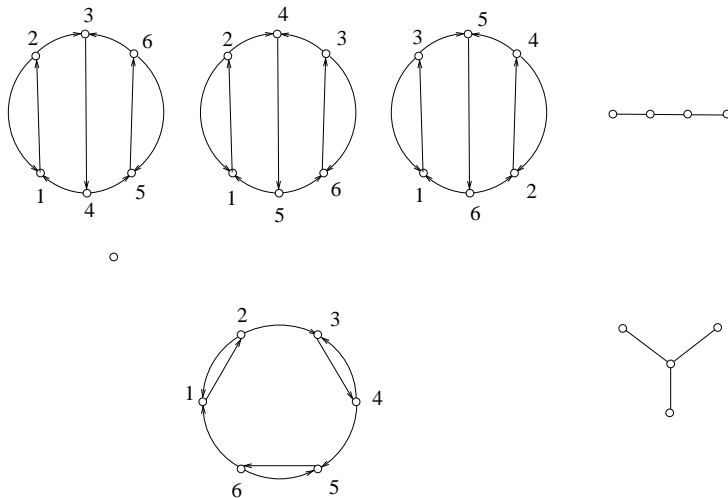


FIG. 2. FOUR C_6 -ORBITS PRESENTING 20 PROPERLY LABELED PLANAR DIAGRAMS FOR $n = 4$

For $n = 5$ there are 3 planar chord diagrams: the one with three square faces, the one with a square face and a hexagonal face, and the one with an octagonal face. The first and the third of them are orientation-symmetric while the second is not. The first diagram has 284 proper labeling split into 32 full orbits of C_8 (i.e. of length 8) and seven C_8 -orbits of length 4. The second diagram has 120 proper labelings split into 15 full orbits of C_8 . Finally, the third diagram has two proper labelings forming a single C_8 -orbit. Therefore, there are 406 distinct real M-functions, and the number of connected components in MRat_5 equals 55.

Another way to get the same number is to use Theorem 1'. There exist 3 planar trees on 5 vertices, which can be planted in 14 different ways, see Fig. 3. The numbers $\sharp(T)$ for the three trees are 71, 15, and 1. We thus get the same numbers 284, 120, and 2.

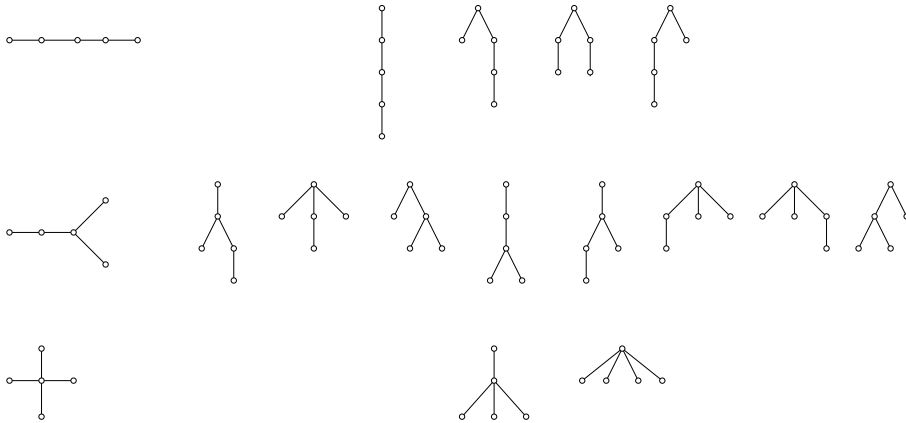


FIG. 3. THREE PLANAR TREES ON 5 VERTICES AND THEIR PLANTINGS

Below we present certain partial results on counting the number of proper labelings of a given planar chord diagram.

We say that a planar chord diagram is of P_k -type if its associated planar tree is a path on $k + 1$ vertices. Let $\sharp(P_n)$ denote the number of nonequivalent properly oriented and labeled planar chord diagrams of P_n -type.

Given a permutation $\sigma \in S_n$ we define its *up-down sequence* as a word of length $n - 1$ in the alphabet $\{U, D\}$ whose i th letter equals U if and only if $\sigma_i < \sigma_{i+1}$. We say that $\sigma \in S_{2n+1}$ is a *2up-2down* permutation if its up-down sequence equals $(UUDD)^k$ for $n = 2k$ or $(UUDD)^kUU$ for $n = 2k + 1$.

Theorem 3. *For any $n \geq 1$ the number $\sharp(P_n)$ equals n times the number of 2up-2down permutations of length $2n - 1$.*

2up-2down and similar permutations were studied in [CaS1, CaS2, CGJN]. Theorem 3 together with results obtained in [CaS2] yields the following corollary. Denote by $F_P(x)$ the modified exponential generating function

$$F_P(x) = \sum_{n=1}^{\infty} \sharp(P_n) \frac{x^{2n}}{(2n)!}.$$

Corollary 1. *The generating function $F_P(x)$ is given by*

$$F_P(x) = \frac{x}{2} \cdot \frac{\varphi_0(x)\varphi_1(x) - \varphi_2(x)\varphi_3(x) + \varphi_3(x)}{\varphi_0^2(x) - \varphi_1(x)\varphi_3(x)},$$

where

$$\varphi_j(x) = \sum_{k=0}^{\infty} \frac{x^{4k+j}}{(4k+j)!} \quad j = 1, 2, 3, 4,$$

is the j th Olivier function.

Combining Corollary 1 with results obtained in [CaS1] one gets the following asymptotic estimate.

Corollary 2. $\#(P_n) \sim \alpha \frac{n(2n-1)!}{\gamma^{2n}}$, where α is a constant and $\gamma = 1.8750\dots$ is the smallest positive solution of the equation $\cos z \cosh z + 1 = 0$.

More generally, consider planar chord diagrams whose associated planar tree is a caterpillar, that is, consists of a path and of an arbitrary number of edges incident to internal vertices of the path. Let us order the vertices in the path of a caterpillar from one endpoint to the other, and let d_1, \dots, d_k be the sequence of the degrees of the internal vertices of the caterpillar; evidently, $d_i \geq 2$ for $1 \leq i \leq k$. In this case we say that the chord diagram is of type $(\delta_1, \dots, \delta_k)$, where $\delta_i = 2(d_i - 1)$. In particular, for the P_k -type one has $\delta_i \equiv 2$.

Let D be a planar chord diagram of type $(\delta_1, \dots, \delta_k)$, and let D^+, D^- be the two properly oriented diagrams corresponding to D . Denote by $\#(D)$ the number of nonequivalent proper labelings of D^+ and D^- . The following result is a generalization of Theorem 3 for caterpillars.

Theorem 4. For any $k \geq 1$, the number $\#(D)$ equals n^* times the number of permutations of length $2e - 1$ whose up-down sequence equals $U^{\delta_1} D^{\delta_2} \dots$, where

$$n^* = \begin{cases} \frac{2e}{|\text{Aut}(D^+)|} & \text{if } D \text{ is orientation-symmetric,} \\ \frac{4e}{|\text{Aut}(D^+)|} & \text{otherwise,} \end{cases}$$

$\text{Aut}(D^+)$ is the group of automorphisms of the oriented diagram D^+ , and $e = 1 + \frac{1}{2} \sum_{i=1}^k \delta_i$ is the number of edges in the corresponding caterpillar.

Permutations with a given up-down sequence were studied in many papers, starting with the classic work of MacMahon (see [MM]). His approach leads to determinantal formulas for the number of such permutations, rediscovered later by Niven [Ni] from very basic combinatorial considerations. For the relations of this approach to the representation theory of the symmetric group see [Fo, St]. Another, purely combinatorial approach to the same problem was suggested by Carlitz [Ca]. However, his general recursive formula for the number of permutations with a given up-down sequence is rather difficult to use.

Using the results of MacMahon and Niven, we can reformulate Theorem 4 as follows. Let $\varepsilon_i = \delta_1 + \delta_2 + \dots + \delta_i$, and let $\{s_1 < s_2 < \dots\}$ denote the sequence

$$\varepsilon_1 + 1, \varepsilon_1 + 2, \dots, \varepsilon_2, \varepsilon_3 + 1, \dots, \varepsilon_4, \dots, \varepsilon_{2i-1} + 1, \dots, \varepsilon_{2i}, \varepsilon_{2i+1} + 1, \dots;$$

extend this sequence by adding $s_0 = 0$ and by appending $2e - 1$.

Corollary 3. For any $k \geq 1$ and any diagram D of type $(\delta_1, \dots, \delta_k)$ one has

$$\#(D) = n^* \cdot \det \left(\begin{pmatrix} s_i \\ s_{j-1} \end{pmatrix} \right).$$

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§2. GENERAL CONSTRUCTIONS AND PROOFS

Proof of Theorem 1. Let $f: \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$ be a real rational function with all distinct real critical values (a generic M -function). Consider the set $C(f)$ of its critical values.

Following [Vi], we define the *net* $\mathcal{N}(f)$ of the function f as the inverse image of the real cycle $\mathbb{R}P^1 \subset \mathbb{C}P^1$. Evidently, $\mathcal{N}(f)$ contains $\mathbb{R}P^1$ and is invariant with respect to the complex conjugation. Moreover, all the critical points of f are real as well; they belong to $\mathbb{R}P^1 \subset \mathcal{N}(f)$ and are the branching points of $\mathcal{N}(f)$ over $\mathbb{R}P^1$. Each critical point is adjacent to exactly 4 arcs (edges) belonging to $\mathcal{N}(f)$. These arcs do not intersect outside $\mathbb{R}P^1 \subset \mathcal{N}(f)$ and split into pairs invariant under complex conjugation. Let us fix some orientation of $\mathbb{R}P^1 \subset \mathbb{C}P^1$ in the image. Then all edges of $\mathcal{N}(f)$ will obtain the induced orientation.

Let us associate to a net $\mathcal{N}(f)$ the properly oriented and labeled planar chord diagram $X(f)$ defined as follows. Roughly speaking, the chord diagram is the part of $\mathcal{N}(f)$ contained in the closed upper halfplane. More exactly, let us cyclically label the critical values along $\mathbb{R}P^1$ by the numbers $1, 2, \dots, 2n-2$ from the labeling set S . The number of vertices in $X(f)$ equals $2n-2$ and they are in a 1-1-correspondence with the critical points of $\mathcal{N}(f)$. Two vertices are connected by a chord if and only if the corresponding critical points are connected by an arc belonging to $\mathcal{N}(f)$. The orientation of any chord (including the boundary arcs) coincides with the induced orientation of the arcs in $\mathcal{N}(f)$. Finally, each vertex is labeled by the label of the corresponding critical value. The connected components in $\mathbb{C}P^1 \setminus \mathcal{N}(f)$ are called the *faces* of the net $\mathcal{N}(f)$. Obviously, the faces of the net are split into complex conjugate pairs, and each such pair corresponds to a single face in $X(f)$. Using a result by Caratheodory on the correspondence of the boundaries under conformal map, one can easily check that

- a) f maps homeomorphically each face to one of the hemispheres in $\mathbb{C}P^1 \setminus \mathbb{R}P^1$;
- b) f maps homeomorphically the boundary of each face of the net to $\mathbb{R}P^1$ (see e.g., [Vi]).

Conditions a) and b) imply that $X(f)$ is properly labeled and oriented.

The following statement accomplishes the proof of Theorem 1.

Realization Theorem. *Let X be a properly oriented and labeled planar chord diagram of order $2n-2$, C be a finite subset on $\mathbb{R}P^1$ consisting of $2n-2$ distinct points with a cyclic order induced by the positive orientation of $\mathbb{R}P^1$. Label points in C by the numbers $1 \prec 2 \prec \dots \prec 2n-2 \prec 1$ preserving the cyclic order. Then there exists a degree n generic M -function f with $C(f) = C$ and $X(f) = X$, whose properly oriented and labeled chord diagram coincides with X .*

The proof of the Realization Theorem is completely parallel to the proofs of similar results in [Vi, SV]. We provide a sketch below.

In order to verify that for any properly oriented and labeled chord diagram X satisfying 1) and 2) there exists a rational function with the equivalent net, we construct, using X , a branched covering satisfying conditions a) and b) in the proof above. Having done that, we induce the complex structure on the preimage from that on the image, and the topological branched covering transforms into a holomorphic one. The construction of the topological covering goes as follows. Note that it suffices to define it on the disc containing the chord diagram, and then to glue the map $\mathbb{C}P^1$ to $\mathbb{C}P^1$ by pasting together the initial map and its complex

conjugate along the real cycle. In order to determine our map on the disc, we first map the vertices of the chord diagram to the corresponding critical values. Then we map the chords connecting the vertices to the arcs of $\mathbb{R}P^1$ between the corresponding critical values, taking into account their orientation. Finally, the faces of the chord diagram are mapped homeomorphically to the upper or lower hemispheres according to their orientations. \square

Proof of Theorem 1'. The proof follows immediately from the proposition below. Let $\pi(D)$ denote the number of all distinct plantings of the trees $T(D)$ and $T'(D)$ obtained from $T(D)$ by a reflection.

Proposition. *The number of all distinct proper labelings of D^+ and D^- equals $\pi(D)\sharp_r(D)$.*

Proof. All the possible proper labelings of D^+ are obtained from the $\sharp_r(D)$ labelings with the root labeled by 1 via the action of the cyclic group C_{2e} on the labels, where e is the number of chords in D . Moreover, each proper labeling is obtained in this way exactly $|\text{Aut}(D^+)|$ times. If D is orientation-symmetric, then the contribution of D^- in the total number of nonequivalent labelings equals 0, since any labeling of D^- is equivalent to some labeling of D^+ . If D is not orientation-symmetric, then the contribution of D^- equals the contribution of D^+ .

On the other hand, there are exactly $2e$ possibilities to plant the tree $T(D)$, and each planting is obtained exactly $|\text{Aut}(T(D))|$ times. Hence, $\pi(D)$ equals $2e/|\text{Aut}(T(D))|$ if there exists an automorphism of $T(D)$ and $T'(D)$ (preserving the orientation of \mathbb{R}^2), and twice this number otherwise.

Introduce

$$\alpha(D) = \begin{cases} 2|\text{Aut}(D^+)| & \text{if } D \text{ is orientation-symmetric,} \\ |\text{Aut}(D^+)| & \text{otherwise,} \end{cases}$$

$$\beta(D) = \begin{cases} 2|\text{Aut}(T(D))| & \text{if there exists an automorphism of } T(D) \text{ and } T'(D), \\ |\text{Aut}(T(D))| & \text{otherwise.} \end{cases}$$

It remains to prove that $\alpha(D) = \beta(D)$. The first of the above numbers can be interpreted as the number of automorphisms φ of the pair $\{D^+, D^-\}$ such that $\varphi(D^+)$ and $\varphi(D^-)$ are two opposite orientations. The second of the above numbers can be interpreted as the number of automorphisms ψ of the pair $\{T(D), T'(D)\}$ such that $\psi(T(D))$ and $\psi(T'(D))$ differ by reflection. It is easy to see that these two numbers coincide. \square

Proof of Theorem 2. The crucial ingredient of this proof is the following modification of the well-known construction sending a branched covering to the set of its branching points, cf. e.g. [A1] or [L]. Denote by Δ_m the connected set of all unordered m -tuples of pairwise distinct points on $\mathbb{R}P^1$. Obviously, the space Δ_m is fibered over $\mathbb{R}P^1$ with $(m-1)$ -dimensional contractible fibers, and therefore $\pi_1(\Delta_m) = \mathbb{Z}$. (In order to convince yourself think of m electrons on $\mathbb{R}P^1$.) Note that the group $PSL_2(\mathbb{R})$ of real linear fractional transformations $x \mapsto \frac{ax+b}{cx+d}$ with $ad-bc=1$ acts freely on the space MRat_n of generic rational M-functions and preserves all the critical values. We now define the *generalized Lyashko-Looijenga mapping*

$$LL: \text{MRat}_n / PSL_2(\mathbb{R}) \rightarrow \Delta_{2n-2}$$

sending the equivalence class of a generic M-function to the unordered set of its branching points. Note that LL is a map between the spaces of the same dimension, and by Theorem 1 the number of inverse images of any point in Δ_{2n-2} is constant.

Lemma 1. *The map $LL: \text{MRat}_n / \text{PSL}_2(\mathbb{R}) \rightarrow \Delta_{2n-2}$ is a finite covering of degree $\frac{\mu_n^{\mathbb{R}}}{\mu_n}$. Moreover the action of $\pi_1(\Delta_{2n-2})$ on the fiber coincides with the action of the group C_{2n-2} of the cyclic shifts of labels on the set of all properly directed and labeled plane chord diagrams of order $2n - 2$.*

Proof. To get a covering we need to show that

- i) the map LL is open, i.e. it is a local homeomorphism onto the image;
- ii) the map LL is surjective.

Property i) is shown in [K] in a more general situation. The above Realization Theorem provides the surjectivity. Finally, Theorem 1 shows that the number of points in each fiber equals $\frac{\mu_n^{\mathbb{R}}}{\mu_n}$. Let us study the action of $\pi_1(\Delta_{2n-2})$ on the fiber of LL . Without loss of generality we can, for example, choose a base point pt in Δ_{2n-2} presenting the configuration of critical values forming some regular $(2n - 2)$ -gon Π on $\mathbb{R}P^1$. The generator of $\pi_1(\Delta_{2n-2})$ can be then geometrically realized as the family $\Pi(t)$, $t \in [0, \frac{2\pi}{2n-2})$ of all regular $(2n - 2)$ -gons. Let us choose some rational function f with the set of critical values given by $\Pi = \Pi(0)$. Let us now rotate Π in the family $\Pi(t)$ and follow the branch of $f(t)$, $f(0) = f$. All topological characteristics of the net $S(f(t))$ (and therefore of $X(f(t))$, see proof of Theorem 1) will be preserved. When t reaches $\frac{2\pi}{2n-2}$ (i.e when we come back to the same base point pt) the labels experience a cyclic shift by 1. \square

Theorem 2 is proven.

§3. CALCULATING THE NUMBER OF PROPER LABELINGS FOR A GIVEN PLANAR CHORD DIAGRAM

Proof of Theorem 3. Consider a properly oriented planar chord diagram of P_n -type. Such a diagram contains two pairs of *corners*, that is, two pairs of vertices connected both by a chord and by a boundary arc (e.g. pairs 1, 2 and 3, 4 on Fig. 1). Let us fix a corner c in such a way that the chord joining c with the adjacent corner in the corresponding pair, c' , is directed from c to c' . We consider all the proper labelings of the diagram satisfying the additional condition: the corner c is labeled by 1.

Let us establish a bijection π between the set of all such labelings and the set of all 2up-2down permutations of length $2n - 1$. First, we start at c and move along the chord to c' . The only way to leave c' is along the boundary arc (the one not leading back to c). We proceed in this way, choosing each time a chord or a boundary arc leading to a vertex not visited previously. Observe that each time there exists a unique possibility to proceed, until we visit two other corners, and the process stops. We thus get a unique linear order on the vertex set of the diagram. Let $\mathcal{L} = (l_1 = 1, l_2, \dots, l_{2n})$ be the sequence of labels of the vertices written in this order.

Recall that all the labels together form a cyclically ordered set $\mathcal{S} = \{1 \prec 2 \prec \dots \prec 2n \prec 1\}$. To find the permutation $\sigma = \pi(\mathcal{L})$, it is convenient to consider its entries as elements of a linearly ordered set $\Sigma = \{1 < 2 < \dots < 2n - 1\}$. The sequence \mathcal{L} can be decomposed into (intersecting) cyclically ordered quadruples $\{l_1 \prec l_2 \prec l_3 \prec l_4 \prec l_1\}$, $\{l_3 \prec l_4 \prec l_5 \prec l_6 \prec l_3\}, \dots$, each representing a face of the chord diagram. On the other hand, any 2up-2down permutation $\sigma =$

$(\sigma_1, \dots, \sigma_{2n-1})$ can be decomposed into (intersecting) linearly ordered triples $\{\sigma_1 < \sigma_2 < \sigma_3\}$, $\{\sigma_3 > \sigma_4 > \sigma_5\}$, \dots . Bijection π takes the k th quadruple in \mathcal{L} to the k th triple in σ . The main building blocks of the bijection π are the following two bijections between quadruples and triples.

Let $\mathcal{R} = \{r_1 < r_2 < \dots < r_m < r_1\}$ be a cyclically ordered set, and let r be an arbitrary element in \mathcal{R} . Consider the set \mathcal{R}_r^4 of all quadruples of the form $\{r < a < b < c < r\}$, $a, b, c \in \mathcal{R}$, and denote by f_1, f_2, f_3, f_4 the numbers of elements in \mathcal{R} lying between r and a , a and b , b and c , and c and r , respectively (see Figure 4). On the other hand, let $\mathcal{T} = \{t_1 < t_2 < \dots < t_{m-1}\}$ be a linearly ordered set, and let \mathcal{T}_+^3 and \mathcal{T}_-^3 be the sets of all increasing and decreasing triples in \mathcal{T} , respectively. Transformation $\pi_{r,m}^+$ takes the quadruple $\{r < a < b < c < r\}$ to the triple $\{\tau_1 < \tau_2 < \tau_3\}$ such that the number of elements in \mathcal{T} less than τ_1 equals f_1 , the number of elements in \mathcal{T} lying between τ_1 and τ_2 equals f_2 , and the number of elements in \mathcal{T} lying between τ_2 and τ_3 equals f_4 (see Fig. 4).

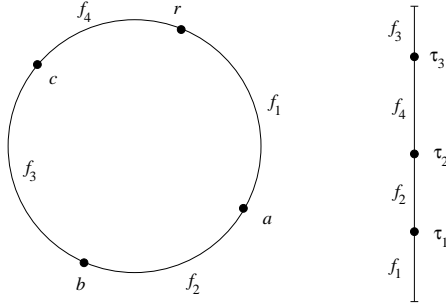


FIG. 4. BIJECTION $\pi_{r,m}^+$

It is easy to see that $\pi_{r,m}^+$ is a bijection between \mathcal{R}_r^4 and \mathcal{T}_+^3 , and that the number of elements in \mathcal{T} greater than τ_3 equals f_3 . Similarly, $\pi_{r,m}^-$ takes $\{r < a < b < c < r\}$ to the triple $\{\tau'_1 > \tau'_2 > \tau'_3\}$ such that the number of elements in \mathcal{T} greater than τ'_1 equals f_1 , the number of elements in \mathcal{T} lying between τ'_1 and τ'_2 equals f_2 , and the number of elements in \mathcal{T} lying between τ'_2 and τ'_3 equals f_4 . It is easy to see that $\pi_{r,m}^-$ is a bijection between \mathcal{R}_r^4 and \mathcal{T}_-^3 , and that the number of elements in \mathcal{T} less than τ'_3 equals f_3 .

To establish the bijection π , we start from the quadruple $\{1 < l_2 < l_3 < l_4 < 1\}$ and use $\pi_{1,2n}^+$ to get $\{\sigma_1, \sigma_2, \sigma_3\}$. We now delete 1 and l_2 from \mathcal{S} , delete σ_1 and σ_2 from Σ , and use $\pi_{l_3,2n-2}^-$ to get $\{\sigma'_3, \sigma'_4, \sigma'_5\}$. Observe that the properties of $\pi_{r,m}^+$ and $\pi_{r,m}^-$ stated above ensure that $\sigma'_3 = \sigma_3$, and hence we can glue the obtained triples into a 2up-2down sequence of length 5. On the next step we delete l_3 and l_4 from \mathcal{S} , as well as σ_3 and σ_4 from Σ , and use $\pi_{l_5,2n-4}^+$, and so on. On the k th step we define $\mathcal{R} = \mathcal{S} \setminus \{l_1, l_2, \dots, l_{2k-2}\}$, $\mathcal{T} = \Sigma \setminus \{\sigma_1, \sigma_2, \dots, \sigma_{2k-2}\}$, $m = 2n - 2k + 2$, $\varepsilon = +$ if k is even and $\varepsilon = -$ otherwise, and use $\pi_{l_{2k-1},m}^\varepsilon$ to transform $\{l_{2k-1} < l_{2k} < l_{2k+1} < l_{2k+2} < l_{2k-1}\} \in \mathcal{R}_{l_{2k-1}}^4$ into $\{\sigma_{2k-1} < \sigma_{2k} < \sigma_{2k+1}\} \in \mathcal{T}_\varepsilon^3$.

The rest of the proof follows immediately from the Proposition. \square

Proof of Corollary 1. We use the following result proved in [CaS2, Theorem 1]. Let $A(k)$ denote the number of 2up-2down permutations of length k ; as mentioned in

§1, such permutations are defined only for odd k . Then

$$\sum_{n=0}^{\infty} A(4n+1) \frac{x^{4n+1}}{(4n+1)!} = \frac{\varphi_0(x)\varphi_1(x) - \varphi_2(x)\varphi_3(x)}{\varphi_0^2(x) - \varphi_1(x)\varphi_3(x)},$$

$$\sum_{n=0}^{\infty} A(4n+3) \frac{x^{4n+3}}{(4n+3)!} = \frac{\varphi_3(x)}{\varphi_0^2(x) - \varphi_1(x)\varphi_3(x)},$$

where $\varphi_j(x)$ is the j th Olivier function (see §1 for the definition). Corollary 1 follows immediately from this result and Theorem 3. \square

Proof of Theorem 4. The proof goes along the same lines as the proof of Theorem 3. The main difference is that the sequence of labels is decomposed into intersecting cyclically ordered subsequences of sizes $2d_1, 2d_2, \dots$. As before, the intersection of two consequent subsequences consists of two adjacent labels. In the P_k -case these two labels formed a suffix of the former subsequence and a prefix of the following one. In the present case, we still maintain that the intersection is a prefix of the following subsequence; however, it may lie somewhere in the middle of the former subsequence.

Accordingly, we decompose the permutation σ into linearly ordered subsequences of sizes $2d_1 - 1, 2d_2 - 1, \dots$, such that each odd subsequence is increasing and each even subsequence is decreasing. As before, we establish bijections between the corresponding cyclical and linear subsequences. Assume that we have a cyclically ordered k -tuple $\{a_1 \prec a_2 \prec \dots \prec a_k \prec a_1\}$, and that the following cyclically ordered subsequence intersects with this k -tuple by the elements a_j and a_{j+1} ; evidently, $3 \leq j \leq k-1$. Let f_i denote the number of elements in \mathcal{R} lying between a_i and a_{i+1} , $i = 1, \dots, k$. Then we map this k -tuple to the sequence $\{\tau_1 < \tau_2 < \dots < \tau_{k-1}\}$ such that the number of elements in \mathcal{T} less than τ_1 equals f_1 , the number of elements in \mathcal{T} greater than τ_{k-1} equals f_j , and the number of elements in \mathcal{T} lying between τ_i and τ_{i+1} equals f_{i+1} for $1 \leq i \leq j-2$ and f_{i+2} for $j-1 \leq i \leq k-2$.

The rest of the proof carries over without substantial changes. \square

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