Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 305 (2025) 106114

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

Full Length Article

In search of a higher Bochner theorem

Emil Horozov[a,b], Boris Shapiro[c,*], Miloš Tater[d]

[a] Department of Mathematics and Informatics, Sofia University, 5 J. Bourchier Blvd., Sofia 1126, Bulgaria
[b] Institute of Mathematics and Informatics, Bulg. Acad. of Sci., Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria
[c] Department of Mathematics, Stockholm University, S-10691, Stockholm, Sweden
[d] Department of Theoretical Physics, Nuclear Physics Institute, Academy of Sciences, 250 68, Řež near Prague, Czech Republic

Received 17 May 2021; received in revised form 27 September 2024; accepted 16 October 2024
Available online 22 October 2024

Communicated by A. Martinez-Finkelshtein

To Salomon Bochner, a mathematical hero

Abstract

We initiate the study of a natural generalisation of the classical Bochner–Krall problem asking which linear ordinary differential operators possess sequences of eigenpolynomials satisfying linear recurrence relations of finite length; the classical case corresponds to the 3-term recurrence relations with real coefficients subject to some extra restrictions. We formulate a general conjecture and prove it in the first non-trivial case of operators of order 3.

MSC: primary 34L20; secondary 30C15; 33E05

Keywords: The Bochner–Krall problem; Finite recurrence relations; Darboux transform; Multi-orthogonal polynomials

1. Introduction

1.1. On the classical problem

In 1929 S. Bochner published a short paper [6] dealing with orthogonal polynomials and Sturm–Liouville problem.[1] Although after writing [6], he left this area for good, his importance

* Corresponding author.
E-mail addresses: horozov@fmi.uni-sofia.bg (E. Horozov), shapiro@math.su.se (B. Shapiro), tater@ujf.cas.cz (M. Tater).

[1] Salomon Bochner made substantial contributions to harmonic analysis, probability theory, differential geometry as well as history of mathematics. Several notions and results such as the Bochner integral, Bochner theorem on

https://doi.org/10.1016/j.jat.2024.106114

for the theory of orthogonal polynomials is difficult to overestimate; at the moment [6] has been cited 453 times.[2]

Namely, the following classification problem was formulated by S. Bochner for differential operators of order 2, and by H. L. Krall for differential operators of general order.

Problem A (*[6,18]*)**.** Describe all linear differential operators with polynomial coefficients of the form:

$$L = L(x, \partial) = \sum_{i=1}^{k} a_i(x)\partial^i, \tag{1.1}$$

such that a) $\deg a_i(x) \leq i$; b) there exists a positive integer $i_0 \leq k$ with $\deg a_{i_0}(x) = i_0$, satisfying the condition that the set of polynomial solutions of the formal spectral problem

$$Lf(x) = \lambda f(x), \quad \lambda \in \mathbb{R}, \tag{1.2}$$

is a sequence of polynomials orthogonal with respect to some real bilinear form. (Here $\partial := \frac{d}{dx}$.)

Following the terminology used in physics, we call linear differential operators given by (1.1) *exactly solvable*, see e.g. [25,26]. Observe that every exactly solvable operator has a unique eigenpolynomial of any sufficiently large degree which makes Problem A well-posed.

Let us denote by $\{P_n^L(x)\}$ the sequence of monic eigenpolynomials of an exactly solvable operator L. (We assume that $\deg P_n^L(x) = n$ where n runs from some positive integer to $+\infty$.) An exactly solvable operator which solves Problem A will be called a *Bochner–Krall operator*.

S. Bochner stated and solved Problem A for the second order differential operators, see [6] and Theorem A. The second order classification contains four families, corresponding to the classical Hermite, Laguerre, Jacobi, and Bessel polynomials. Eleven years after that H. L. Krall settled the order four case, see [19]. The order four classification contains seven families: the four classical families corresponding to the case when an order four operator is a function of an operator of order two, and three new families which are polynomial eigenfunctions of differential operators that do not reduce to operators of order two, see [18] and Theorem B.

An assortment of families corresponding to order six operators has been found in the ensuing decades, see e.g. [20, Chapter XVI]. W. Hahn showed that the four classical families are the only orthogonal polynomial sequences $\{P_n(x)\}_{n=0}^{\infty}$ for which $\{\frac{d}{dz}P_n(x)\}_{n=0}^{\infty}$ is also an orthogonal polynomial sequence, see [15]. An analog of this was proved for the order 4 case by K. H. Kwon, L. L. Littlejohn, J. K. Lee, and B. H. Yoo [22]. The most general form of Problem A is still open for operators of order six or higher, but K. H. Kwon and J. K. Lee have found a satisfactory solution if the polynomials are required to be orthogonal with respect to a compactly supported positive measure, see [21]. (A weaker result in the same direction was somewhat earlier obtained in [5].)

Fourier transforms, Bochner–Riesz means, Bochner–Martinelli formula bear at present his name. His mathematical production consists of 6 books on various subjects including history of mathematics and about 140 research papers. Among these papers 32 were published in Proc. Nat. Acad. USA, 46 in Annals of Mathematics, 3 in Acta Math., and 4 in Duke Math. J. He belonged to a sizeable group of European mathematicians of Jewish origin who moved to US before or during the WWII and contributed to an enormous development of mathematics in their new motherland.

[2] As was pointed out by an anonymous referee the main result of [6] is essentially contained in an earlier paper [24] with a very fuzzy title.

In Problem A, one may equivalently seek a sequence of moments $\{\mu_j\}_{j=0}^{\infty}$ which permits construction of the orthogonal sequence of polynomials, see [20, p. 223]. Let $\langle\cdot,\cdot\rangle$ be a candidate bilinear form, and define a weight on the set of polynomials with respect to the inner product by requiring that $\langle w(x), x^n\rangle = \mu_n$. (Note that such a weight $w(x)$ is not necessarily positive or unique.) One can show that solutions to Problem A exist only if the product wL is equal to its formal adjoint when acting on polynomials, and this immediately implies that the order of L must be even, see [20], p. 228.

Recall that by a trivial generalisation of Favard's theorem, every sequence of monic orthogonal polynomials satisfies a 3-term recurrence relation of the form:

$$xP_n(x) = P_{n+1}(x) + u(n)P_n(x) + v(n)P_{n-1}(x), \tag{1.3}$$

where $u(n)$ and $v(n)$ are real numbers. Observe that the weight function $w(x)$ is non-negative in $\mathbb{R}$ if and only if $v(n) > 0$ for all $n > 0$.

1.2. Algebraisation and generalisation

The following natural algebraic version of the classical Bochner–Krall problem was studied in substantial details in [13,14].

Problem 1 (*Algebraic Bochner–Krall Problem*)**.** Describe all linear differential operators (1.1) such that their sequence of monic eigenpolynomials satisfies (1.3) with some complex-valued $u(n)$ and $v(n)$.

Observe that contrary to the case of the classical Bochner–Krall problem, Problem 1 is purely algebraic and (hopefully) has an easier solution compared to the classical one, see Conjecture 1.7. On the other hand, the connection between the algebraic Bochner–Krall problem and the classical one is rather straight-forward.

Proposition 1.1. *A differential operator solves the classical Bochner–Krall problem with a positive weight function $w(x)$ if and only if it solves Problem 1 and all its eigenpolynomials are real-rooted and the roots of any two consecutive eigenpolynomials are strictly interlacing.*

Remark 1.2. Which additional conditions on an exactly solvable operator can guarantee the real-rootedness of its eigenpolynomials is unclear at the moment, but this could be related to the major results in the Pólya–Schur theory, see e.g. [7].

Remark 1.3. Observe that if we allow coefficients of the operator to depend on n, then the whole problem trivialises, i.e., more or less any sequence of polynomials can be obtained in such a way.

In what follows, we also discuss the following natural generalisation of Problem 1.

Problem 2 (*Generalised Algebraic Bochner–Krall Problem*)**.** Describe the set of linear differential operators (1.1) such that their sequence of monic eigenpolynomials satisfies a finite recurrence relation of the form:

$$xP_n(x) = P_{n+1}(x) + \sum_{j=0}^{d} b_j(n)P_{n-j}(x) = \Lambda P_n(x),\ n = 0, 1, \ldots \tag{1.4}$$

where d is independent of n and the coefficients $b_j(n)$ are independent of x. Here Λ is a difference operator acting on infinite column vectors $(\mu_0, \mu_1, \dots)^T$; interpreting Λ as an infinite matrix we get that its nth row corresponds to

$$(\Lambda)_n = T + \sum_{j=0}^{d} b_j(n) T^{-j}$$

and $T(\mu_0, \mu_1, \dots) := (\mu_1, \mu_2, \dots)$ is the shift operator.

Remark 1.4. By a theorem of Maroni [23], the latter condition means that the sequence $\{P_n(x)\}$ of polynomials is d-orthogonal, see e.g., [27]. The study of d-orthogonal and multiple orthogonal polynomials has been a popular area of research during the last 3 decades, see e.g., [2].

Remark 1.5. It is clear that if an operator L solves Problems 1 or 2, then for any univariate polynomial s of positive degree, the operator $s(L)$ also solves the same problem with the same sequence of eigenpolynomials $\{P_n^L(x)\}$ and the sequence of eigenvalues coinciding with $\{s(\lambda_n)\}$. Therefore in what follows, we will always assume that L is the operator of the lowest order with a given sequence of eigenpolynomials.

Remark 1.6. Another way to produce new solutions to Problems 1 and 2 from the existing ones is to apply a bispectral Darboux transformation to the difference operator Λ. This will produce a new difference operator $\widehat{\Lambda}$ and a new differential operator $\widehat{L}$. However the order of $\widehat{L}$ could increase and, in fact, this is what happens in all known examples. We will call a differential operator L *irreducible* if it cannot be obtained by applying a bispectral Darboux transformation to a differential operator of a lower order. (For general information about Darboux transformation consult e.g. [8]).

Most of the above situations allow application of Darboux transformations on the side of the difference operator. More precisely, let us present the operator Λ as a matrix, acting on the column vector $\mathcal{P}(x) = (P_0(x), P_1(x), \dots)^T$ of polynomials in the variable x. Then in certain situations there exists a factorisation of Λ

$$\Lambda = D_1 \circ D_2,$$

where D_1 and D_2 are rational functions of n, and, additionally, D_1 is an upper triangular matrix while D_2 is a lower triangular matrix. Set

$$\widehat{\Lambda} = D_2 \circ D_1$$

and define a new column vector $\widehat{\mathcal{P}}(x) = (\widehat{P}_0(x), \widehat{P}_1(x), \dots)^T$ of polynomials by using

$$\widehat{\mathcal{P}}(x) = D_2 \mathcal{P}(x).$$

It is easy to see that $\deg \widehat{P}_n(x) = n$ and also that

$$x\widehat{\mathcal{P}}(x) = \widehat{\Lambda}\widehat{\mathcal{P}}(x).$$

In such a situation a general procedure described in [3] allows us to obtain a univariate differential operator $\widehat{L}$, such that

$$\widehat{L}\widehat{P}_n(x) = \nu_n \widehat{P}_n(x), \; n = 0, 1, \dots.$$

To carry out this procedure in concrete cases is a nontrivial problem. However in [13,14] it has been successfully performed in the situations when one starts either from the Laguerre or from the Jacobi polynomials. In particular, using a 1-step Darboux transformation one obtains Krall's polynomials. Analogously, many-step Darboux transformations lead to differential operators of higher order, but they do not increase the order of the difference operator.

In this language, the above mentioned result of S. Bochner [6] can be stated as follows.

Theorem A. *Every orthogonal polynomial system $\{P_n(x)\}$ obtained as a sequence of monic eigenpolynomials of a linear differential operator of order 2 coincides (up to the action of the affine group on the variable x and scaling) with one of the sequences of Hermite, Laguerre, Jacobi or Bessel orthogonal polynomials, i.e., with one of the classical orthogonal polynomial systems.*

Similarly, the above mentioned result of H. Krall claims the following, see [19].

Theorem B. *Every linear differential operator of order 4 solving the classical Bochner–Krall problem and which cannot be obtained as a function of a linear differential operator of order 2 is given either by*
(a) *a 1-step Darboux transformation applied to an operator of order 2 corresponding to a system of Jacobi polynomials with at least one of the parameters α, β being an integer,*
or by
(b) *a 1-step Darboux transformation applied to an operator corresponding to a system of Laguerre polynomials with an integer value of its parameter α.*

Let us now present our conjectures and results related to Problems 1 and 2. Concerning Problem 1, we propose the following general guess supported by the results of [13,14].[3]

Conjecture 1.7. *Any sequence $\{P_n\}$ of monic polynomials obtained as a sequence of eigenpolynomials of a linear differential operator solving Problem 1 belongs to one of the following 2 classes:*

(1) *A classical sequence of orthogonal polynomials with, in general, complex-valued parameters, i.e.,*
(a) *Hermite polynomials H_n;*
(b) *Laguerre polynomials $L_n^{(\alpha)}$;*
(c) *Jacobi polynomials $P_n^{(\alpha,\beta)}$;*
(d) *Bessel polynomials Y_n.*
(2) *A sequence of polynomials which can be obtained from*
(e) *Laguerre polynomials $L_n^{(\alpha)}$ with the positive integer values of α by applying a finite number of Darboux transformations to the difference operator Λ with*

$$(\Lambda)_n := T + u(n)Id + v(n)T^{-1}$$

occurring in the right-hand side of (1.3);
(f) *Jacobi polynomials $P_n^{(\alpha,\beta)}$ with the positive integer values of either α, β or both α and β by applying a finite number of Darboux transformations to the difference operator Λ.*

Remark 1.8. The fact that the families (a)–(d) solve Problem 1 is classical. Analogous statement for the families (e)–(f) has been proven in [13,14]. The major difficulty in settling

[3] As was pointed out by an anonymous referee Conjecture 1.7 can be found on p. 153 of [12].

Conjecture 1.7 is to show that the above list of cases is exhaustive which is possible in principle, but computationally is quite a challenge.

Concerning Problem 2, we have the following two conjectural claims.

Conjecture 1.9. *For any irreducible differential operator L of order k solving Problem 2, the order of the corresponding difference operator Λ is also k.*

The next claim similar to Conjecture 1.7 gives a description of all irreducible operators solving Problem 2.

Conjecture 1.10. *For any positive integer k, the irreducible differential operators of order k solving Problem 2 belong to one of the following two types:*

(1) $$L = \sum_{j=1}^{k} a_j x^{j-1} \partial^j + x\partial, \quad a_j \in \mathbb{C},\ a_k \neq 0,$$

generating the so-called $(k-1)$-orthogonal polynomials, see [4];

(2) $$L = q'(G)G + x\partial,$$

where $q(t)$ is any complex polynomial of degree k/ℓ without a constant term with ℓ being any divisor of k. Additionally, $G := (\sum_{m=0}^{\ell-1} a_m (x\partial)^m)\partial, \quad a_m \in \mathbb{C}, a_{\ell-1} \neq 0$, see Theorem 2.3 of [17].

Remark 1.11. Both types of operators appearing in Conjecture 1.10 together with their properties were discussed in some detail in publications [16,17].

Remark 1.12. Notice that the operators

$$L = \sum_{j=1}^{k} a_j \partial^j + x\partial$$

generating the Appell polynomials [1] are included in Type 2 with $\ell = 1$, see [9,17].

Example 1.13. Conjecturally, all the irreducible operators of order 4 solving Problem 2 are given by:

(1) $$L = \sum_{j=1}^{4} a_j x^{j-1} \partial^j + x\partial, \quad a_j \in \mathbb{C}, a_4 \neq 0$$

generating the so-called 3-orthogonal polynomials;

(2′) $$L = \sum_{j=1}^{4} a_j \partial^j + x\partial, \quad a_j \in \mathbb{C}, a_4 \neq 0$$

generating the Appell polynomials, see [1,11];

(2″) $$L = b_2 G^2 + b_1 G + x\partial, \quad b_2 \neq 0,$$

where $G = (a_1 x\partial + a_0)\partial,\ a_1 \neq 0$. The difference operator Λ in all these cases is of order 4, i.e., $d = 3$.

The main result of the present paper is a proof of Conjecture 1.10 in the first non-trivial case of differential operators L of order 3.

Theorem 1.14. *All irreducible differential operators L of order 3 solving Problem 2 whose corresponding difference operators Λ also have order 3 are given by:*

(1) $$L = \sum_{j=1}^{3} a_j x^{j-1} \partial^j + x\partial, \quad a_j \in \mathbb{C}, a_3 \neq 0,$$

generating the 2-orthogonal monic polynomials, see [4]. They satisfy the 4-term recurrence relation:

$$x P_n(x) = P_{n+1}(x) - (a_1 + 2na_2 + 3n(n-1)a_3)P_n(x) + n(a_2 + (3n-3)a_3)$$
$$(a_1 + (n-1)a_2 + (n-1)(n-2)a_3)P_{n-1}(x) - n(n-1)a_3(a_1 + (n-1)a_2 + (n-1)(n-2)a_3)$$
$$(a_1 + (n-1)a_2 + (n-1)(n-2)a_3)P_{n-2}(x),$$

with the standard initial conditions: $P_{-2}(x) = P_{-1}(x) = 0$, $P_0(x) = 1$;

(2) $$L = \sum_{j=1}^{3} a_j \partial^j + x\partial, \quad a_j \in \mathbb{C}, a_3 \neq 0$$

generating the monic Appell polynomials, see [1,11]. They satisfy the 4-term recurrence relation:

$$x P_n(x) = P_{n+1}(x) - a_1 P_n(x) - a_2 n P_{n-1}(x) - a_3 n(n-1) P_{n-2}(x),$$

with the standard initial conditions: $P_{-2}(x) = P_{-1}(x) = 0$, $P_0(x) = 1$.

Remark 1.15. Theorem 1.14 will follow from the detailed study of three special cases presented in § 3, § 4, and § 5 respectively. Its proof contains several new ideas, but also a substantial amount of explicit calculations which we were able to carry out by hands in § 3 and § 5. In § 4 we had to use Wolfram Mathematica package since these calculations were too heavy. It is of course highly desirable to find an alternative proof avoiding such an amount of explicit calculations and which might be applicable for a possible attack on our general Conjecture 1.10, but at the moment we do not see such a possibility.

2. Preliminary facts

Let us write the three-term recurrence relation for monic polynomials in the form

$$x P_n = P_{n+1} + u_n P_n + v_{n-1} P_{n-1}. \tag{2.5}$$

Proof of Proposition 1.1. It is a well-known fact that polynomials in a sequence $\{P_n\}$ of (monic orthogonal) polynomials which satisfy a 3-term recurrence relation (2.5) with real $\{u_n\}$ and positive $\{v_n\}$ are all real-rooted and having simple interlacing zeros. As a special case the same property holds for the sequences of eigenpolynomials solving the classical Bochner–Krall problem. The converse claim that a sequence of monic polynomials satisfying (2.5) with real $\{u_n\}$ and $\{v_n\}$ which have all real and interlacing roots necessarily has all v_n positive is an immediate consequence of Wendroff's theorem which, in its turn, is a simple consequence of Sturm's algorithm and should be better known, see [28] . □

Problem 2 asks to describe all exactly solvable linear differential operators L that have a sequence $\mathcal{P}(x) = \{P_n(x)\}$ of monic eigenpolynomials with eigenvalues λ_n, i.e.,

$$LP_n(x) = \lambda_n P_n(x) \tag{2.6}$$

and which at the same time satisfy a difference equation of the form

$$\Lambda\mathcal{P}(x) = x\mathcal{P}(x). \tag{2.7}$$

In the next lemma we deduce a simple, but very useful necessary condition for the solvability of the latter problem for differential operators of order 3. Let us assume that a polynomial sequence $\mathcal{P}(x) = \{P_n(x)\}$ satisfying (2.6) and (2.7) exists. Define

$$\mathrm{ad}_x L := [x, L]$$

and

$$\mathrm{ad}_x^{j+1} L := \mathrm{ad}_x(\mathrm{ad}_x^j L).$$

It is obvious that $\mathrm{ad}_x L$ is a univariate differential operator of order less than the order of L. Below we assume that the order of L equals 3, i.e.,

$$L = a_3(x)\partial^3 + a_2(x)\partial^2 + a_1(x)\partial. \tag{2.8}$$

Let us deduce that

$$\mathrm{ad}_x^4 L \equiv 0. \tag{2.9}$$

Indeed,

$$\mathrm{ad}_x L(u(x)) = xL(u(x)) - L(xu(x)) = -2a_2(x)u'(x) - 3a_3(x)u''(x) - a_1(x)u(x).$$

Similarly,

$$\mathrm{ad}_x^2 L(u(x)) = L\left(x^2u(x)\right) + x(xL(u(x)) - L(xu(x))) - xL(xu(x)) = 6a_3(x)u'(x) + 2a_2(x)u(x).$$

Finally,

$$\mathrm{ad}_x^3 L(u(x)) = -6a_3(x)u(x) \quad \text{and} \quad \mathrm{ad}_x^4 L(u(x)) \equiv 0. \tag{2.10}$$

The following result obtained first in [10] will be crucial in all our calculations. We state it in a form suitable for the present paper. For a column vector $\mathcal{M} = (\mu_0, \mu_1, \dots)^T$, in what follows we use the notation

$$\mathrm{ad}_\Lambda^{j+1}\mathcal{M} := \mathrm{ad}_\Lambda(\mathrm{ad}_\Lambda^j \mathcal{M})$$

with $\mathrm{ad}_\Lambda \mathcal{M}$ meaning $[\Lambda, \mathcal{M}]$, i.e., the commutator of the operator Λ and the operator $\mathcal{M} \cdot \mathbb{1}$ acting diagonally on a sequence of numbers by multiplication of its jth entry by μ_j, $j = 0, 1, \dots$. (However slightly abusing our notation we will omit the symbol $\mathbb{1}$ below).

Lemma 2.1. *Assuming that the pair (L, Λ) solves Problem 2 with L being a differential operator of order 3 and denoting by $\mathcal{L} = (\lambda_0, \lambda_1, \dots)^T$, the sequence of its eigenvalues, one has the relation*

$$\mathrm{ad}_\Lambda^4 \mathcal{L} \equiv 0, \tag{2.11}$$

where the left-hand side is understood as a difference operator acting on column vectors.

Proof. By definition, $[x, L](P_n(x)) = xL(P_n(x)) - L(xP_n(x))$. Introducing the column vector $\mathcal{P} = (P_0(x), P_1(x), \ldots)^T$ of monic eigenpolynomials of L, we have the equality of vectors of functions

$$xL(\mathcal{P}(x)) - L(x\mathcal{P}) = x\mathcal{L}(\mathcal{P}(x)) - L(\Lambda(\mathcal{P}(x))),$$

which follows from (2.7). Here (as above) the expression $\mathcal{L}(\mathcal{P}(x))$ means that each monic eigenpolynomial $P_j(x),\ \ j = 0, 1, \ldots$ is multiplied by its own eigenvalue λ_j. Using the fact that both $\mathcal{L}$ and Λ are independent of x, we can rewrite the above as

$$x\mathcal{L}(\mathcal{P}(x)) - L(\Lambda(\mathcal{P}(x))) = \mathcal{L}(x\mathcal{P}(x)) - \Lambda(L(\mathcal{P}(x))) = \mathcal{L}(\Lambda(\mathcal{P}(x))) - \Lambda(\mathcal{L}(\mathcal{P}(x))).$$

Summarising, we obtain

$$[x, L](\mathcal{P}(x)) = \mathcal{L}(\Lambda(\mathcal{P}(x))) - \Lambda(\mathcal{L}(\mathcal{P}(x))) = [\mathcal{L}, \Lambda](\mathcal{P}(x)),$$

which is equivalent to

$$\mathrm{ad}_{\Lambda}\mathcal{L}(\mathcal{P}(x)) = -\mathrm{ad}_x L(\mathcal{P}(x)). \tag{2.12}$$

Observe that the operator in the left-hand side originally acts on column vectors of numbers while the operator in the right-hand side acts on functions in the variable x. The above equality means that their extended action on the sequence $\mathcal{P}$ of monic eigenpolynomials coincide.

Notice that (2.12) can be also proven directly by calculating its left-hand and right-hand sides explicitly. Namely, both give the following expression

$$(\lambda_n - \lambda_{n+1})P_{n+1}(x) + \sum_{j=0}^{d} b_j(n)(\lambda_n - \lambda_{n-j})P_{n-j}(x)$$

for the nth coordinate in the image.

Formula (2.12) implies that for $j = 1, 2, \ldots,$

$$\mathrm{ad}_x^j L(\mathcal{P}(x)) = (-1)^j \mathrm{ad}_{\Lambda}^j \mathcal{L}(\mathcal{P}(x)). \tag{2.13}$$

Now the identity (2.9) together with (2.13) imply

$$\mathrm{ad}_{\Lambda}^4 \mathcal{L} \equiv 0$$

since the left-hand side of (2.13) for $j = 4$ is vanishing identically and $\mathrm{ad}_{\Lambda}^4\mathcal{L}$ is a difference operator whose kernel contains number sequences $\mathcal{P}(x)$ for every fixed real x. (Formula (2.13) can be obtained by explicit calculations similar to that following (2.12), but we omit them here). □

Remark 2.2. Eq. (2.11) will be referred to as the ad-*condition* and will be our main technical tool. Appendix A contains a purely computational proof of Lemma 2.1 which we decided to include after a number of questions raised by an anonymous referee.

Notation. Below we will impose the restriction that Λ is a monic difference operator of order 3, i.e.,

$$(\Lambda)_n = T + \sum_{i=0}^{2} b_i(n)T^{-i}, \tag{2.14}$$

comp. Conjecture 1.9. In what follows we will use the following notation for the coefficients of the polynomials $a_3(x), a_2(x)$ and $a_1(x)$ respectively:

$$\begin{cases} a_3(x) = a_{33}x^3 + a_{32}x^2 + a_{31}x + a_{30}; \\ a_2(x) = a_{22}x^2 + a_{21}x + a_{20}; \\ a_1(x) = a_{11}x + a_{10}. \end{cases}$$

2.1. The ad-condition

Notice that if L is a differential operator of order 3 then the sequence $\mathcal{L} = \{\lambda_n\}_{n=0}^{\infty}$ is the sequence of values of a polynomial of degree at most 3 in the variable n at consecutive non-negative integer points. Indeed, the entries of λ_n come from the differentiation of the term x^n in $P_n(x)$. More exactly,

$$\lambda_n = (n)_3 \cdot a_{33} + (n)_2 \cdot a_{22} + n \cdot a_{11}.$$

Lemma 2.3. *The operator* $\mathrm{ad}^3_{\Lambda}\mathcal{L}$ *is of the form:*

$$\mathrm{ad}^3_{\Lambda}\mathcal{L} = \alpha\Lambda^3 + \beta\Lambda^2 + \gamma\Lambda + \delta I = 6a_3(\Lambda), \tag{2.15}$$

where the coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *are independent of* n.

Proof. From the condition $\mathrm{ad}^4_{\Lambda}\mathcal{L} \equiv 0$ it follows that $\mathrm{ad}^3_{\Lambda}\mathcal{L}$ is a function of the difference operator Λ which we denote by $q(\Lambda)$. Moreover in our situation $q(\Lambda)$ is a polynomial of degree at most 3 in Λ since it contains T in at most the third degree. Also notice that

$$\mathrm{ad}^3_x L = (-1)^3 6a_3(x).$$

Having in mind that

$$\mathrm{ad}^3_x L(\mathcal{P}(x)) = (-1)^3 \mathrm{ad}^3_{\Lambda}\mathcal{L}(\mathcal{P}(x)),$$

we obtain

$$(-1)^3 6a_3(\Lambda)(\mathcal{P}(x)) = (-1)^3 \mathrm{ad}^3_{\Lambda}\mathcal{L}(\mathcal{P}(x)),$$

i.e., $q(t) = 6a_3(t)$. □

Remark 2.4. Formula (2.15) says that the leading polynomial coefficient $a_3(x)$ coincides with $\frac{1}{6}(\alpha x^3 + \beta x^2 + \gamma x + \delta)$, i.e., in the above notation $a_{33} = \alpha/6,\ a_{32} = \beta/6, a_{31} = \gamma/6,\ a_{30} = \delta/6$ which we will use below.

Remark 2.5. Observe that if two finite-dimensional linear operators commute and one of them has a simple spectrum then the other is a (polynomial) function of the first one. This fact is unknown and probably false for commuting infinite-dimensional operators. However the latter Lemma provides such a statement in our particular case.

3. Case when $\mathcal{L}$ is given by a linear polynomial

As we mentioned above, the sequence $\mathcal{L} = \{\lambda_n\}_{n=0}^{\infty}$ is given by a polynomial in n of degree at most 3. Here we will discuss the simplest case when λ_n is given by a linear polynomial. In terms of the differential operator L given by (2.8), the linearity of λ_n means that $\deg a_3(x) < 3$

(which is equivalent to $a_{33} = \alpha/6 = 0$), $\deg a_2(x) < 2$ (which is equivalent to $a_{22} = 0$) and $\deg a_1(x) = 1$ (which is equivalent to $a_{11} \neq 0$). Additionally, observe that in this case we can without loss of generality assume that $\lambda_n \equiv n$. Indeed, if $\lambda_n = \mu n + \nu$, then we can consider $L - \nu$ instead of L. Finally, we can divide both sides by μ, which results in $\lambda_n \equiv n$.

Assuming that $\lambda_n \equiv n$, we will separately study the following 3 (sub)cases:

(i) $a_3(x)$ is a quadratic polynomial, i.e., $a_{33} = 0$; $a_{32} = \beta/6 \neq 0$;

(ii) $a_3(x)$ is a linear polynomial, i.e., $a_{33} = a_{32} = 0$, $a_{31} = \gamma/6 \neq 0$;

(iii) $a_3(x)$ is a non-vanishing constant, i.e., $a_{33} = a_{32} = a_{31} = \gamma/6 = 0$, $a_{30} = \delta/6 \neq 0$.

3.1. *Case* $\deg a_3(x) = 2 \Leftrightarrow \beta \neq 0$

Since $\lambda_n \equiv n$, we get

$$L = (a_{32}x^2 + a_{31}x + a_{30})\partial_x^3 + (a_{21}x + a_{20})\partial^2 + (x + a_{10})\partial_x,$$

where $a_{32} \neq 0$. Rescaling L we can achieve $a_{32} = 1$. Shifting x we can additionally assume that $a_{10} = 0$. (Both changes only result in somewhat shorter formulas, but otherwise are not essential.) Thus without loss of generality, we can assume that the operator L is of the form

$$L = (x^2 + a_{31}x + a_{30})\partial_x^3 + (a_{21}x + a_{20})\partial^2 + x\partial_x. \tag{3.16}$$

To prove the required result, we do the following. Expanding the polynomial $P_n(x)$ as

$$P_n(x) = x^n + p_1(n)x^{n-1} + \cdots + p_k(n)x^{n-k} + \cdots ,$$

we are going to compute the coefficients $p_k(n)$ for the first few values of k. Then we compute the coefficients $b_j(n)$ of the recurrence relation Λ. The condition that the polynomials $P_n(x)$ satisfy a 4-term relation implies that $b_3(n) \equiv 0$. We will use the following identities:

$$\begin{cases} p_1(n) = p_1(n+1) + b_0(n); \\ p_2(n) = p_2(n+1) + b_0(n)p_1(n) + b_1(n); \\ p_3(n) = p_3(n+1) + b_0(n)p_2(n) + b_1(n)p_1(n-1) + b_2(n); \\ p_4(n) = p_4(n+1) + b_0(n)p_3(n) + b_1(n)p_2(n-1) + b_2(n)p_1(n-2) + b_3(n). \end{cases} \tag{3.17}$$

These relations imply the following expressions for $b_j(n)$:

$$\begin{cases} b_0(n) = p_1(n) - p_1(n+1); \\ b_1(n) = p_2(n) - p_2(n+1) - b_0(n)p_1(n); \\ b_2(n) = p_3(n) - p_3(n+1) - b_0(n)p_2(n) - b_1(n)p_1(n-1); \\ b_3(n) = p_4(n) - p_4(n+1) - b_0(n)p_3(n) - b_1(n)p_2(n-1) - b_2(n)p_1(n-2). \end{cases} \tag{3.18}$$

We want to conclude that under our assumptions, we get $a_{31} = a_{30} = 0$. The following statement being the major result of this section claims exactly this.

Proposition 3.1. *If a differential operator L of the form*

$$L = (x^2 + a_{31}x + a_{30})\partial_x^3 + (a_{21}x + a_{20})\partial^2 + x\partial_x \tag{3.19}$$

has a sequence of polynomial eigenfunctions $\{P_n(x)\}$, $n = 0, 1, \dots$ which satisfies a 4-term recurrence relation, then $a_{31} = a_{30} = a_{20} = 0$. Thus the operator L will have the form (1) *from Theorem 1.14, i.e.,*

$$L = \sum_{j=1}^{3} a_j x^{j-1}\partial^j + x\partial.$$

Proof. We will actually show that if $b_3(n) \equiv 0$ (which is a necessary condition for the sequence $\{P_n(x)\}$ to satisfy a 4-term recurrence), then all three parameters a_{31}, a_{30}, a_{20} vanish. To do this, we will not need the explicit form of $b_3(n)$ which is very cumbersome, but it would be enough to obtain polynomial coefficients at a_{31}, a_{30} and a_{20} in the presentation of $b_3(n)$.

The reason for the latter sufficiency is as follows. It has been shown in [17] that if $a_{31} = a_{30} = a_{20} = 0$, then $b_3(n) \equiv 0$. Below we will refer to a_{31}, a_{30}, a_{20} as *parameters*.

We will calculate several terms in the expansion of $b_3(n)$ (as well as other $b_i(n)$'s) in powers of parameters with coefficients being powers of n. Namely, we denote by " $\sim$ " the following 2-step truncation of the expansion of $b_i(n)$ in the latter powers. Firstly, we discard all terms containing powers of parameters exceeding 1. Secondly, for each parameter, we only keep the highest power of n in its coefficient. For example,

$$b_3(n) \sim n^{m_1} \cdot a_{31} + n^{m_2} \cdot a_{30} + n^{m_3} \cdot a_{20},$$

where the numbers m_j are some nonnegative integers and the relation itself means that $b_3(n)$ is equal to the r.h.s. modulo some terms that contain higher powers of parameters. Here n^{m_j} is the leading power of n in the coefficient of the corresponding parameter.

To compute these leading terms, we first calculate the corresponding terms in $4p_4(n)$. Namely, set

$$4p_4(n) = cn^{q_0} + a_{31}n^{q_1} + a_{30}n^{q_2} + a_{20}n^{q_3},$$

where $c \in \mathbb{C}$ does not depend on the parameters. (Later we are going to use a similar expansion for $b_3(n)$).

Using the formulas (3.18) and applying the operator (3.16) to $P_n(x)$, we get

$$\begin{aligned} L(P_n) &= (x^2 + a_{31}x + a_{30})((n)_3 \cdot x^{n-3} + p_1(n)(n-1)_3 \cdot x^{n-4} + \dots + p_k(n)(n-k)_3 \cdot x^{n-k-3} + \dots) \\ &+ \quad (a_{21}x + a_{20})((n)_2 \cdot x^{n-2} + p_1(n)(n-1)_2 \cdot x^{n-3} + \dots + p_k(n)(n-k)_2 \cdot x^{n-k-2} + \dots) \\ &+ \quad x(n \cdot x^{n-1} + p_1(n)(n-1) \cdot x^{n-2} + \dots + p_k(n)(n-k) \cdot x^{n-k-1} + \dots) \\ &= \quad n(x^n + p_1(n)x^{n-1} + \dots + p_k(n)x^{n-k} + \dots). \end{aligned}$$

Comparing the coefficients of x^{n-1} in both sides of the latter equality, we obtain

$$p_1(n) = (n)_3 + (n)_2 \cdot a_{21}.$$

From (3.18) we obtain

$$b_0(n) = -\Delta(p_1(n)) = -3(n)_2 - 2n \cdot a_{21}$$

which means that both $p_1(n)$ and $b_0(n)$ do not contain our parameters. We observe that in order to calculate $b_3(n)$, we need first to compute $b_1(n)$ and $b_2(n)$.

Starting with $p_2(n)$ we obtain

$$2p_2(n) = (n)_3 \cdot a_{31} + (n)_2 \cdot a_{20} + p_1(n-1)p_1(n).$$

For later use, notice that

$$p_2(n) \sim \frac{1}{2}\left[n^3 \cdot a_{31} + n^2 \cdot a_{20} + n^6\right].$$

Using (3.18) again, we get

$$\begin{aligned} b_1(n) &= -\Delta(p_2(n)) - b_0(n)p_1(n) \\ &= -\frac{3(n)_2}{2} \cdot a_{31} - n \cdot a_{20} + \frac{1}{2}p_1(n)\Delta^2(p_1(n-1)) \end{aligned}$$

which gives

$$b_1(n) \sim -\frac{3n^2}{2} \cdot a_{31} - n \cdot a_{20} + 3n^4.$$

Let us now apply a similar procedure to find $p_3(n)$. From the above expression for $L(P_n(x))$ we get

$$\begin{aligned} 3p_3(n) &= (n)_3 \cdot a_{30} + p_1(n)(n-1)_3 \cdot a_{31} + p_1(n)(n-1)_2 \cdot a_{20} + p_1(n-2)p_2(n) \\ &= \frac{1}{2}p_1(n-2)\left((n)_3 \cdot a_{31} + (n)_2 \cdot a_{20} + p_1(n-1)p_1(n)\right) \\ &+ (n)_3 \cdot a_{30} + p_1(n)(n-1)_3 \cdot a_{31} + p_1(n)(n-1)_2 \cdot a_{20}. \end{aligned}$$

The latter relation implies that

$$p_3(n) \sim \frac{n^3}{3} \cdot a_{30} + \frac{n^6}{2} \cdot a_{31} + \frac{n^5}{2} \cdot a_{20} + \frac{1}{3!}n^9.$$

For $b_2(n)$, we get

$$b_2(n) = -\Delta(p_3(n)) - b_0(n)p_2(n) - b_1(n)p_1(n)$$

which implies that

$$b_2(n) \sim n^4 \cdot a_{31} - n^2 \cdot a_{30} + \frac{2n^3}{3} \cdot a_{20}.$$

Further, computing $p_4(n)$, we get

$$4p_4(n) = p_2(n)(n-2)_3 \cdot a_{31} + p_1(n)(n-1)_3 \cdot a_{30} + p_2(n)(n-2)_2 \cdot a_{20} + p_1(n-3) \cdot p_3(n)$$

which implies that

$$p_4(n) \sim \frac{n^9}{4} \cdot a_{31} + \frac{n^6}{3} \cdot a_{30} + \frac{n^8}{4} \cdot a_{20} + \frac{1}{4!}n^{12}.$$

Finally, we obtain

$$b_3(n) = -\Delta(p_4(n)) - b_0(n)p_3(n) - b_1(n)p_2(n) - b_2(n)p_1(n)$$

by plugging the expressions for $p_j(n), b_j(n)$ from the above formulas. This gives

$$b_3(n) \sim \frac{9n^7}{2} \cdot a_{31} + \frac{9n^4}{2} \cdot a_{30} + 3n^6 \cdot a_{20}.$$

As the consequence of the latter expansion, we see that if we assume that $b_3(n) \equiv 0$, then, in particular, we get that $a_{31} = a_{30} = a_{20} = 0$. □

3.2. *Case* $\deg a_3(x) \le 1 \Leftrightarrow a_{32} = 0$

In this subsection our goal is to prove the following statement.

Proposition 3.2. *Solution of Problem 2 for* $\deg a_3(x) \le 1$ *is possible, if and only if* $a_{31} = a_{21} = 0$ *and* $a_{30} \neq 0$. *In this case we obtain the Appell polynomials as eigenpolynomials of* L.

Proof. We use the same approach as in Section 3.1. The operator L is of the form

$$L = (a_{31}x + a_{30})\partial_x^3 + (a_{21}x + a_{20})\partial_x^2 + (x + a_{10})\partial_x.$$

By translation of x we can make $a_{10} = 0$ and work with

$$L = (a_{31}x + a_{30})\partial_x^3 + (a_{21}x + a_{20})\partial_x^2 + x\partial_x.$$

(Again such change of variables only results in somewhat shorter formulas, but otherwise is not essential.)

Expanding

$$P_n(x) = x^n + p_1(n)x^{n-1} + \cdots + p_k(n)x^{n-k} + \cdots ,$$

let us apply L to the polynomial $P_n(x)$. Straightforward calculation gives for $L(P_n)$ the expression

$$\begin{aligned} L(P_n) &= (a_{31}x + a_{30})((n)_3 \cdot x^{n-3} + p_1(n)(n-1)_3 \cdot x^{n-4} + \cdots + p_k(n)(n-k)_3 \cdot x^{n-k-3} + \cdots) \\ + &\quad (a_{21}x + a_{20})((n)_2 \cdot x^{n-2} + p_1(n)(n-1)_2 \cdot x^{n-3} + \cdots + p_k(n)(n-k)_2 \cdot x^{n-k-2} + \cdots) \\ + &\quad x(nx^{n-1} + p_1(n)(n-1)x^{n-2} + \cdots + p_k(n)(n-k)x^{n-k-1} + \cdots) \\ = &\quad n(x^n + p_1(n)x^{n-1} + \cdots + p_k(n)x^{n-k} + \cdots). \end{aligned}$$

Comparing the coefficients of x^{n-1} in both sides, we find

$$p_1(n) = (n)_2 \cdot a_{21}.$$

From the expansion of Λ we conclude that

$$b_0(n) = -2n \cdot a_{21}. \tag{3.20}$$

From the coefficients of x^{n-2} we get

$$p_2(n) = \frac{1}{2}\left((n)_3 \cdot a_{31} + (n)_2 \cdot a_{20} + p_1(n)p_1(n-1)\right).$$

Evaluating $b_1(n), b_2(n), b_3(n)$ from (3.18) we arrive at

$$b_1(n) = \frac{1}{2}n\left((n-1)\left(2a_{21}^2 - 3a_{31}\right) - 2a_{20}\right), \tag{3.21}$$

$$b_2(n) = \frac{1}{3}n(n-1)\left(2(n-2)a_{21}a_{31} - 3a_{30}\right), \tag{3.22}$$

and

$$b_3(n) = \frac{1}{8}n\left(n^2 - 3n + 2\right)\left(2(n-3)a_{31}a_{21}^2 + a_{31}\left(3(n-3)a_{31} + 4a_{20}\right) - 8a_{30}a_{21}\right). \tag{3.23}$$

The last factor in the right-hand side of (3.23) equals $a_{31}(n-3)(2a_{21}^2+3a_{31})+(4a_{31}a_{20}-8a_{30}a_{21})$. For $b_3(n)\equiv 0$ as a polynomial in n one has the following two options

$$\begin{cases} a_{31}=0 \\ 4a_{31}a_{20}-8a_{30}a_{21}=0 \end{cases} \quad \text{or} \quad \begin{cases} 2a_{21}^2+3a_{31}=0 \\ 4a_{31}a_{20}-8a_{30}a_{21}=0. \end{cases}$$

To refute the second option we have to involve the expressions for $b_4(n)$ which is as follows:

$$b_4(n)=\frac{1}{15}a_{21}(n-3)(n-2)(n-1)n\left(2a_{31}a_{21}^2(n-4)+2a_{31}\left(3a_{31}(n-4)+5a_{20}\right)-15a_{30}a_{21}\right). \tag{3.24}$$

One can check that the equations of the second option lead to $b_4(n)\not\equiv 0$ which is a contradiction. Finally, the first option splits into two possibilities: $a_{31}=a_{21}=0$ and $a_{31}=a_{30}=0$.

The case $a_{31}=a_{21}=0$ leads to $b_3(n)\equiv 0$ as claimed above. (The case $a_{31}=a_{30}=0$ is forbidden since then L becomes a second order differential operator.) □

4. Case when $\mathcal{L}$ is given by a quadratic polynomial

This case corresponds to $\deg a_3(x)<3$, $\deg a_2(x)=2$, and $\deg a_1(x)\le 1$. We want to show that in this situation there are no differential operators whose eigenpolynomials satisfy a finite recurrence relation.

Proposition 4.1. *The case when λ_n is quadratic is impossible, i.e., no linear differential operator L of order 3 satisfying this condition can solve Problem 2.*

Proof. Our arguments are partially computer-aided due to the complexity of calculations. (The corresponding Mathematica code and its results can be requested from the third author). The scheme of what we are doing is presented below.

We are going to compute the coefficients of the 4-term recurrence in terms of the coefficients of the operator L. In the case under consideration, we can write the operator L in the form

$$L=(a_{32}x^2+a_{31}x+a_{30})\partial^3+(x^2+a_{21}x+a_{20})\partial^2+(a_{11}x+a_{10})\partial.$$

We can also assume that $a_{21}=0$, which can be achieved by a translation of x. (By a slight abuse of notation we use the same letters for the coefficients of L.) From the above form of L we get that

$$\lambda_n=n(n-1)+\nu n.$$

As above, introduce

$$P_n(x)=x^n+p_1(n)x^{n-1}+\cdots+p_k(n)x^{n-k}+\cdots.$$

We are going to compute the coefficients $p_k(n)$ of $P_n(x)$ by considering $L(P_n(x))$ and using the formulas (3.17) and (3.18). $L(P_n(x))$ satisfies the relation

$$\begin{aligned} L(P_n)= {} & (a_{32}x^2+a_{31}x+a_{30})((n)_3x^{n-3}+p_1(n)(n-1)_3x^{n-4}\ldots+p_k(n)(n-k)_3x^{n-k-3}+\cdots) \\ + {} & (x^2+a_{20})((n)_2x^{n-2}+p_1(n)(n-1)_2x^{n-3}\ldots+p_k(n)(n-k)_2x^{n-k-2}+\cdots) \\ + {} & (a_{11}x+a_{10})(nx^{n-1}+p_1(n)(n-1)x^{n-2}\ldots+p_k(n)(n-k)x^{n-k-1}+\cdots) \\ = {} & (n(n-1)+\nu n)(x^n+p_1(n)x^{n-1}+\cdots+p_k(n)x^{n-k}+\cdots). \end{aligned}$$

We now equalise the coefficients at the same powers of x in the right-hand and the left-hand sides of the latter equation. For our purposes it would be enough to find expressions for $p_1(n)$, $p_2(n)$, $p_3(n)$, and $p_4(n)$ only. Knowing $p_1(n)$, $p_2(n)$, $p_3(n)$, $p_4(n)$ and using the above formulas, we can express $b_0(n), b_1(n), b_2(n)$ and $b_3(n)$. As before, to obtain a recurrence relation of order at most 4 for the eigenpolynomials, we need to ensure that $b_3(n) \equiv 0$.

We will normalise the above expression for L using the action of the affine group on x. First, let us consider the case when $a_{32} \neq 0$. By rescaling we can obtain $a_{32} = 1$. Then the leading coefficient of the polynomial of degree 9 in the numerator of $b_3(n)$ equals 8 which implies that $b_3(n)$ cannot vanish identically, see (3.23).

If $a_{32} = 0$, but $a_{31} \neq 0$ by rescaling we can assume that $a_{31} = 1$. In this case the leading coefficient of the polynomial of degree 7 in the numerator of $b_3(n)$ equals 24 which again implies that $b_3(n)$ cannot vanish identically.

Next assume that $a_{32} = a_{31} = 0$. Let us rescale x to obtain $a_{30} = 1$.

We still need to find if there exist values of a_{20}, a_{11}, a_{10} for which $m_9 = m_8 = m_7 = m_6 = 0$, where m_i is the coefficient of n^i in the expression (B.29) for $b_3(n)$.

First, we get that $m_9 = 8(96a_{20}^2 + 288a_{10})$. Assuming that $m_9 = 0$ we obtain $a_{10} = -\frac{a_{20}^2}{3}$. Inserting the latter expression for a_{10} in $b_3(n)$, and using $m_8 = 0$, we get that either $a_{11} = 2$ or $a_{20} = 0$. In the former case, setting $a_{10} = -\frac{a_{20}^2}{3}$ and $a_{11} = 2$ in the expressions for m_7 and m_6, we get

$$\begin{cases} m_6 = -\frac{64}{3}a_{20}^2(27 + 4a_{20}^3); \\ m_7 = \frac{64}{3}a_{20}^2(9 + 2a_{20}^3). \end{cases}$$

Thus, if $a_{20} \neq 0$, then the equations for $m_6 = 0$ and $m_7 = 0$ are obviously incompatible, i.e., have no common solutions.

Now let us consider the case when $a_{32} = a_{31} = a_{21} = a_{20} = 0$. We will show that $b_3(n) \equiv 0$, which however is not enough to claim that all $b_j(n) \equiv 0$ for $j > 2$. For this reason we are going to compute them up to $j = 5$. We know that in this case also $a_{10} = 0$. Again applying the operator L to the polynomials $P_n(x)$, we obtain

$$\begin{aligned} L(P_n) = {} & (n)_3 \cdot x^{n-3} + p_1(n)(n-1)_3 \cdot x^{n-4} \ldots + p_k(n)(n-k)_3 \cdot x^{n-k-3} + \cdots \\ + \quad & x^2((n)_2 \cdot x^{n-2} + p_1(n)(n-1)_2 \cdot x^{n-3} \ldots + p_k(n)(n-k)_2 \cdot x^{n-k-2} + \cdots) \\ + \quad & a_{11}x(nx^{n-1} + p_1(n)(n-1)x^{n-2} \ldots + p_k(n)(n-k)x^{n-k-1} + \cdots) \\ = \quad & (n(n-1) + a_{11}n)(x^n + p_1(n)x^{n-1} + \cdots + p_k(n)x^{n-k} + \cdots). \end{aligned}$$

Then, for $k > 3$, we find the following formulas for $p_j(n)$:

$$\begin{cases} p_1(n)(2 - 2n - a_{11}) & = 0; \\ p_2(n)(6 - 4n - 2a_{11}) & = 0; \\ p_3(n)(12 - 6n - 3a_{11}) & = -(n)_3; \\ \vdots \quad \vdots \quad \vdots \quad \vdots & \vdots \\ p_k(n)(k(k+1) - 2kn - ka_{11}) & = -(n-k+3)_3 \cdot p_{k-3}(n). \end{cases}$$

By induction, we see that $p_{3j}(n) \not\equiv 0$ while $p_i(n) \equiv 0$ for $i \neq 3j$. In particular,

$$p_6(n)(42 - 12n - 6a_{11}) = -(n-3)_3 \cdot p_3(n)$$

i.e.,

$$p_6(n) = \frac{(n)_5}{18(2n + a_{11} - 4)(2n + a_{11} - 7)}.$$

From this we easily compute several first $b_j(n)$'s getting

$$\begin{cases} b_0(n) &= 0; \\ b_1(n) &= 0; \\ b_2(n) &= -\Delta p_3(n) \neq 0; \\ b_3(n) &= -\Delta p_4(n) - b_0(n)p_3(n) - b_1(n)p_2(n) - b_2(n)p_1(n) = 0; \\ b_4(n) &= -\Delta p_5(n) - b_0(n)p_4(n) - b_1(n)p_3(n) - b_2(n)p_2(n) - b_3(n)p_1(n) = 0; \\ b_5(n) &= -\Delta p_6(n) - b_0(n)p_5(n) - b_1(n)p_4(n) - b_2(n)p_3(n) - b_3(n)p_2(n) - b_4(n)p_1(n). \end{cases}$$

In the expression for $b_5(n)$ there are two non-vanishing terms: $\Delta p_6(n)$ and $b_2(n)p_3(n)$. Both terms are of order n^3. We only need to show that their sum is not identically zero. Straight-forward calculations give

$$b_2(n)p_3(n) = \frac{1}{9}\frac{(n)_3 \cdot (n)_2 \cdot (4n - a_{11})}{(2n + a_{11} - 2)(2n + a_{11} - 4)^2}$$

and

$$\Delta p_6(n) = \frac{(n)_4 \cdot R(n)}{(2n + a_{11} - 2)(2n + a_{11} - 4)(2n + a_{11} - 5)(2n + a_{11} - 7)},$$

where $R(n)$ is a polynomial of degree at most 2, whose explicit expression is irrelevant for our purposes. We see that the sum of $b_2(n)p_3(n)$ and $\Delta p_6(n)$ cannot vanish since they contain different factors.

In fact, the explicit expression for $b_5(n)$ can be obtained even without computer. Our symbolic computations give

$$b_5(n) = \frac{n(n-1)(n-2)(n-3)(n-4)\,(3a_{11} + 5n - 16)}{9\,(a_{11} + 2n - 2)\,(a_{11} + 2n - 7)\,(a_{11} + 2n - 5)\,(a_{11} + 2n - 4)}. \quad \square \tag{4.25}$$

5. Case when $\mathcal{L}$ is given by a cubic polynomial

When $\mathcal{L} = \{\lambda_n\}$ is given by a cubic polynomial in n, by using translation and scaling, we can without loss of generality assume that

$$\lambda_n = n(n-1)(n-2) + \nu n(n-1) + \mu n.$$

As before, one can easily observe that $\nu = a_{22}$ and $\mu = a_{11}$.

Using this ansatz, we first formulate some important preliminary statements about the coefficients of Λ and L.

Lemma 5.1. *The coefficient α in (2.15) equals 6.*

Proof. To prove this, we compute the term of the highest degree in T in the two expressions for $\mathrm{ad}^3_\Lambda \mathcal{L}$ which we presented earlier. On one hand, from (2.15) we obtain that $(\mathrm{ad}^3_\Lambda \mathcal{L})_n = \alpha T^3 + \dots$. Let us compute the term containing T^3 in $(\mathrm{ad}^3_\Lambda \mathcal{L})_n$. Simple computation shows that

$$\begin{cases} (\mathrm{ad}^1_\Lambda \mathcal{L})_n = (\Delta(\mathcal{L}))_n \cdot T + \cdots ; \\ (\mathrm{ad}^2_\Lambda \mathcal{L})_n = (\Delta^2(\mathcal{L}))_n \cdot T^2 + \cdots ; \\ (\mathrm{ad}^3_\Lambda \mathcal{L})_n = (\Delta^3(\mathcal{L}))_n \cdot T^3 + \cdots . \end{cases}$$

Since $\Delta^3(\mathcal{L}) \equiv 6$, the result follows. □

Lemma 5.2. *The coefficient $b_2(n)$ in the expansion (2.14) of Λ vanishes identically.*

Proof. Suppose that $b_2(n) \not\equiv 0$. Then again we can use the above approach and compute the term in $(\mathrm{ad}^3_\Lambda \mathcal{L})_n$ containing T^{-6} in two different ways. On one hand, if taken from $6\Lambda^3$, this term is given by

$$\alpha \cdot b_2(n)b_2(n-2)b_2(n-4)T^{-6} = 6b_2(n)b_2(n-2)b_2(n-4)T^{-6}.$$

On the other hand, the computation of the same term from $(\mathrm{ad}^3_\Lambda \mathcal{L})_n$ shows that it coincides with

$$b_2(n)b_2(n-2)b_2(n-4)(-\lambda_{n-6} + 3\lambda_{n-4} - 3\lambda_{n-2} + \lambda_n)T^{-6}.$$

However, $-\lambda_{n-6} + 3\lambda_{n-4} - 3\lambda_{n-2} + \lambda_n = 48$ which is only possible if $b_2(n) \equiv 0$. Indeed, since $b_2(n)$ is a polynomial in the variable n, the equation

$$b_2(n)b_2(n-2)b_2(n-4) = 0$$

cannot have infinitely many solutions unless $b_2(n) \equiv 0$. □

Lemma 5.3. *Under our current assumptions, the coefficient $b_1(n)$ in the expansion (2.14) of Λ vanishes identically.*

Proof. The argument is similar to the above computation of $b_2(n)$. Assume that $b_1(n) \not\equiv 0$. Notice that by Lemma 5.2,

$$(\Lambda)_n = T + b_0(n)T^0 + b_1(n)T^{-1}.$$

The coefficient of T^{-3} in the expression for $6\Lambda^3$ is given by $6b_1(n)b_1(n-1)b_1(n-2)$.

From the expression for $(\mathrm{ad}^3_\Lambda \mathcal{L})_n$, we find that the same term coincides with

$$b_1(n)b_1(n-1)b_1(n-2)(\lambda_{n-3} - 3\lambda_{n-2} + 3\lambda_{n-1} - \lambda_n) = -6b_1(n)b_1(n-1)b_1(n-2).$$

Again we get a contradiction. □

Lemma 5.4. *The coefficient $b_0(n)$ in the expansion (2.14) of Λ is an arbitrary constant.*

Proof. By Lemmas 5.2 and 5.3, $\Lambda = T + b_0(n)$ which makes the computation of both Λ^3 and $\mathrm{ad}^3_\Lambda \mathcal{L}$ very easy. Moreover, we only need the coefficient at $T^0 = Id$. We have

$$(\Lambda^j)_n = \cdots + b_0^j(n)T^0, \;\; j = 0, 1, \ldots .$$

For $(\mathrm{ad}^j_\Lambda \mathcal{L})_n$, $j = 0, 1, \ldots$, it is obvious that it does not contain T^0. Comparing the coefficients at T^0, we obtain

$$\alpha b_0^3(n) + \beta b_0^2(n) + \gamma b_0(n) + \delta = 0.$$

However this is only possible if $b_0(n)$ is a constant. □

Denoting this above constant by p we get the following claim.

Proposition 5.5. *The only linear differential operators of order* 3 *solving Problem 2 which have polynomial eigenfunctions with* $\lambda_n = n(n-1)(n-2) + \nu n(n-1) + \mu n$ *are of the form*

$$L = 6(x-p)^3\partial_x^3 + \nu(x-p)^2\partial_x^2 + \mu(x-p)\partial.$$

Hence any such operator L is reducible.

Proof. By the last lemma we have

$$xP_n(x) = P_{n+1}(x) + pP_n(x).$$

Hence $P_{n+1} = (x-p)^n$. The corresponding differential operator of order 3 is as claimed. □

6. Final remarks

I. To the best of our knowledge, Conjecture 1.10 containing a complete conjectural description of the set of all solutions to the algebraic version of the classical Bochner–Krall Problem 1 and its generalisation Problem 2 has never previously appeared in the literature. However the way our conjectures are formulated, it is difficult to verify them for operators of somewhat high order unless one finds an alternative description. On the other hand, the special case of operators of order 4 seems to be doable by using the methods of the present paper.
II. One additional restriction of the validity of the results obtained in the present paper comes from the fact that we are using the assumptions of Conjecture 1.9. To actually claim that we have solved Problem 2 for all operators of order 3, we need to settle this conjecture at least in this case. We hope to return to this project in a future publication.
III. Our proof of the main Theorem 1.14 is based on consideration of a large number of special subcases and is partially computer-aided. Such method is not very effective in order to be able to approach our general conjectures and a more conceptual understanding of our proof is needed.
IV. In connection with Proposition 1.1 and Wendroff's theorem we want to ask under which conditions on the sequences $\{u_n\}$ and $\{v_n\}$ the polynomial sequence $\{P_n(x)\}$ consists of real-rooted polynomials.

Acknowledgements

The first and the third authors are sincerely grateful to the Mathematics Department of Stockholm University for the hospitality in December 2014 and April 2015, when this project was initiated. The first and the second authors want to thank the Department of Theoretical Physics, Nuclear Physics Institute, Academy of Sciences, 250 68 Řež near Prague for the hospitality in December 2016. The research of the first author has been partially supported by Grant KP-06-N 62/5 of Bulgarian National Science Fund, Republic of Bulgaria. The third author has been supported by the Czech Science Foundation (GAČR) within the project

21-07129S. Finally, the authors are sincerely grateful to both anonymous referees for their constructive criticism and detailed suggestions which helped us to improve the exposition.

Appendix A. Proving Lemma 2.1 by explicit computations

Here us present an alternative approach to evaluation of (2.9). To this end, we find explicit form of $(\Lambda)_n$, i.e. we express $b_j(n)$, $j = 0, 1, 2$ in terms of the coefficients a_{ij} of the differential operator $L(x, \partial)$. The operator $(\Lambda)_n$ is represented as a four-diagonal infinite matrix and $\mathcal{L}$ as an infinite matrix with nonzero main diagonal $\{\mu_n\}$.

More explicitly, one has

$$\mu_n = na_{11} + (n-1)na_{22} + (n-2)(n-1)na_{33} \quad \text{and} \quad b_0(n) = \rho/\sigma,$$

where

$$\begin{aligned}\rho =& -3(n-2)n(n-1)^2 a_{32}a_{33} - 2n(n-1)a_{21}a_{22} - 3n(n-1)a_{11}a_{32} - \\ & 2n(2n-1)(n-1)a_{22}a_{32} + 3(n+2)(n-1)a_{10}a_{33} + 6n(n-1)a_{21}a_{33} - \\ & 2na_{11}a_{21} - a_{10}a_{11} + 2a_{10}a_{22};\end{aligned}$$

$$\sigma = (a_{11} + 2(n-1)a_{22} + 3(n-2)(n-1)a_{33})\,(a_{11} + 2na_{22} + 3(n-1)na_{33})\,.$$

The other two coefficients are given by cumbersome formulae which can be sent to an interested reader upon request. (In particular, the expressions for the coefficients b_1 consists of 53 and for b_2 of 113 terms respectively).

The fourth power of commutator (2.9) is then a matrix with at most 13 non-vanishing diagonals (eight below and four above the main diagonal). However, several of them actually vanish once the explicit expressions for the entries are substituted. These vanishing diagonals are the fourth, the third, the second and the first above the main diagonal as well as the eighth below the main diagonal. More explicitly one has the following.

The fourth diagonal above the main one is given by:

$$\mu_i - 4\mu_{i+1} + 6\mu_{i+2} - 4\mu_{i+3} + \mu_{i+4} \equiv 0, \qquad \text{for } i = 0, \ldots.$$

The third diagonal above the main one is given by:

$$\begin{aligned}&(-3\mu_i + 6\mu_{i+1} - 4\mu_{i+2} + \mu_{i+3})b_0(i) + (\mu_i + 2\mu_{i+1} - 4\mu_{i+2} + \mu_{i+3})b_0(i+1) + \\ &(\mu_i - 4\mu_{i+1} + 2\mu_{i+2} + \mu_{i+3})b_0(i+2) + \\ &(\mu_i - 4\mu_{i+1} + 6\mu_{i+2} - 3\mu_{i+3})b_0(i+3) \equiv 0, \qquad \text{for } i = 0, \ldots.\end{aligned}$$

The second diagonal above the main one is given by:

$$\begin{aligned}&(-4\mu_i + 7\mu_{i+1} - 4\mu_{i+2} + \mu_{i+3})b_1(i+1) + (3\mu_{i+1} - 4\mu_{i+2} + \mu_{i+3})b_0(i+1)^2 + \\ &(7\mu_{i+1} - 8\mu_{i+2} + \mu_{i+3})b_1(i+2) + (\mu_{i+1} - 2\mu_{i+2} + \mu_{i+3})b_0(i+2)^2 + \\ &(\mu_{i+1} - 8\mu_{i+2} + 7\mu_{i+3})b_1(i+3) + (\mu_{i+1} + 2\mu_{i+2} - 3\mu_{i+3})b_0(i+2)b_0(i+3) + \\ &(\mu_{i+1} - 4\mu_{i+2} + 3\mu_{i+3})b_0(i+3)^2 + b_0(i+1)((-3\mu_{i+1} + 2\mu_{i+2} + \mu_{i+3})b_0(i+2) + \\ &(-3\mu_{i+1} + 6\mu_{i+2} - 3\mu_{i+3})b_0(i+3)) + \\ &(\mu_{i+1} - 4\mu_{i+2} + 7\mu_{i+3} - 4\mu_{i+4})b_1(i+4) \equiv 0, \qquad \text{for } i = 0, \ldots.\end{aligned}$$

The first diagonal above the main one is given by:

$$
\begin{aligned}
&-\mu_{i+2}b_0(i+2)^3+\mu_{i+3}b_0(i+2)^3+3\mu_{i+2}b_0(i+3)b_0(i+2)^2-\\
&3\mu_{i+3}b_0(i+3)b_0(i+2)^2-3\mu_{i+2}b_0(i+3)^2b_0(i+2)+3\mu_{i+3}b_0(i+3)^2b_0(i+2)-\\
&4\mu_{i+2}b_1(i+2)b_0(i+2)+2\mu_{i+3}b_1(i+2)b_0(i+2)-4\mu_{i+2}b_1(i+3)b_0(i+2)+\\
&4\mu_{i+3}b_1(i+3)b_0(i+2)-3\mu_{i+2}b_1(i+4)b_0(i+2)+7\mu_{i+3}b_1(i+4)b_0(i+2)-\\
&4\mu_{i+4}b_1(i+4)b_0(i+2)+\mu_{i+2}b_0(i+3)^3-\mu_{i+3}b_0(i+3)^3-4\mu_i b_2(i+2)-\\
&3\mu_{i+2}b_2(i+2)+\mu_{i+3}b_2(i+2)-3\mu_{i+2}b_0(i+1)b_1(i+2)+\\
&\mu_{i+3}b_0(i+1)b_1(i+2)-3\mu_{i+2}b_2(i+3)-3\mu_{i+3}b_2(i+3)+\\
&7\mu_{i+2}b_1(i+2)b_0(i+3)-3\mu_{i+3}b_1(i+2)b_0(i+3)+4\mu_{i+2}b_1(i+3)b_0(i+3)-\\
&4\mu_{i+3}b_1(i+3)b_0(i+3)+2\mu_{i+1}\left(3b_2(i+2)+b_0(i+1)b_1(i+2)+\right.\\
&\left.b_1(i+2)b_0(i+2)+3b_2(i+3)-2b_1(i+2)b_0(i+3)\right)-3\mu_{i+2}b_2(i+4)-\\
&3\mu_{i+3}b_2(i+4)+6\mu_{i+4}b_2(i+4)+2\mu_{i+2}b_0(i+3)b_1(i+4)-\\
&4\mu_{i+3}b_0(i+3)b_1(i+4)+2\mu_{i+4}b_0(i+3)b_1(i+4)+\mu_{i+2}b_1(i+4)b_0(i+4)-\\
&3\mu_{i+3}b_1(i+4)b_0(i+4)+2\mu_{i+4}b_1(i+4)b_0(i+4)+\mu_{i+2}b_2(i+5)-\\
&3\mu_{i+3}b_2(i+5)+6\mu_{i+4}b_2(i+5)-4\mu_{i+5}b_2(i+5)\equiv 0,\qquad \text{for } i=0,\dots.
\end{aligned}
$$

Finally, the eighth diagonal below the main one is given by:

$$
\begin{aligned}
&(\mu_i-4\mu_{i+2}+6\mu_{i+4}-4\mu_{i+6}+\mu_{i+8})\,b_2(i+2)b_2(i+4)b_2(i+6)b_2(i+8)\equiv 0,\\
&\qquad \text{for } i=0,\dots.
\end{aligned}
$$

The rest of diagonals are non-vanishing and their explicit expressions are rather complicated functions of the coefficients a_{ij}.

However, if we use coefficients of polynomials that solve both the eigenvalue problem (1.2) and satisfy the four-term recurrence relation (1.4) then the whole fourth commutator operator given by (2.9) becomes the zero matrix; this fact can be verified by plugging the coefficients from the recurrences listed in Theorem 1.14 into $(\Lambda)_n$ and $\mathcal{L}$.

Appendix B. Computer-aided formulas relevant for Section 4

$$L_3=(a_{32}x^2+a_{31}x+a_{30})\partial^3+(x^2+a_{21}x+a_{20})\partial^2+(a_{11}x+a_{10})\partial.$$

$$xP_n(x)=P_{n+1}(x)+\sum_{j=0}^{n}b_j(n)P_{n-j}(x). \tag{B.26}$$

Putting $a_{32}=1$ we get

$$b_3(n)=\frac{\left(8n^9+\mathcal{O}\left(n^8\right)\right)}{8\left(a_{11}+2n-6\right)\left(a_{11}+2n-5\right)\left(a_{11}+2n-4\right)\left(a_{11}+2n-3\right)\left(a_{11}+2n-2\right)}. \tag{B.27}$$

With $a_{32}=0$ and $a_{31}=1$ we get

$$b_3(n)=\frac{\left(24n^7+\mathcal{O}\left(n^6\right)\right)}{8\left(a_{11}+2n-6\right)\left(a_{11}+2n-5\right)\left(a_{11}+2n-4\right)\left(a_{11}+2n-3\right)\left(a_{11}+2n-2\right)}. \tag{B.28}$$

Finally, with $a_{32} = 0$, $a_{31} = 0$, and $a_{30} = 1$ (recall that $a_{21} = 0$) we arrive at

$$b_3(n) = \frac{n\left(n^2 - 3n + 2\right) a_{10}\left((7n - 15)a_{11} + 2a_{11}^2 + 6n^2 - 26n + 28\right)}{(a_{11} + 2n - 6)(a_{11} + 2n - 5)(a_{11} + 2n - 4)(a_{11} + 2n - 3)(a_{11} + 2n - 2)}. \tag{B.29}$$

Data availability

No data was used for the research described in the article.

References

[1] P. Appell, Sur une classe de polynômes, Annales Scientifiques de l'É.N.S. 2me série 9 (1880) 119–144.

[2] A.I. Aptekarev, A.B.J. Kuijlaars, Hermite–Padé approximations and multiple orthogonal polynomial ensembles, Russian Math. Surveys 66 (6) (2011) 1133–1199.

[3] B. Bakalov, E. Horozov, M. Yakimov, General methods for constructing bispectral operators, Phys. Lett. A 222 (1–2) (1996) 59–66.

[4] Y. Ben Cheikh, K. Douak, A generalized hypergeometric d-orthogonal polynomial set, C.R. Acad. Sci Paris, t. 331, Série I (2000) 349–354.

[5] T. Bergkvist, H. Rullgård, B. Shapiro, On Bochner-Krall orthogonal polynomial systems, Math. Scand. 94 (1) (2004) 148–154.

[6] S. Bochner, Über Sturm–Liouvillesche polynomsysteme, Math. Z. 29 (1929) 730–736.

[7] J. Borcea, P. Brändén, Pólya-Schur master theorems for circular domains and their boundaries, Ann. of Math. (2) 170 (1) (2009) 465–492.

[8] A.M. Bruckner, J.B. Bruckner, Darboux transformations, Trans. Amer. Math. Soc. 128 (1967) 103–111.

[9] Kh. Douak, The relation of the d-orthogonal polynomials to the Appell polynomials, J. Comput. Appl. Math. 70 (1996) 279–295.

[10] J.J. Duistermaat, F.A. Grünbaum, Differential equation in the spectral parameter, Comm. Math. Phys. 103 (1998) 177–240.

[11] H.W. Gould, A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, Duke Math. J. 29 (1962) 51–63.

[12] F.A. Grünbaum, L. Haine, Orthogonal polynomials satisfying differential equations: The role of the darboux transformation, in: D. Levi, L. Vinet, P. Winternitz (Eds.), Symmetries and Integrability of Difference Equations, in: CRM Proc. Lecture Notes 9, Amer. Math. Soc. Providence, 1996, pp. 143–154.

[13] F.A. Grünbaum, L. Haine, E. Horozov, Some functions that generalize the Krall-Laguerre polynomials, Jour. Comp. Appl. Math. 106 (2) (1999) 271–297.

[14] F.A. Grünbaum, M. Yakimov, Discrete bispectral Darboux transformations from Jacobi operators, Pacific J. Math. 204 (2) (2002) 395–431.

[15] W. Hahn, Über die Jacobischen Polynome und zwei verwandte Polynomklassen, Math. Z. 39 (1935) 634–638.

[16] E. Horozov, Vector orthogonal polynomials with Bochner's property, Constr. Approx. (2017) http://dx.doi.org/10.1007/s00365-017-9410-6.

[17] E. Horozov, d-Orthogonal analogs of classical orthogonal polynomials, SIGMA 14 (2018) 063, 27pages.

[18] H.L. Krall, Certain differential equations for Tchebycheff polynomials, Duke Math. J. 4 (1938) 705–718.

[19] H.L. Krall, On orthogonal polynomials satisfying a certain fourth order differential equation, in: The Pennsylvania State College Studies, No. 6, The Pennsylvania State College, State College, PA, 1940.

[20] A.M. Krall, Hilbert Space, Boundary Value Problems, and Orthogonal Polynomials (Operator Theory, Advances and Applications, V. 133), Birkhauser, Berlin, 2002.

[21] K.H. Kwon, D.W. Lee, Characterizations of -Krall orthogonal polynomials of Jacobi type, Constr. Approx. 19 (2003) 599–619.

[22] K.H. Kwon, L.L. Littlejohn, J.K. Lee, B.H. Yoo, Characterization of classical type orthogonal polynomials, Proc. Amer. Math. Soc. 120 (1994) 485–493.

[23] P. Maroni, L'orthogonalité et les récurrences de polynômes d'ordre supérieur à deux, Annales de la faculté des sciences de Toulouse 10 (1) (1989) 105–139.

[24] E.J. Routh, On some properties of certain solutions of a differential equation of the second order, Proc. LMS 16 (1884) 245–261.
[25] W. Rühl, A. Turbiner, Exact solvability of the Calogero and Sutherland models, Modern Phys. Lett. A 10 (29) (1995) 2213–2221.
[26] A. Turbiner, On polynomial solutions of differential equations, J. Math. Phys. 33 (1992) 3989–3994.
[27] J. Van Iseghem, Vector orthogonal relations. Vector QD-algorithm, J. Comput. Appl. Math. 19 (1) (1987) 141–150.
[28] B. Wendroff, On orthogonal polynomials, Proc. Amer. Math. Soc. 12 (1961) 554–555.