A FEW RIDDLES BEHIND ROLLE'S THEOREM

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Abstract. First year undergraduates usually learn about classical Rolle's theorem which says that between two consecutive zeros of a smooth univariate function f, one can always find at least one zero of its derivative f'. In this paper, we study a generalization of Rolle's theorem dealing with zeros of higher derivatives of smooth univariate functions enjoying a natural additional property. Namely, we call a smooth function whose nth derivative does not vanish on some interval $I \subseteq \mathbb{R}$ a *polynomial-like function of degree n on I*. We conjecture that for polynomial-like functions of degree n with n real distinct roots there exists a non-trivial system of inequalities completely describing the set of possible locations of their zeros together with their derivatives of order up to $n-1$. We describe the corresponding system of inequalities in the simplest non-trivial case $n = 3$.

1. Getting started

In what follows we only consider smooth real-valued functions defined on a subinterval of the real axis. Consider a smooth function f with n distinct real zeros $x_1^{(0)} < x_2^{(0)} < \ldots < x_n^{(0)}$ on some interval $I \subseteq \mathbb{R}$. Then, by Rolle's theorem, f' has at least $(n-1)$ zeros, f'' has at least $(n-2)$ zeros, ... , $f^{(n-1)}$ has at least one zero on the open interval $(x_1^{(0)}, x_n^{(0)})$. In what follows, we will be interested in smooth functions f with n real simple zeros on I such that for all $i = 1, \ldots, n$ the ith derivative $f^{(i)}$ has on I exactly $n-i$ real simple zeros denoted by $x_1^{(i)} < x_2^{(i)} < ... < x_{n-i}^{(i)}$. Note, in particular, that $f^{(n)}$ is non-vanishing on I.

Main Definition. A smooth function f defined on an interval I is called *polynomial-like of degree n* if $f^{(n)}$ does not vanish on I. A polynomial-like function of degree n on I with n simple real zeros is called *real-rooted*.

Notice that by Rolle's theorem n is the maximal possible number of real zeros of a polynomial-like function of degree n. An obvious example of a real-rooted polynomial-like function of degree n on $\mathbb{R}$ is a usual real polynomial of degree n with all real and distinct zeros. Observe also that if a polynomial-like function f of degree n is real-rooted on I, then for all $i < n$ its derivatives $f^{(i)}$ are also real-rooted of degree $n-i$ on the same interval. In the above notation the following system of inequalities holds:

$$x_l^{(i)} < x_l^{(j)} < x_{l+j-i}^{(i)} \quad \text{for } i < j \leq n-l. \tag{1}$$

We call (1) the system of *standard Rolle's restrictions.*

With any real-rooted polynomial-like function f of degree n, one can associate its configuration $\mathcal{A}_f$ of $\binom{n+1}{2}$ zeros $\{x_l^{(i)}\}$ of $f^{(i)}$ (for $i = 0, \ldots, n-1$ and $1 \leq l \leq n-i$), say, by taking first all $x_l^{(0)}$, then all $x_l^{(1)}$, etc.

The main problem we address in this short note is as follows.

Question. What additional restrictions besides (1) exist on configurations $\mathcal{A}_f = \{x_l^{(i)}\}$ coming from real-rooted polynomial-like functions of a given degree n? Or, even more ambitiously, given a configuration $\mathcal{A} = \{x_l^{(i)} \mid i = 0, \dots, n-1; l = 1, \dots n-i\}$ of $\binom{n+1}{2}$ real numbers satisfying standard Rolle's restrictions, is it possible to determine if there exists a real-rooted polynomial-like f of degree n such that $\mathcal{A}_f = \mathcal{A}$?

Our motivation to consider the class of real-rooted polynomial-like functions is twofold. First, this class is a natural generalization of the well-studied and important class of real-rooted polynomials. Second, one can easily see that as soon as one allows several real zeros of f' between two consecutive zeros of f, then no interesting additional restrictions are possible. We will soon show that the fact that a smooth function f is real-rooted polynomial-like implies additional inequalities on the components of $\mathcal{A}_f$.

Let $\mathcal{RR}_n(I)$ denote the set of all real-rooted polynomial-like functions of degree n on an interval I (since the particular choice of I is unimportant in the formulation below, we will often use $\mathcal{RR}_n$ instead of $\mathcal{RR}_n(I)$). Our result below answers the posed question in the first non-trivial case $n = 3$. In order to simplify our notation, we denote the three zeros of a real-rooted function f of degree 3 by $x_1 < x_2 < x_3$, the two zeros of f' by $y_1 < y_2$, and the only zero of f'' by z_1.

Main Theorem. The configuration $\mathcal{A}_f = (x_1, x_2, x_3, y_1, y_2, z_1)$ of any real-rooted polynomial-like function f of degree 3 satisfies the following inequalities:

$$\begin{cases} x_1 < y_1 < x_2 < y_2 < x_3, \quad y_1 < z_1 < y_2, \\ y_1 - x_1 < \min\left(x_2 - y_1, \sqrt{(z_1 - y_1)^2 + 2(z_1 - y_1)|z_1 - x_2|}\right), \\ x_3 - y_2 < \min\left(y_2 - x_2, \sqrt{(y_2 - z_1)^2 + 2(y_2 - z_1)|z_1 - x_2|}\right). \end{cases} \tag{2}$$

And, conversely, for any 6-tuple $(x_1, x_2, x_3, y_1, y_2, z_1)$ satisfying (2), there exists a real-rooted function f of degree 3 whose configuration $\mathcal{A}_f$ of the zeros of (f, f', f'') coincides with the given 6-tuple.

The inequalities on the first line of (2) are standard Rolle's restrictions. The geometrical meaning of the additional inequalities will be clear from the proof below (notice that two new inequalities above interchange places under the substitution $x \mapsto -x$).

Proof. Take some $f \in \mathcal{RR}_3$. Without loss of generality, we can assume $f''' > 0$ which implies that f' is convex (otherwise multiply f by -1). As above, the three zeros of f are denoted by $x_1 < x_2 < x_3$, the two zeros of f' by $y_1 < y_2$ and the only zero of f'' by z_1. Let us first consider the case $x_2 < z_1$, see Fig. 1.

We now derive the additional inequalities:

$$y_1 - x_1 < x_2 - y_1 \quad \text{and} \quad x_3 - y_2 < \sqrt{(y_2 - z_1)^2 + 2(y_2 - z_1)(z_1 - x_2)}. \tag{3}$$

The case $z_1 < x_2$ is completely analogous and leads to:

$$x_3 - y_2 < y_2 - x_2 \quad \text{and} \quad y_1 - x_1 < \sqrt{(z_1 - y_1)^2 + 2(z_1 - y_1)(x_2 - z_1)}. \tag{4}$$

Combining (3) and (4), one gets exactly the required new inequalities in (2). Indeed, the convexity of f' implies that:

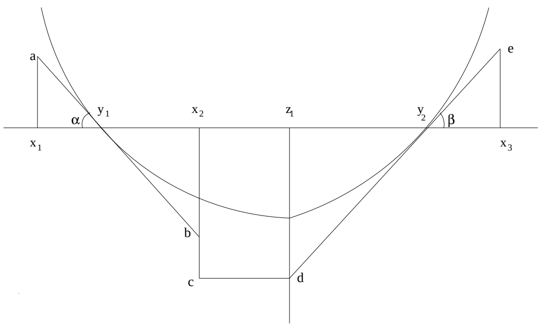

FIGURE 1. f' and its accompanying elementary configuration.

i) on (x_1, y_1) the graph of f' lies above the line segment $\{a, y_1\}$ which is tangent to it at y_1;
ii) on (y_1, x_2) the graph of f' lies between the x-axis and the line segment $\{y_1, b\}$ which is tangent to it at y_1;
iii) on (x_2, y_2) the graph of f' lies between the x-axis and the broken line segment $\{c, d, y_2\}$ where (c, d) is horizontal;
iv) on (y_2, x_3) the graph of f' lies above the line segment $\{y_2, e\}$ which is tangent to it at y_2.

Note that since $x_1 < x_2 < x_3$ are consecutive real zeros of f one has

$$\int_{x_1}^{x_2} f'dx = \int_{x_2}^{x_3} f'dx = 0$$

which implies that

$$\int_{x_1}^{y_1} f'dx = -\int_{y_1}^{x_2} f'dx \quad \text{and} \quad \int_{x_2}^{y_2} f'dx = -\int_{y_2}^{x_3} f'dx.$$

Therefore,

$$Ar(\triangle_{ax_1y_1}) < Ar(\triangle_{y_1bx_2}) \quad \text{and} \quad Ar(\triangle_{ey_2x_3}) < Ar(\square_{x_2cdy_2}),$$

where Ar stands for the area of the corresponding figures (the figure $\square_{x_2cdy_2}$ is a trapezoid). Using standard expressions for the area of a triangle and a trapezoid, we get

$$Ar(\triangle_{ax_1y_1}) = \frac{1}{2}(y_1-x_1)^2 \tan\alpha,\ Ar(\triangle_{ey_2x_3}) = \frac{1}{2}(x_3-y_2)^2 \tan\beta,\ Ar(\triangle_{y_1bx_2}) = \frac{1}{2}(x_2-y_2)^2 \tan\alpha,$$

and,

$$Ar(\square_{x_2cdy_2}) = Ar(\triangle_{z_1dy_2}) + Ar(\square_{x_2cdz_1}) = \frac{1}{2}(y_2-z_1)^2 \tan\beta + (y_2-z_1)(z_1-x_2)\tan\beta =$$

$$= \frac{1}{2}\left((y_2-z_1)^2 + 2(y_2-z_1)(z_1-x_2)\right)\tan\beta.$$

These relations immediately imply the required inequalities. To finish the proof, pick any 6-tuple of real numbers $x_1, x_2, x_3, y_1, y_2, z_1$ satisfying (2). Again we can assume that $x_2 < z_1$ (the case $z_1 \leq x_2$ is analogous). Draw a piecewise linear

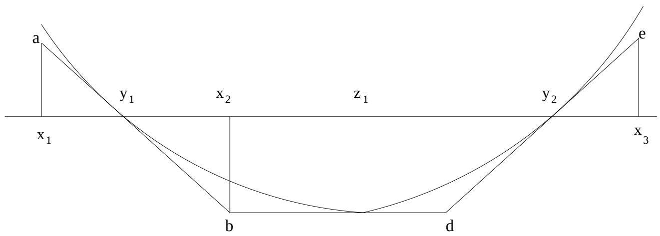

FIGURE 2. Constructing an appropriate function.

function g as shown on Fig. 2 whose only difference with Fig. 1 is that we force $c = b$.

Inequalities (2) imply that $Ar(\triangle_{ax_1y_1}) < Ar(\triangle_{y_1bx_2})$ and $Ar(\triangle_{ey_2x_3}) < Ar(\square_{x_2bdy_2})$. It is easy to approximate g by a convex function h such that:
i) y_1 and y_2) are the zeros of h and z_1 is the zero of h';
ii) $\int_{x_1}^{y_1} hdx < -\int_{y_1}^{x_2} hdx$ and $\int_{x_2}^{y_2} hdx < -\int_{y_2}^{x_3} hdx$.

Keeping the function h as it is on the interval (y_1, y_2), we can increase it on the intervals (x_1, y_1) and (y_2, x_3) and construct a new convex function $\tilde{h}$ with the properties:
i) y_1 and y_2 are the zeros of $\tilde{h}$ and z_1 is the zero of $\tilde{h}'$;
ii) $\int_{x_1}^{y_1} \tilde{h}dx = -\int_{y_1}^{x_2} \tilde{h}dx$ and $\int_{x_2}^{y_2} \tilde{h}dx = -\int_{y_2}^{x_3} \tilde{h}dx$.

The function $f \in \mathcal{RR}_3$ we were looking for is now obtained as $f = -\int_{y_1}^{x} \tilde{h}(t)dt$. □

Example. As an illustration of the above theorem, consider the real-rooted cubic polynomial $p(x) = x(x-1)(x-4)$. An elementary calculation gives that in the notation of the above theorem we have

$$x_1 = 0,\ \ x_2 = 1,\ \ x_3 = 4,\ \ y_1 = \frac{5-\sqrt{13}}{3},\ \ y_2 = \frac{5+\sqrt{13}}{3},\ \text{and } z_1 = \frac{5}{3}.$$

Therefore, $x_2 < z_1$, and we are interested in checking the validity of (3). Indeed, $y_1 - x_1 = \frac{5-\sqrt{13}}{3} \simeq 0.464816$ which is smaller than $x_2 - y_1 = \frac{\sqrt{13}-2}{3} \simeq 0.535184$. Next, $x_3 - y_2 = \frac{7-\sqrt{13}}{3} \simeq 1.13148$ which is smaller than $\sqrt{(z_1-y_1)^2 + 2(z_1-y_1)(x_2-z_1)} = \frac{\sqrt{13+4\sqrt{13}}}{3} \simeq 1.74554$.

Some further results extending the above main theorem can be found in [3] and [4].

2. Usual polynomials

Let us now discuss what meaningful restrictions there might exist on configurations $\mathcal{A}_p$ for the usual real-rooted polynomials (note that inequalities (2) remain valid, but they clearly do not give all the restrictions for the usual real-rooted polynomials since in this case the zeros of a polynomial define all the zeros of all its higher derivatives in a unique way). Let us assume that all real zeros $x_1^{(0)} < x_2^{(0)} < \ldots < x_n^{(0)}$ of a polynomial p of degree n are distinct. Assume additionally that all $x_l^{(i)}$ are pairwise distinct as well (one can check that the latter conditions hold for almost all real-rooted polynomials). We call such polynomials *strictly real-rooted*. For a strictly real-rooted polynomial p, its whole configuration $\mathcal{A}_p$ is naturally ordered on the real line. Substituting each zero of p by the symbol 0, each zero of p' by 1, ... , the unique zero of $p^{(n-1)}$ by $(n-1)$ respectively, we get a *symbolic sequence of* p of length $\binom{n+1}{2}$ with n occurrences of 0, $(n-1)$ occurrences of 1,..., one occurence of $(n-1)$. Standard Rolle's restrictions result in the condition that between any two consecutive occurrences of the symbol i in such a sequence one has exactly one occurrence of the symbol $i+1$.

For example, there are only two possible symbolic sequences for $n = 3$, namely, 012010 and 010210. For $n = 4$ there are 12 such sequences 0123012010, 0120312010, 0120132010, 0102312010, 0102132010, 0123010210, 0120310210, 0120130210, 0120103210, 0123010210, 0102130210, 0102103210. A patient reader will find that for $n = 5$ there are 286 such sequences.

If we denote by $\flat_n$ the number of all possible symbolic sequences of length n, then it is possible to calculate this number explicitly. It turns out to be equal to

$$\flat_n = \binom{n+1}{2}!\frac{1!2!...(n-1)!}{1!3!...(2n-1)!}.$$

Since this calculation requires a separate treatment we refer an interested reader to [7].

We can now formulate a natural discrete analog of the main problem from the previous section which makes sense for usual polynomials.

Question. What symbolic sequences can occur for strictly real-rooted polynomials of degree n? We call such sequences *realizable.*

As we will see shortly, the first nontrivial case $n = 4$, already shows that the number $\sharp_n$ of all realizable symbolic sequences is strictly smaller than the corresponding $\flat_n$, namely, $10 = \sharp_4 \neq \flat_4 = 12$.

The fact that $\sharp_n \neq \flat_n$ was apparently observed by a number of authors but the only relevant reference we found is [1] which was published in 1993. An explanation of this phenomenon for $n = 4$ is as follows (see further generalizations in [5]).

Theorem (see [1]). A polynomial p of degree 4 with real zeros $x_1 < x_2 < x_3 < x_4$ satisfying the inequalities $x_2 < z_1$ and $x_3 < z_2$, satisfies additionally the inequality $y_2 < t_1$. Here $y_1 < y_2 < y_3$ are the zeros of f'; the zeros of f'' are $z_1 < z_2$ and t_1 is the zero of f'''. In other words, the symbolic sequences 0102310210 and 0120132010 are non-realizable.

This is easy to check once you know what to prove! Indeed, any monic polynomial of degree 4 with all real zeros can be put in the form $x^4 - x^2 + ux + v$ by a linear change of x and scaling. Namely, by shifting $x \mapsto x + \alpha$ we can always get rid of the x^3-term. Since the second derivative of the obtained polynomial has two real zeros, the coefficient at x^2 should be negative. Appropriate scaling now puts p in the above form. Note that $p'' = 12x^2 - 2$, its zeros being $\pm\sqrt{\frac{1}{6}}$. The assumptions $x_2 < z_1$, $x_3 < z_2$ together with p having real zeros imply $p(-\sqrt{\frac{1}{6}}) > 0$, $p(\sqrt{\frac{1}{6}}) < 0$ (draw the graph of p). Noting that $p''' = 24x$, what we need to prove is that $p'(0) < 0$ (draw the graph of p'). The last inequality is equivalent to $u < 0$. Expanding

$p(-\sqrt{\frac{1}{6}}) > 0$ and $p(\sqrt{\frac{1}{6}}) < 0$, we get $\frac{1}{36} - \frac{1}{6} - \frac{u}{\sqrt{6}} + v > 0$ and $\frac{1}{36} - \frac{1}{6} + \frac{u}{\sqrt{6}} + v < 0$. Subtraction of the former inequality from the latter implies that $\frac{2u}{\sqrt{6}} < 0$. □

The next case $n = 5$ was considered in [2]. V. Kostov was able to show that among 286 possible symbolic sequences, only 116 are realizable by strictly real-rooted polynomials. Later, the same author also considered which symbolic sequences are realizable in the case of real-rooted polynomial-like functions, see [3]. It turned out that for $n = 4$ all 12 symbolic sequences are realizable, but already for $n = 5$ there are non-realizable sequences. The situation does not seem to change much if we instead of actual real-rooted polynomials consider real-rooted polynomial-like functions.

Finally, let us present a tempting problem posed by the famous Russian mathematician Vladimir Arnold (who unfortunately passed away in June 2010) after a talk given by V. Kostov in 2007.

Problem 1. Is it true that $\lim_{n\to\infty} \frac{\sharp_n}{\flat_n} = 0$? If yes, how fast does the quotient $\frac{\sharp_n}{\flat_n}$ decrease?

3. Periodic functions

Let us briefly discuss what happens with periodic functions, i.e., functions defined on a circle. In the previous sections, we defined the class of real-rooted polynomial-like functions – a generalization of real-rooted polynomials of degree n with distinct real zeros – and found some inequalities involving the zeros of their higher derivatives valid for any such function of degree 3. It seems quite natural to try to develop a similar concept for periodic functions. The periodic analog of polynomials of degree n are trigonometric polynomials of degree n, i.e., expressions of the form $a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$. Any trigonometric polynomial of degree n has at most $2n$ real zeros on a period.

Observe that if we take such a trigonometric polynomial with exactly $2n$ real and distinct zeros, then its derivative of *any* order will also be a trigonometric polynomial of the same degree n. Moreover, by Rolle's theorem, it will also have exactly $2n$ real and distinct zeros on a period. So it seems tempting to define a periodic analog of a real-rooted polynomial-like function as a periodic function such that it and its derivatives of any order have exactly $2n$ real zeros. But the situation with periodic functions turns out to be much more rigid than that with functions on an interval and it is not completely clear why.

To explain the situation, we have to invoke the famous classical result of G. Polya and N. Wiener, see [6].

Theorem. Any periodic function f such that as $i \to \infty$ the number of real zeros of the ith derivative $f^{(i)}$ on a period remains bounded is a trigonometric polynomial.

In particular, any (conjectural) real-rooted polynomial-like periodic function must necessarily be an actual trigonometric polynomial. On the other hand, one can define a periodic analog of symbolic sequences from the previous section and ask which of those are realizable by trigonometric polynomials with all real zeros. Namely, for a positive integer k, consider a sequence of integers of length nk written on a circle and containing n zeros, n ones, n twos,..., n copies of $(k-1)$. We call such a sequence *periodic* if for any $i = 0, 1, ..., k-2$ in between any two consecutive (on the circle) copies of i, the sequence contains exactly one copy of $i+1$.

Problem 2. What periodic sequences can occur as the sequences of zeros of f and its higher derivatives of order up to $k-1$, where f is a trigonometric polynomial of degree n with all real and distinct zeros (here the integer i substitutes a real root of the ith derivative of f)?

Unfortunately, there is no interesting information about the latter problem available at present.

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