

# ROOT ASYMPTOTICS OF SPECTRAL POLYNOMIALS FOR THE LAMÉ OPERATOR

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ABSTRACT. The study of polynomial solutions to the classical Lamé equation in its algebraic form, or equivalently, of double-periodic solutions of its Weierstrass form has a long history. Such solutions appear at integer values of the spectral parameter and their respective eigenvalues serve as the ends of bands in the boundary value problem for the corresponding Schrödinger equation with finite gap potential given by the Weierstrass  $\wp$ -function on the real line. In this paper we establish several natural (and equivalent) formulas in terms of hypergeometric and elliptic type integrals for the density of the appropriately scaled asymptotic distribution of these eigenvalues when the integer-valued spectral parameter tends to infinity. We also show that this density satisfies a Heun differential equation with four singularities.

## 1. INTRODUCTION AND MAIN RESULTS

The algebraic form of the classical Lamé equation [21, Chap. 23] was introduced by Lamé in the 1830's in connection with the separation of variables in the Laplace equation by means of elliptic coordinates in  $\mathbb{R}^l$ . Lamé's equation is given by

$$\left\{ Q(z) \frac{d^2}{dz^2} + \frac{1}{2} Q'(z) \frac{d}{dz} + V(z) \right\} S(z) = 0, \quad (1)$$

where  $Q_l(z)$  is a real degree  $l$  polynomial with all real and distinct roots and  $V(z)$  is a polynomial of degree at most  $l-2$  whose choice depends on the type of solution to (1) one is looking for. In the second half of the nineteenth century several famous mathematicians including Bôcher, Heine, Klein and Stieltjes studied the number and various properties of the so-called *Lamé solutions of the first kind* (of given degree and type) to equation (1). These are also known as *Lamé polynomials* of a certain type. Such solutions exist for special choices of  $V(z)$  and are characterized by the property that their logarithmic derivative is a rational function. For a given  $Q(z)$  of degree  $l \geq 2$  with simple roots there exist  $2^l$  different possibilities for Lamé solutions depending on whether these solutions are smooth at a given root of  $Q(z)$  or have a square root singularity, see [14, 21] for more details.

A *generalized Lamé equation* [21] is a second order differential equation of the form

$$\left\{ Q(z) \frac{d^2}{dz^2} + P(z) \frac{d}{dz} + V(z) \right\} S(z) = 0, \quad (2)$$

where  $Q(z)$  is a complex polynomial of degree  $l$  and  $P(z)$  is a complex polynomial of degree at most  $l-1$ . As it was first shown by Heine [10] for a generic equation of either of the forms (1) or (2) and an arbitrary positive integer  $n$  there are exactly  $\binom{n+l-2}{n}$  polynomials  $V(z)$  such that  $S(z)$  is a polynomial of degree  $n$ . Below we concentrate on the most classical case of a cubic polynomial  $Q(z)$  with all real roots as treated in e.g. [21]. Already Lamé and Liouville knew that in this case the

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(unique) root of  $V(z)$  is real and located between the minimal and the maximal roots of  $Q(z)$ . The latter property was further generalized by Heine, Van Vleck and other authors. Note that if  $\deg Q(z) = 3$  then  $V(z)$  is at most linear and that for a given value of the positive integer  $n$  there are at most  $n + 1$  such polynomials.

Before formulating our main results let us briefly review – following mostly [13, §3] and [14] – some necessary background on the version of the Lamé equation used below and its spectral polynomials. Setting, as one traditionally does,  $Q(z) = 4(z - e_1)(z - e_2)(z - e_3)$  with  $e_1 > e_2 > e_3$  we can rewrite equation (1) as

$$\left\{ \frac{d^2}{dz^2} + \frac{1}{2} \sum_{i=1}^3 \frac{1}{z - e_i} \frac{d}{dz} - \frac{n(n+1)z + E}{4 \prod_{i=1}^3 (z - e_i)} \right\} S(z) = 0. \quad (3)$$

(The chosen representation for the linear polynomial  $V(z)$  will shortly become clear.) Notice that several equivalent forms of the Lamé equation are classically known. Among those one should mention two algebraic forms, the Jacobian form and the Weierstrassian form, respectively, see [21, §23.4] and Remark 1 below. Equation (3) presents the most commonly used real algebraic form of the Lamé equation which is smooth while the other (real) algebraic form is singular.<sup>1</sup>

A *Lamé solution of the first kind* to equation (3) is a solution of the form  $S(z) = (z - e_1)^{\kappa_1} (z - e_2)^{\kappa_2} (z - e_3)^{\kappa_3} \tilde{S}(z)$ , where each  $\kappa_i$  is either 0 or  $\frac{1}{2}$  and  $\tilde{S}(z)$  is a polynomial. Lamé solutions which are pure polynomials – i.e., for which  $\kappa_1 = \kappa_2 = \kappa_3 = 0$  – are said to be of *type 1*, those with a single square root are of *type 2*, those with two square roots are of *type 3* and, finally, those involving three square roots are said to be of *type 4*. (“Types” are sometimes called “species”, see, e.g., [21].) One can easily check that (3) has a Lamé solution if and only if  $n$  is a nonnegative integer. Moreover, if  $n$  is even then only solutions of types 1 and 3 exist. Namely, for appropriate choices of the energy constant  $E$  one gets exactly  $\frac{n+2}{2}$  distinct independent solutions of type 1 and  $\frac{3n}{2}$  independent solutions of type 3. If  $n$  is odd then only solutions of types 2 and 4 exist. In this case, for appropriate choices of the energy constant  $E$  one gets exactly  $\frac{3(n+1)}{2}$  distinct independent solutions of type 2 and  $\frac{n-1}{2}$  independent solutions of type 4. In both cases the total number of Lamé solutions equals  $2n + 1$ , which coincides with the number of independent spherical harmonics of order  $n$ .

Let  $R_n(E) = \prod_{j=0}^{2n+1} (E - E_j)$ ,  $n \in \mathbb{Z}_+$ , denote the monic polynomial of degree  $2n + 1$  whose roots are exactly the values of the energy  $E$  at which equation (3) has a Lamé solution. These polynomials are often referred to as *spectral polynomials* in the literature; their study goes back to Hermite and Halphen. The most recent results in this direction can be found in [3, 13, 16, 17, 20] and [8, 9]. In particular, [13] contains an excellent survey of this topic as well as a comprehensive table with these polynomials (and their modified versions) correcting several mistakes that occurred in previous publications. Article [9] is apparently the first attempt to give a (somewhat) closed formula for  $R_n(E)$ ; for this the authors use yet another family of polynomials which they call *elliptic Bernoulli polynomials*. Since explicit formulas for  $R_n(E)$  seem to be rather complicated and difficult to handle, in this paper we study the asymptotics of the root distribution of appropriately scaled versions of  $R_n(E)$  (Corollary 1). In spite of a more than 150 years long history of the Lamé equation the only source discussing questions similar to ours that we were able to locate is [6]. We should also mention that the results below are actually much more precise and therefore supersede those of *loc. cit.*

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<sup>1</sup>The authors thank the anonymous referee for this observation.

Let us now introduce eight spectral polynomials  $R_n^{\kappa_1, \kappa_2, \kappa_3}(E)$  related to the eight types of Lamé solutions mentioned above, namely

$$R_n(E) = \begin{cases} R_n^{0,0,0}(E)R_n^{\frac{1}{2},\frac{1}{2},0}(E)R_n^{\frac{1}{2},0,\frac{1}{2}}(E)R_n^{0,\frac{1}{2},\frac{1}{2}}(E) & \text{when } n \text{ is even,} \\ R_n^{\frac{1}{2},0,0}(E)R_n^{0,\frac{1}{2},0}(E)R_n^{0,0,\frac{1}{2}}(E)R_n^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}(E) & \text{when } n \text{ is odd.} \end{cases} \quad (4)$$

Note for example that if  $n$  is even then  $R_n^{0,0,0}(E)$  is the (unique) monic polynomial of degree  $\frac{n+2}{2}$  whose roots are precisely the values of  $E$  for which equation (3) has a pure polynomial solution.

The results of the present paper actually hold not only for equation (3) but also for generalized Lamé equations (cf. (2)) of the form

$$\left\{ \frac{d^2}{dz^2} + \sum_{i=1}^3 \frac{\alpha_i}{z - e_i} \frac{d}{dz} + \frac{V(z)}{\prod_{i=1}^3 (z - e_i)} \right\} S(z) = 0, \quad (5)$$

where  $\alpha_i > 0$ ,  $i = 1, 2, 3$ , and  $V(z)$  is an undetermined linear polynomial. This case was thoroughly treated by Stieltjes in [15]. In particular, he proved that for any positive integer  $n$  there exist exactly  $n + 1$  polynomials  $V_{n,j}(z)$ ,  $1 \leq j \leq n + 1$ , such that (5) has a polynomial solution  $S(z)$  of degree  $n$ . Moreover, the unique root  $t_{n,j}$  of  $V_{n,j}(z)$  lies in the interval  $(e_3, e_1)$  and these  $n + 1$  roots are pairwise distinct. Consider now the polynomial

$$Sp_n(t) = \prod_{j=1}^{n+1} (t - t_{n,j}),$$

which is the scaled version of the spectral polynomial  $R_n^{0,0,0}(E)$ . More exactly, for any even  $n$  one has that  $Sp_n(t) = \frac{R_n^{0,0,0}(n(n+1)t)}{(n(n+1))^{\frac{n+2}{2}}}$ . Assume further that the set  $\{t_{n,j} : 1 \leq j \leq n + 1\}$  is ordered so that  $t_{n,1} < t_{n,2} < \dots < t_{n,n+1}$  and associate to it the finite measure

$$\mu_n = \frac{1}{n+1} \sum_{j=1}^{n+1} \delta(z - t_{n,j}),$$

where  $\delta(z - a)$  is the Dirac measure supported at  $a$ . The measure  $\mu_n$  thus obtained is clearly a real probability measure that one usually refers to as the root-counting measure of  $Sp_n(t)$ .

Below we shall make use of some well-known notions from the theory of special functions that can be found in e.g. [1]. In fact several of our formulas rely on various integral representations and transformation properties of the Gauss hypergeometric series

$$F(a, b, c; z) = {}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)} \frac{z^m}{m!},$$

where  $\Gamma$  denotes Euler's Gamma-function, as well as the complete elliptic integral of the first kind

$$\mathbb{K}(\zeta) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\zeta^2 t^2)}}. \quad (6)$$

Recall from [1, §15.3.1] that if  $\Re(c) > \Re(b) > 0$  then a convenient way of rewriting  $F(a, b, c; z)$  is given by

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt. \quad (7)$$

The above integral represents a one-valued analytic function in the  $z$ -plane cut along the real axis from 1 to  $\infty$  and thus it gives the analytic continuation of

$F(a, b, c; z)$ . To simplify our formulas let us also define a real-valued function of three real variables  $(x_1, x_2, x_3) \mapsto \omega(x_1, x_2, x_3)$  by setting

$$\omega(x_1, x_2, x_3) = \frac{(|x_1| + |x_2|)|x_3|}{|x_1||x_2 - x_3| + |x_2||x_1 - x_3|}. \quad (8)$$

Note that  $\omega(x_1, x_2, x_3)$  is symmetric in its first two arguments and that

$$0 \leq \omega(x_1, x_2, x_3) \leq 1 \text{ whenever } x_1 < 0, x_2 > 0 \text{ and } x_1 \leq x_3 \leq x_2. \quad (9)$$

We should emphasize that the latter conditions on  $x_1, x_2, x_3$  – hence also the bounds for  $\omega(x_1, x_2, x_3)$  given in (9) – always hold in the present context.

We are now ready to state our main results.

**Theorem 1.** *For any choice of real numbers  $e_1 > e_2 > e_3$  and positive numbers  $\alpha_1, \alpha_2, \alpha_3$  the sequence of measures  $\{\mu_n\}_{n \in \mathbb{Z}_+}$  strongly converges to the probability measure  $\mu_Q$  supported on the interval  $[e_3, e_1]$  with density  $\rho_Q$  given by any of the following equivalent expressions:*

(i)

$$\rho_Q(s) = \frac{1}{\pi} \sqrt{\frac{1 + \sqrt{1 - \omega(e_1 - e_2, e_3 - e_2, s - e_2)^2}}{(e_1 - e_3)|s - e_2|\omega(e_1 - e_2, e_3 - e_2, s - e_2)}} \times \mathbb{K} \left( i \sqrt{\frac{2\sqrt{1 - \omega(e_1 - e_2, e_3 - e_2, s - e_2)^2}}{1 - \sqrt{1 - \omega(e_1 - e_2, e_3 - e_2, s - e_2)^2}}} \right),$$

where  $\omega$  is the function defined in (8).

(ii)

$$\rho_Q(s) = \frac{1}{2} \sqrt{\frac{1 + \sqrt{1 - \omega(e_1 - e_2, e_3 - e_2, s - e_2)^2}}{(e_1 - e_3)|s - e_2|\omega(e_1 - e_2, e_3 - e_2, s - e_2)}} \times F \left( \frac{1}{2}, \frac{1}{2}, 1; -\frac{2\sqrt{1 - \omega(e_1 - e_2, e_3 - e_2, s - e_2)^2}}{1 - \sqrt{1 - \omega(e_1 - e_2, e_3 - e_2, s - e_2)^2}} \right).$$

(iii)

$$\rho_Q(s) = \sqrt{\frac{\omega(e_1 - e_2, e_3 - e_2, s - e_2)}{2(e_1 - e_3)|s - e_2|(1 + \omega(e_1 - e_2, e_3 - e_2, s - e_2))}} \times F \left( \frac{1}{2}, \frac{1}{2}, 1; \frac{1 - \omega(e_1 - e_2, e_3 - e_2, s - e_2)}{1 + \omega(e_1 - e_2, e_3 - e_2, s - e_2)} \right).$$

(iv)

$$\rho_Q(s) = \frac{1}{\sqrt{2[(e_1 + e_3 - 2e_2)(s - e_2) + 2(e_1 - e_2)(e_2 - e_3) + (e_1 - e_3)|s - e_2|]}} \times F \left( \frac{1}{2}, \frac{1}{2}, 1; \frac{(e_1 + e_3 - 2e_2)(s - e_2) + 2(e_1 - e_2)(e_2 - e_3) - (e_1 - e_3)|s - e_2|}{(e_1 + e_3 - 2e_2)(s - e_2) + 2(e_1 - e_2)(e_2 - e_3) + (e_1 - e_3)|s - e_2|} \right),$$

that is,

$$\rho_Q(s) = \frac{1}{2\pi} \int_{e_2}^{e_1} \frac{dx}{\sqrt{(e_1 - x)(x - e_2)(x - e_3)(x - s)}} \text{ when } e_3 < s < e_2,$$

$$\rho_Q(s) = \frac{1}{2\pi} \int_{e_3}^{e_2} \frac{dx}{\sqrt{(e_1 - x)(e_2 - x)(x - e_3)(s - x)}} \text{ when } e_2 < s < e_1.$$

**Theorem 2.** *The density function  $\rho_Q$  defined in Theorem 1 satisfies the following Heun differential equation*

$$8Q(s)\rho_Q''(s) + 8Q'(s)\rho_Q'(s) + Q''(s)\rho_Q(s) = 0, \quad (10)$$

where  $Q(s) = (s - e_1)(s - e_2)(s - e_3)$ . Both indices of this equation at the finite regular singularities  $e_1, e_2, e_3$  vanish while its indices at  $\infty$  equal  $\frac{1}{2}$  and  $\frac{3}{2}$ .

**Corollary 1.** *The root-counting measures for each of the eight normalized spectral polynomials  $R_n^{\kappa_1, \kappa_2, \kappa_3}(n(n+1)t)$  as well as that of the normalized spectral polynomial  $R_n(n(n+1)t)$  converge to the measure  $\mu_Q$  defined in Theorem 1, see Figure 1 below.*

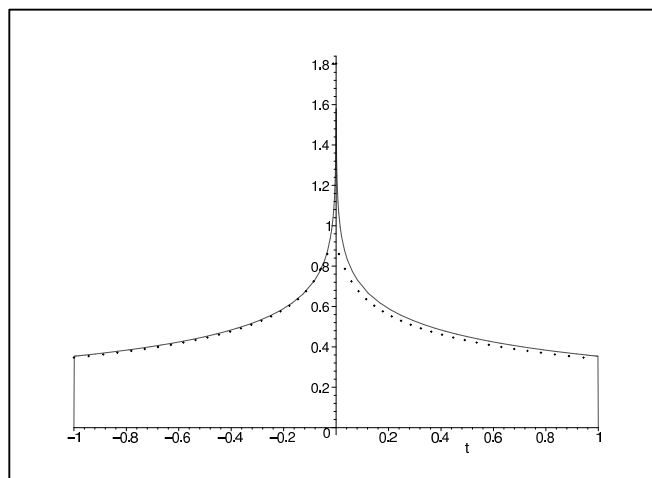


FIGURE 1. Comparison of the theoretical density with the numerical density of the measure  $\mu_Q$  for  $Q(z) = z^3 - z$ .

*Remark 1.* Note that equation (3) is usually lifted to the elliptic curve  $\Psi$  defined by  $y^2 = Q(z)$  and that on  $\Psi$  this equation takes the so-called Weierstrassian form

$$\left\{ \frac{d^2}{du^2} - [n(n+1)\wp(u) + E] \right\} S(u) = 0,$$

where  $u$  is the canonical coordinate on the universal covering of  $\Psi$  given by

$$u = \int_z^\infty \frac{dw}{\sqrt{Q(w)}}.$$

If  $Q(z)$  has all real roots then  $n(n+1)\wp(u)$  becomes a finite gap real periodic potential on  $\mathbb{R}$ , see [11]. Depending on the parity on  $n$  four of the eight spectral polynomials  $R_n^{\kappa_1, \kappa_2, \kappa_3}(E)$  determine the ends of bands of the spectrum of the corresponding Schrödinger operator on  $\mathbb{R}$ . Hopefully our results will find applications to the spectral theory of finite gap potentials.

To end this introduction let us mention that in the present context one can actually “guess” the last two formulas in Theorem 1 (iv) by WKB-type considerations, as we were kindly informed by K. Takemura in the final stages of this work. However, such arguments fail to provide an accurate mathematical proof. By contrast, our methods use solely rigorous results involving orthogonal polynomials, elliptic integrals and hypergeometric functions, and we are currently unaware of any possible “shortcuts” in this set-up. Finally, we should also note that so far neither our

methods nor WKB-type “guesses” could provide an answer to these questions in more general situations, see §3 below.

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## 2. PROOFS

*Proof of Theorem 1.* Theorem 1 essentially follows from the main result of [7] after some amount of work. First we express the polynomial  $Sp_n(t)$  as the characteristic polynomial of a certain matrix. In order to make this matrix tridiagonal so as to simplify the calculations we assume wlog that

$$Q(z) = (z - e_3)z(z - e_1) = z^3 + vz^2 + wz$$

with  $e_3 < e_2 = 0 < e_1$ . Set

$$T = (z^3 + vz^2 + wz) \frac{d^2}{dz^2} + (\alpha z^2 + \beta z + \gamma) \frac{d}{dz} - \theta_n(z - t),$$

where  $v, w, \alpha, \beta, \gamma$  are fixed constants and  $\theta_n, t$  are variables. Assuming that  $S(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n$  with undetermined coefficients  $a_i$ ,  $0 \leq i \leq n$ , we are looking for the values of  $\theta_n, t$  and  $a_i$ ,  $0 \leq i \leq n$ , such that  $T(S(z)) = 0$ . Note that  $T(S(z))$  is in general a polynomial of degree  $n + 1$  whose leading coefficient equals  $a_0[n(n-1) + \alpha n - \theta_n]$ . To get a non-trivial solution we therefore set

$$\theta_n = n(n-1 + \alpha).$$

Straightforward computations show that the coefficients of the successive powers  $z^n, z^{n-1}, \dots, z^0$  in  $T(S(z))$  can be expressed in the form of a matrix product  $M_n A$ , where  $A = (a_0, a_1, \dots, a_n)^T$  and  $M_n$  is the following tridiagonal  $(n+1) \times (n+1)$  matrix

$$M_n := \begin{pmatrix} t - \xi_{n,1} & \alpha_{n,2} & 0 & 0 & \cdots & 0 & 0 \\ \gamma_{n,2} & t - \xi_{n,2} & \alpha_{n,3} & 0 & \cdots & 0 & 0 \\ 0 & \gamma_{n,3} & t - \xi_{n,3} & \alpha_{n,4} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \alpha_{n,n} & 0 \\ 0 & 0 & 0 & \cdots & \gamma_{n,n} & t - \xi_{n,n} & \alpha_{n,n+1} \\ 0 & 0 & 0 & \cdots & 0 & \gamma_{n,n+1} & t - \xi_{n,n+1} \end{pmatrix}$$

with

$$\begin{aligned} \xi_{n,i} &= -\frac{v(n-i)(n-i+1) + \beta(n-i+1)}{\theta_n}, \quad i \in \{1, \dots, n+1\}, \\ \alpha_{n,i} &= \frac{(n-i)(n-i+1) + \alpha(n-i+1)}{\theta_n} - 1, \quad i \in \{2, \dots, n+1\}, \\ \gamma_{n,i} &= \frac{w(n-i+1)(n-i+2) + \gamma(n-i+2)}{\theta_n}, \quad i \in \{2, \dots, n+1\}. \end{aligned} \tag{11}$$

A similar matrix can be found in [10] and also in [19]. The matrix  $M_n$  depends linearly on the indeterminate  $t$  which appears only on its main diagonal. If the linear system  $M_n A = 0$  is to have a nontrivial solution  $A = (a_0, a_1, \dots, a_n)^T$  the determinant of  $M_n$  has to vanish. This gives the polynomial equation

$$Sp_n(t) = \det(M_n) = 0.$$

The sequence of polynomials  $\{Sp_n(t)\}_{n \in \mathbb{Z}_+}$  does not seem to satisfy any reasonable recurrence relation. In order to overcome this difficulty and to be able to use the technique of 3-term recurrence relations with variable coefficients (which is applicable since  $M_n$  is tridiagonal) we extend the above polynomial sequence by introducing an additional parameter. Namely, define

$$Sp_{n,i}(t) = \det M_{n,i}, \quad i \in \{1, \dots, n+1\},$$

where  $M_{n,i}$  is the upper  $i \times i$  principal submatrix of  $M_n$ . One can easily check (see, e.g., [2, p. 20]) that the following 3-term relation holds

$$Sp_{n,i}(t) = (t - \xi_{n,i})Sp_{n,i-1}(t) - \psi_{n,i}Sp_{n,i-2}(t), \quad i \in \{1, \dots, n+1\}, \quad (12)$$

where  $\xi_{n,i}$  is as in (11) and

$$\psi_{n,i} = \alpha_{n,i}\gamma_{n,i}, \quad i \in \{2, \dots, n+1\}. \quad (13)$$

Here we used the (standard) initial conditions  $Sp_{n,0}(t) = 1$ ,  $Sp_{n,-1}(t) = 0$ . It is well-known that if all  $\xi_{n,i}$ 's are real and all  $\psi_{n,i}$ 's are positive then the polynomials  $Sp_{n,i}(t)$ ,  $i \in \{0, \dots, n+1\}$ , form a (finite) sequence of orthogonal polynomials. In particular, all their roots are real. Under the assumptions of Theorem 1 the reality of  $\xi_{n,i}$  is obvious. Assuming that the positivity of  $\psi_{n,i}$  is also settled (see Lemma 1 below) let us complete the proof of Theorem 1. For this we invoke [7, Theorem 1.4] which translated in our notation claims that if there exist two continuous functions  $\xi(\tau)$  and  $\psi(\tau)$ ,  $\tau \in [0, 1]$ , such that

$$\lim_{i/(n+1) \rightarrow \tau} \xi_{i,n} = \xi(\tau), \quad \lim_{i/(n+1) \rightarrow \tau} \psi_{i,n} = \psi(\tau), \quad \psi(\tau) \geq 0 \quad \forall \tau \in [0, 1],$$

then the density of the asymptotic root-counting measure of the polynomial sequence  $\{Sp_n(t)\}_{n \in \mathbb{Z}_+} = \{Sp_{n,n+1}(t)\}_{n \in \mathbb{Z}_+}$  is given by

$$\int_0^1 \omega_{[\xi(\tau) - 2\sqrt{\psi(\tau)}, \xi(\tau) + 2\sqrt{\psi(\tau)}]} d\tau,$$

where for any  $x < y$  one has

$$\omega_{[x,y]}(s) = \begin{cases} \frac{1}{\pi\sqrt{(y-s)(s-x)}} & \text{if } s \in [x, y], \\ 0 & \text{otherwise.} \end{cases}$$

From the explicit formulas for  $\xi_{n,i}$  and  $\psi_{n,i}$  (see (11) and (13)) one easily gets

$$\begin{aligned} \xi(\tau) &= \lim_{i/(n+1) \rightarrow \tau} \xi_{i,n} = -v(1 - \tau)^2, \\ \psi(\tau) &= \lim_{i/(n+1) \rightarrow \tau} \psi_{i,n} = -w(1 - (1 - \tau)^2)(1 - \tau)^2. \end{aligned}$$

Notice that the above limits are independent of the coefficients  $\alpha, \beta, \gamma$  and that by the assumption on  $Q(z)$  made at the beginning of this section one has  $w = e_3 e_1 < 0$ , which in its turn implies that  $\psi(\tau) \geq 0$  for  $\tau \in [0, 1]$ . The required density  $\rho_Q(s)$  is therefore given by

$$\begin{aligned} \rho_Q(s) &= \int_0^1 \frac{d\tau}{\pi \sqrt{(\xi(\tau) + 2\sqrt{\psi(\tau)} - s) (s - \xi(\tau) + 2\sqrt{\psi(\tau)})}} \\ &= \int_0^1 \frac{d\tau}{\pi \sqrt{-4w(1 - (1 - \tau)^2)(1 - \tau)^2 - (v(1 - \tau)^2 + s)^2}}, \end{aligned}$$

where  $\sqrt{\cdot}^+$  is meant to remind that the integrand vanishes whenever the expression under the square root becomes negative. Introducing  $\sqrt{\nu} = 1 - \tau$  we get

$$\rho_Q(s) = \frac{1}{2\pi} \int_0^1 \frac{d\nu}{\sqrt{\nu[(4w - v^2)\nu^2 - (4w + 2vs)\nu - s^2]}^+}.$$

In order to get rid of  $\sqrt{\cdot}^+$  we rewrite

$$\rho_Q(s) = \frac{1}{2\pi\sqrt{v^2 - 4w}} \int_{\nu_{min}(s)}^{\nu_{max}(s)} \frac{d\nu}{\sqrt{\nu(\nu - \nu_{min}(s))(\nu_{max}(s) - \nu)}}, \quad (14)$$

where  $\nu_{min}(s)$  and  $\nu_{max}(s)$  are the minimal and maximal roots of of the equation

$$(4w - v^2)\nu^2 - (4w + 2vs)\nu - s^2 = 0$$

with respect to  $\nu$ , that is,

$$\begin{cases} \nu_{min}(s) = -(v^2 - 4w)^{-1}[2w + vs + 2\sqrt{w(w + vs + s^2)}], \\ \nu_{max}(s) = -(v^2 - 4w)^{-1}[2w + vs - 2\sqrt{w(w + vs + s^2)}]. \end{cases}$$

A few remarks are in order at this stage. First, by the real-rootedness of  $Q(z)$  one has  $v^2 - 4w = (e_1 - e_3)^2 > 0$ . Second, we claim that

$$[\xi(\tau) - 2\sqrt{\psi(\tau)}, \xi(\tau) + 2\sqrt{\psi(\tau)}] \subseteq [e_3, e_1]$$

for  $\tau \in [0, 1]$ . Indeed,

$$\begin{aligned} e_1 - \xi(\tau) - 2\sqrt{\psi(\tau)} &= e_1 + v(1 - \tau)^2 - 2\sqrt{-w(1 - (1 - \tau)^2)(1 - \tau)^2} \\ &= e_1(1 - (1 - \tau)^2) - 2\sqrt{e_1(1 - (1 - \tau)^2)}\sqrt{-e_3(1 - \tau)^2 - e_3(1 - \tau)^2} \\ &= \left[ \sqrt{e_1(1 - (1 - \tau)^2)} - \sqrt{-e_3(1 - \tau)^2} \right]^2 \geq 0 \end{aligned}$$

and similarly

$$\begin{aligned} \xi(\tau) - 2\sqrt{\psi(\tau)} - e_3 &= -v(1 - \tau)^2 - 2\sqrt{-w(1 - (1 - \tau)^2)(1 - \tau)^2} - e_3 \\ &= e_1(1 - \tau)^2 - 2\sqrt{e_1(1 - \tau)^2}\sqrt{-e_3(1 - (1 - \tau)^2)} - e_3(1 - (1 - \tau)^2) \\ &= \left[ \sqrt{e_1(1 - \tau)^2} - \sqrt{-e_3(1 - (1 - \tau)^2)} \right]^2 \geq 0. \end{aligned}$$

It follows that whenever  $s$  is such that the integrand in the above formulas is non-vanishing one has  $e_3 \leq s \leq e_1$  hence

$$\begin{aligned} w(w + vs + s^2) &= -e_3e_1(s - e_3)(e_1 - s) \geq 0 \text{ and} \\ 2w + vs &= e_1(e_3 - s) + e_3(e_1 - s) < 0 \end{aligned}$$

for all  $s$  as above. Therefore

$$\begin{aligned} \nu_{min}(s) + \nu_{max}(s) &= -2(2w + vs)(v^2 - 4w) > 0 \text{ and} \\ \nu_{min}(s)\nu_{max}(s) &= s^2(v^2 - 4w) \geq 0 \end{aligned}$$

from which we conclude that  $\nu_{max}(s) > \nu_{min}(s) \geq 0$  for  $e_3 < s < e_1$ .

Recall the definition and properties of the function  $\omega$  from (8)-(9) and note that

$$\omega(e_1, e_3, s) = \frac{(e_1 - e_3)|s|}{e_1(s - e_3) - e_3(e_1 - s)} = -\frac{|s|\sqrt{v^2 - 4w}}{2w + vs} \quad (15)$$

so that  $\nu_{min}(s)$  and  $\nu_{max}(s)$  may actually be rewritten as follows:

$$\nu_{min}(s) = \frac{|s|\left(1 - \sqrt{1 - \omega(e_1, e_3, s)^2}\right)}{(e_1 - e_3)\omega(e_1, e_3, s)}, \quad \nu_{max}(s) = \frac{|s|\left(1 + \sqrt{1 - \omega(e_1, e_3, s)^2}\right)}{(e_1 - e_3)\omega(e_1, e_3, s)}.$$

Using these expressions combined with the fact that for any  $x < y$  one has

$$\int_x^y \frac{d\nu}{\sqrt{\nu(\nu-x)(y-\nu)}} = \frac{2}{\sqrt{x}} \mathbb{K}\left(\sqrt{\frac{x-y}{x}}\right)$$

(which readily follows from (6)) we deduce from (14) that

$$\rho_Q(s) = \frac{1}{\pi} \sqrt{\frac{1 + \sqrt{1 - \omega(e_1, e_3, s)^2}}{(e_1 - e_3)|s|\omega(e_1, e_3, s)}} \mathbb{K}\left(i \sqrt{\frac{2\sqrt{1 - \omega(e_1, e_3, s)^2}}{1 - \sqrt{1 - \omega(e_1, e_3, s)^2}}}\right). \quad (16)$$

Now from (6)-(7) and the well-known identities  $\Gamma(1) = 1$ ,  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$  one easily gets

$$\mathbb{K}(\zeta) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; \zeta^2\right)$$

hence

$$\rho_Q(s) = \frac{1}{2} \sqrt{\frac{1 + \sqrt{1 - \omega(e_1, e_3, s)^2}}{(e_1 - e_3)|s|\omega(e_1, e_3, s)}} F\left(\frac{1}{2}, \frac{1}{2}, 1; -\frac{2\sqrt{1 - \omega(e_1, e_3, s)^2}}{1 - \sqrt{1 - \omega(e_1, e_3, s)^2}}\right) \quad (17)$$

by (16). It is a remarkable fact due to Kummer and Goursat that for special choices of the numbers  $a, b, c$  the hypergeometric series  $F(a, b, c; z)$  obeys certain quadratic transformation laws. To complete the proof we need precisely such transformation properties, namely formulas 15.3.19-20 in [1] with  $a = \frac{1}{4}$ ,  $b = \frac{1}{2}$  and  $c = 1$ :

$$\frac{1}{\sqrt{1-\sqrt{z}}} F\left(\frac{1}{2}, \frac{1}{2}, 1; -\frac{2\sqrt{z}}{1-\sqrt{z}}\right) = \sqrt{\frac{2}{1+\sqrt{1-z}}} F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right).$$

The above identity for  $z = 1 - \omega(e_1, e_3, s)^2$  together with (17) then yields

$$\rho_Q(s) = \sqrt{\frac{\omega(e_1, e_3, s)}{2(e_1 - e_3)|s|(1 + \omega(e_1, e_3, s))}} F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{1 - \omega(e_1, e_3, s)}{1 + \omega(e_1, e_3, s)}\right) \quad (18)$$

which by (15) amounts to

$$\rho_Q(s) = \frac{1}{\sqrt{2[(e_1 + e_3)s - 2e_1e_3 + (e_1 - e_3)|s|]}} \times F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{(e_1 + e_3)s - 2e_1e_3 - (e_1 - e_3)|s|}{(e_1 + e_3)s - 2e_1e_3 + (e_1 - e_3)|s|}\right). \quad (19)$$

To prove the last two formulas of Theorem 1 (iv) we use (19) and (7) with  $a = b = \frac{1}{2}$  and  $c = 1$  in order to get an integral representation of  $\rho_Q(s)$  in which one makes the following variable substitutions:

$$t = -\frac{e_3(e_1 - x)}{e_1(x - e_3)} \text{ if } e_3 < s < 0, \quad t = -\frac{e_1(x - e_3)}{e_3(e_1 - x)} \text{ if } 0 < s < e_1.$$

The desired expressions are then obtained by straightforward computations.

We thus established all formulas stated in Theorem 1 in the special case when  $e_3 < e_2 = 0 < e_1$ . The general case reduces to this one by noticing that if  $e_3 < e_2 < e_1$  and  $Q(z) = (z - e_3)(z - e_2)(z - e_1)$  then all the above arguments may be used for the polynomial  $Q(z + e_2)$ . Hence the expressions for  $\rho_Q(s)$  given in Theorem 1 in the general case are obtained from (16)-(19) simply by replacing  $e_1, e_3$  and  $s$  with  $e_1 - e_2, e_3 - e_2$  and  $s - e_2$ , respectively.  $\square$

**Lemma 1.** *Let  $Q(z) = z^3 + vz^2 + wz$  and  $P(z) = \alpha z^2 + \beta z + \gamma$  be two polynomials such that  $Q(z)$  has three real distinct roots  $e_1 > e_2 = 0 > e_3$  and*

$$\frac{P(z)}{Q(z)} = \frac{\alpha_1}{z - e_1} + \frac{\alpha_2}{z - e_2} + \frac{\alpha_3}{z - e_3}$$

with  $\alpha_1, \alpha_2, \alpha_3 > 0$ . Then the coefficients  $\psi_{n,i}$  defined in (13) are all positive.

*Proof.* One immediately gets  $\alpha > 0$ , which implies that  $\theta_n > 0$  for any positive integer  $n$ . Recall from (11) and (13) that  $\psi_{n,i} = \alpha_{n,i}\gamma_{n,i}$ . Now

$$\alpha_{n,i} = \theta_n^{-1} \{[(n-i)(n-i+1) - n(n-1)] + \alpha[(n-i+1) - n]\}$$

and since both  $\alpha$  and  $\theta_n$  are positive it follows that  $\alpha_{n,i} < 0$ . Let us show that under our assumptions one also has  $\gamma_{n,i} < 0$ . For this it is clearly enough to show that both  $w$  and  $\gamma$  are negative. Obviously,  $w = e_3 e_1 < 0$  and since  $\alpha_1, \alpha_2, \alpha_3 > 0$  the quadratic polynomial  $P(z)$  has two real roots interlacing with  $e_3, e_2, e_1$ . Therefore  $P(z)$  has one positive and one negative root and positive leading coefficient hence  $\gamma = P(0) < 0$ .  $\square$

*Proof of Theorem 2.* In order to deduce the differential equation satisfied by  $\rho_Q(s)$  we note first that the restrictions of  $\rho_Q(s)$  to  $(e_3, e_2)$  and  $(e_2, e_1)$ , respectively, are two branches of the same multi-valued analytic function (this can be seen e.g. from the last two expressions in Theorem 1). Thus it suffices to derive the linear differential equation satisfied by  $\rho_Q(s)$  restricted to, say,  $(e_2, e_1)$ . Specializing formula (iv) of Theorem 1 to this case we get

$$\rho_Q(s) = \frac{I_Q(s)}{2\pi\sqrt{(e_1 - e_2)(s - e_3)}},$$

where

$$I_Q(s) = \int_0^1 \frac{dw}{\sqrt{w(1-w) \left(1 - \frac{(e_2 - e_3)(e_1 - s)}{(e_1 - e_2)(s - e_3)} w\right)}}$$

and  $s \in (e_2, e_1)$ . By (7) we see that up to a constant factor  $I_Q(s)$  is the same as the hypergeometric series  $F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{(e_2 - e_3)(e_1 - s)}{(e_1 - e_2)(s - e_3)}\right)$ , which is known to satisfy the following Riemann differential equation (see, e.g., [1, §15.6]):

$$I_Q''(s) + \left(\frac{1}{s - e_1} + \frac{1}{s - e_2}\right) I_Q'(s) + \frac{(e_3 - e_2)(e_3 - e_1)}{4(s - e_3)Q(s)} I_Q(s) = 0.$$

Substituting  $I_Q(s) = K\sqrt{s - e_3}\rho_Q(s)$ , where  $K := 2\pi\sqrt{e_1 - e_2}$  is but a constant, in the latter equation we get after some straightforward algebraic manipulations

$$\rho_Q''(s) + \left(\sum_{i=1}^3 \frac{1}{s - e_i}\right) \rho_Q'(s) + \frac{3s - \sum_{i=1}^3 e_i}{4\prod_{i=1}^3 (s - e_i)} \rho_Q(s) = 0, \quad (20)$$

which after multiplication by  $8Q(s)$  coincides with the required equation (10). To calculate the indices recall from e.g. [12] that for a second order linear differential equation  $\rho''(s) + a_1(s)\rho'(s) + a_2(s)\rho(s) = 0$  its indicial equation at a finite regular or regular singular point  $\tilde{s}$  has the form

$$\zeta(\zeta - 1) + \alpha_1\zeta + \alpha_2 = 0,$$

where  $\alpha_1 = \lim_{s \rightarrow \tilde{s}} (s - \tilde{s})a_1(s)$  and  $\alpha_2 = \lim_{s \rightarrow \tilde{s}} (s - \tilde{s})^2 a_2(s)$ . Thus for (20) the indicial equation at each  $e_i$ ,  $i = 1, 2, 3$ , has the form  $\zeta^2 = 0$ , which implies that both corresponding indices vanish and that a solution to (20) might have a logarithmic singularity at any of these points, see Remark 2 below. The indicial equation of  $\rho''(s) + a_1(s)\rho'(s) + a_2(s)\rho(s) = 0$  at  $\infty$  has the form

$$\zeta(\zeta + 1) - \tilde{\alpha}_1\zeta + \tilde{\alpha}_2 = 0,$$

where  $\tilde{\alpha}_1 = \lim_{s \rightarrow \infty} s a_1(s)$  and  $\tilde{\alpha}_2 = \lim_{s \rightarrow \infty} s^2 a_2(s)$ . Therefore the indicial equation at  $\infty$  for (20) is  $\zeta(\zeta + 1) - 3\zeta + \frac{3}{4} = 0$  whose roots are  $\frac{1}{2}$  and  $\frac{3}{2}$ .  $\square$

*Remark 2.* It is not difficult to show that

$$\rho_Q(s) \approx \frac{1}{2\pi} \frac{\log\left(\frac{16(e_1-e_2)(e_2-e_3)}{(e_1-e_3)|s-e_2|}\right)}{\sqrt{(e_1-e_2)(e_2-e_3)}} \text{ as } s \rightarrow e_2,$$

so that  $\rho_Q(s)$  always has a logarithmic singularity at  $e_2$ .

*Proof of Corollary 1.* The main idea of the proof of Corollary 1 is that the polynomial part of any Lamé solution to (3) itself satisfies a very similar differential equation. Let us illustrate this in the case of Lamé solutions of type 2. (The other cases can be dealt with in the same way.) Assume that  $\tilde{S}(z)$  is a polynomial of degree  $n$  such that  $S(z) := (z - e_1)^{\frac{1}{2}}(z - e_2)^{\frac{1}{2}}\tilde{S}(z)$  solves the equation

$$4Q(z) \left[ S''(z) + \frac{1}{2} \left( \sum_{i=1}^3 \frac{1}{z - e_i} \right) S'(z) \right] + V(z)S(z) = 0$$

for some linear polynomial  $V(z)$ . Recall that the (unique) root of each such  $V(z)$  is also a root of the scaled spectral polynomial  $R_n^{\frac{1}{2}, \frac{1}{2}, 0}(n(n+1)t)$ . Substituting  $S(z) = (z - e_1)^{\frac{1}{2}}(z - e_2)^{\frac{1}{2}}\tilde{S}(z)$  one gets after some straightforward calculations that  $\tilde{S}(z)$  satisfies the equation

$$4Q(z) \left[ \tilde{S}''(z) + \frac{1}{2} \left( \frac{3}{z - e_1} + \frac{3}{z - e_2} + \frac{1}{z - e_3} \right) \tilde{S}'(z) \right] + \tilde{V}(z)\tilde{S}(z) = 0, \quad (21)$$

where  $\tilde{V}(z) = V(z) + (z - e_1) - (z - e_2) + 4(z - e_3)$ . Notice that in the coefficient in front of  $\tilde{S}'(z)$  the numerators of the first two simple fractions are increased by 1 while the remaining one is unchanged. Thus we are practically in the situation covered by Theorem 1. The only (slight) difference is that we are not considering the asymptotic distribution of the roots of polynomials  $\tilde{V}(z)$  but that of  $V(z)$ . However, since  $\tilde{V}(z) = V(z) + (z - e_1) - (z - e_2) + 4(z - e_3)$  and the leading coefficient  $\theta_n$  of  $V(z)$  tends to  $\infty$  as  $n \rightarrow \infty$  it follows that both these families of polynomials actually have the same asymptotic root distribution. Finally, note that by (4) the scaled spectral polynomial  $R_n(n(n+1)t)$  equals the product of four corresponding polynomials  $R_n^{\kappa_1, \kappa_2, \kappa_3}(n(n+1)t)$  and that all eight polynomial families  $\{R_n^{\kappa_1, \kappa_2, \kappa_3}(n(n+1)t)\}_{n \in \mathbb{Z}_+}$  have the same asymptotic root distribution when  $n \rightarrow \infty$ . This implies that the family  $\{R_n(n(n+1)t)\}_{n \in \mathbb{Z}_+}$  itself has the same limiting root distribution.  $\square$

### 3. REMARKS AND CONJECTURES

**1.** So far we were unable to extend our method of proving Theorem 1 to the case of a cubic polynomial  $Q(z)$  with complex roots. The main difficulty comes from the fact that Theorem 1.4 of [7] seems to fail in this case. Indeed, if it were true then the support of the resulting root-counting measure would be two-dimensional. However, numerical experiments strongly suggest that this support is one-dimensional, see Figure 2.

**Problem 1.** Generalize Theorem 1 to the case of a complex cubic polynomial  $Q(z)$ .

Based on our previous experience (comp. [4, 5]) we conjecture that:

- (i) The asymptotic root distribution of the Van Vleck polynomials for equation (5) with complex coefficients is independent of the  $\alpha_i$ 's and depends only on the leading polynomial  $Q(z)$ ;
- (ii) Denoting the above limiting distribution by  $\mu_Q$  we claim that its support is straightened out in the canonical local coordinate  $w(z) = \int_{z_0}^z \frac{dt}{\sqrt{Q(t)}}$ .

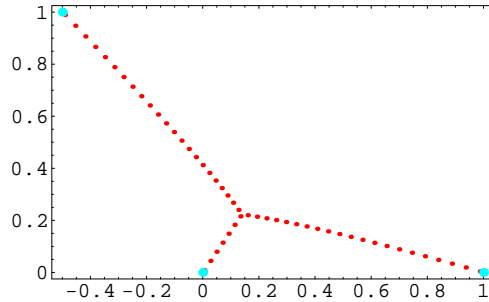


FIGURE 2. The roots of the scaled spectral polynomial  $Sp_{50}(t)$  for  $Q(z) = z(z-1)(z + \frac{1}{2} - i)$ .

**2.** Even more difficulties occur when dealing the more general Lamé equation (2) since in this case Van Vleck polynomials and their roots can not be found by means of a determinantal equation. They are in fact related to a more complicated situation when the rank of a certain non-square matrix is less than the maximal one, see [5].

**Problem 2.** Describe the asymptotic distribution of the roots of all Van Vleck polynomials for equation (2) when  $n \rightarrow \infty$ .

An illustration of this asymptotic distribution is given in the next picture.

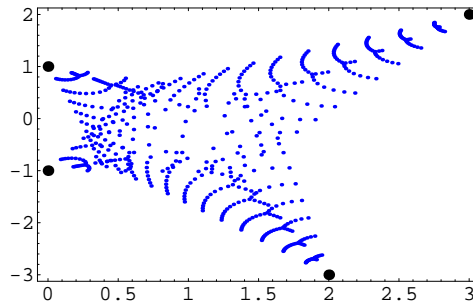


FIGURE 3. The union of the roots of 861 quadratic Van Vleck polynomials corresponding to Stieltjes polynomials of degree 40 for the classical Lamé equation  $Q(z)S''(z) + \frac{Q'(z)}{2}S'(z) + V(z)S(z) = 0$  with  $Q(z) = (z^2 + 1)(z - 3i - 2)(z + 2i - 3)$ .

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